Introduction: Many seismic data processing steps can be viewed as nonstationary shifts in the time or frequency domain. Nonstationary filtering theory was presented in a geophysical context by Margrave (1998), and has since been shown to be useful for a variety of data processing steps from inverse Q-filtering to pre-stack depth migration (Ferguson and Margrave, 2002). Combining nonstationary convolution with basic properties of Fourier transforms leads to the formulation of a set of reversible integral transforms, presented here. Matrix-vector multiplication is shown to be a valuable tool for implementing these integral transforms.

Performing seismic data processing steps and Fourier transforms by matrix-vector multiplication separately are not new ideas (Claerbout, 1992). However, extending these ideas to construct general forward and inverse integral transforms is now possible under the theory of nonstationary filtering. When implemented as nonstationary shifts, data processing steps are exactly reversible, because they do not require interpolation. By constructing discrete matrix operators for these integral transforms, they can be applied and removed exactly within computational accuracy. Although less efficient than conventional algorithms, implementation of these transforms by matrix-vector multiplication reveals symmetry in the matrix operators that may provide means for improving their efficiency.

Theory: Nonstationary convolution is a general form of convolution (Margrave, 1998) that has stationary convolution as a special case. Nonstationary convolution has the form,

$$g(\tau) = \int_{-\infty}^{\infty} a(\tau - t, t) h(t) dt \,. \tag{1}$$

Discretely, convolution can be viewed as a matrix-vector multiplication (Karl 1989, Claerbout 1992),

$$\mathbf{g} = \mathbf{Ah}, \tag{2}$$

where **g** and **h** are column vectors of N time elements, and **A** is an NxN matrix. In the integral form, t is the input time-coordinate, and τ is the output time-coordinate. The filter matrix **A** has its horizontal axis as input time, and its vertical axis as output time. Each column of **A** is the output-time-shifted filter on **h** applied at a single input time. Unlike stationary convolution, the filter is not only shifted along the output time axis, but also changes with input time. Matrix-vector multiplication discretely replaces the integral as it sums the product of the filter and the trace over the input index of **h**, leaving the output index of **g**.

Nonstationary convolution can be implemented in the Fourier domain, but unlike stationary convolution, with no immediate reduction in computational cost (Margrave, 1998). The mixed-domain allows a general framework for constructing nonstationary filter matrices by convolution, while quantifying the frequency-content distortions of nonstationary filtering.

To develop a general form of mixed-domain data processing transforms, we first define a convention to use for the forward and inverse Fourier transform,

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \quad \text{and} \quad h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{-i\omega t} d\omega \,, \tag{3}$$

where *h(t)* is a single seismic data trace that is a function of input time, *t*. Often, a seismic processing step is simply a shift of the data from input time to output time,

$$\Delta = \tau(t) - t \,. \tag{4}$$

For even a single trace, this shift value is nonstationary, as it changes depending on input time. The shift-theorem allows a time-shift to be applied to a series easily in the Fourier domain, by multiplication with an exponential. Nonstationary shifts can be applied in a simliar manner. As a shift is applied, the spectrum of the original series is altered, so inverse Fourier transforming yields a trace that is now a function of output time, τ:

$$h(\tau) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega\Delta} e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega\tau(t)} d\omega \,. \tag{5}$$

Since (5) does not integrate over time, nothing new is needed to apply a nonstationary shift versus a stationary one. Implementing this integral tranform by matrix-vector multiplication is done by performing a regular discrete Fourier transform to the trace, constructing the mixed-domain operator matrix, **A**, and then mulitplying **A** with the spectrum of the trace. The horizontal axis of **A** will be discrete values of frequency, ω, and the vertical axis will be output time, τ. Each element of **A** is given as,

$$a_{ij} \equiv \frac{e^{i\omega_j t_i}}{2\pi} e^{i\omega_j \Delta_i} = \frac{e^{i\omega_j \tau_i}}{2\pi}, \tag{6}$$

with no summation. This matrix can be constructed as a Hadamard matrix product (entry-wise, that is, $(\mathbf{A}\circ\mathbf{B})_{ij}=a_{ij}\, b_{ij}$, no summation (Horn and Johnson, 1994)) of a discrete Fourier transform matrix with a shifting matrix. This approach isolates the shifting matrix, which is all that changes between traces or processing steps.

The magnitude of the nonstationary shift for the inverse process must be kept the same as it was for the forward process. We simply reverse the direction of the shift, again using the shift-theorem, by changing the sign in the shifting exponential. The inverse integral transform has the goal of recovering the spectrum of the original signal. Since the current input time axis, τ, can also be expressed in terms of t using equation (4), an integration-variable substitution is all that is necessary to integrate over the un-distorted time axis, t, in terms of τ. Integrating over t allows the inverse transform to be expressed as a normal forward Fourier transform over t, alongside the reversed shifting exponential. This can be shown to ensure that ω is the output axis. The inverse tranform is then expressed as:

$$H(\omega) \equiv \int_{-\infty}^{\infty} h(\tau) e^{-i\omega\tau(t)} d\tau = \int_{-\infty}^{\infty} h(\tau)\alpha e^{-i\omega\Delta} e^{-i\omega t} dt, \tag{7}$$

where,

$$\alpha = \frac{d\tau}{dt}. \tag{8}$$

A standard inverse Fourier transform of the spectrum in (7), using the convention defined in equation (3), will yield the original trace as a function of t.

The factor, α, can be used to quantify image distortions caused by the forward processing step. The forward transform does not alter the amplitudes of seismic events seen in the original input trace. This is a desireable feature of the forward transform for seismic data processing, because often subsequent processes, such as stacking or AVO analysis, require the true arrival amplitude to be in the corrected position on the output trace. However, maintaining the amplitudes of the events while applying a nonstationary shift distorts the frequency content of the data. If the amplitude spectrum is shifted towards lower frequencies, this results in a non-physical adding of energy to the spectrum, as in the case of the NMO example below. In the NMO case, the frequency distortion is commonly known as "NMO stretch", and in all cases, the distortion is exactly predicted by α. During the inverse transform, as the events are shifted back to their original frequencies by summation over time, the added energy must be accounted for, which is done by α.

To implement the inverse transform with matrix-vector multiplication, a new filter matrix, **B**, is constructed. **B** removes the processing step while simultaneously Fourier transforming the trace. Each element of **B** is given by:

$$b_{ij} \equiv \alpha_{ij} e^{-i\omega_i \Delta_j} e^{-i\omega_i t_j} = \alpha_{ij} e^{-i\omega_i \tau_j}, \tag{9}$$

with no summation. **B** is matrix-multiplied with the corrected trace to output the original spectrum. The original trace can be recovered with an inverse Fourier transform algorithm.

Examples: The well-known second order NMO equation used here, demonstrates the non-reversible nature of interpolation-based algorithms:

$$t_x = \sqrt{t_0^2 + \frac{x^2}{v(t_0)^2}}. \tag{10}$$

The calculated NMO arrival time, t_x, does not often coincide with an integer multiple of the time-sampling interval. Conventionally, some method of interpolation in either the time or frequency domain is used to extract the value of the trace at the predicted time. A popular option is interpolation in the time-domain using, for example, an N-point sinc interpolator (Karl, 1989). For finite sinc-methods, NMO is not completely reversible, as the interpolation for the forward process requires a non-unique redistribution for the reverse process.

To apply and remove the NMO correction using the integral transform, we must construct the matrix operators **A** and **B** for each trace. It may seem counter-intuitive, but t_x will represent the output time, while t_0 will represent input time. Data is indeed shifted from t_x to t_0, but the NMO transform actually treats each absolute coordinate in the input trace as a possible t_0 value, and outputs the data at t_x to that coordinate in the output trace. Only two values are calculated to construct each element of the NMO and NMO^{-1} matrices,

$$\alpha \equiv \frac{dt_x}{dt_0} = \frac{t_0 - \frac{x^2}{v^3}\frac{dv}{dt_0}}{t_x}, \text{ and } \Delta = t_x - t_0. \quad (11)$$

The distortion factor here, α, represents NMO-stretch for a variable velocity model. Barnes (1992) derived the exact same expression for NMO-stretch in terms of instantaneous frequency (Yilmaz, 2001). Example NMO and NMO^{-1} shifting matrices for a single trace are shown in figures 1a and 1b, respecitively. Figure 2b shows the results of applying and removing NMO from a shot gather (figure 2a) using an 8-point sinc-function algorithm, and then taking the difference from the input data. A simple vertical-gradient velocity model was used for the process, but the sinc-function algorithm cannot recover the exact section. The same forward and inverse process is performed on the same gather, with the same velocity model using the full matrix-vector multiplication. The difference section is shown in figure 2c, with noticeable improvement over the conventional results. The difference plots are both scaled equally by 100, but figure 2c confirms that a small amount of data is indeed lost during the inverse process of the sinc-function interpolation algorithm.

Discussion: The general form of one-dimensional forward and reverse data processing transforms has been developed, and then demonstrated for a variable-velocity NMO correction. This general form also provides insight to distortions in seismic images caused by nonstationary filtering. The transforms here demonstrate a general method for predicting and quantifying the frequency-content distortions for any one-dimensional processing step.

Under the theory of nonstationary filtering, many 1-D processing steps can now be viewed as reversible transforms. This opens the door to quantitative processing techniques that may be more effective on data viewed in a corrected domain (i.e., NMO-corrected). If that correction is not required or ideal for subsequent processing, the data can be manipulated in that corrected-domain, the effects of the correction can be exactly removed by inverse transform, and the processing flow can continue. Extending this method to more processing steps could essentially provide a new set of domains in which to work with seismic data.

As seen in figure 1, the shifting matrices display symmetry along the frequency axes, which will immediately allow computational efficiency of the integral transforms to be improved. Additionally, approximating the transform kernels should improve efficiency, while providing a means to quantify and predict error. With the increasing demand for processing steps that accommodate amplitude preservation, reversible processing by integral transform and matrix-vector multiplication could provide an efficient means for novel non-linear processing flows that are quantitatively valid.

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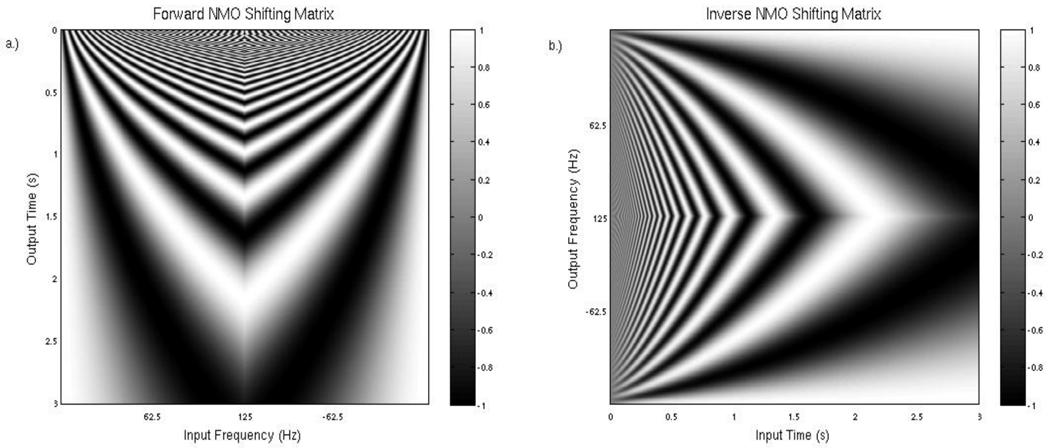

Figure 1. The shifting matrices for the forward (a.) and reverse (b.) NMO shifting exponentials. Only the real part of each matrix is displayed, and the colorbar is normalized. The operator matrices, **A** and **B**, are constructed by the Hadamard matrix product of the Fourier kernel matrix with (a.) and the inverse Fourier kernel matrix. The shifting matrices are calculated for each trace; in this case the trace was at 2 km offset, with 4 ms sampling, for a velocity model that increased linearly from 2-3 km/s.

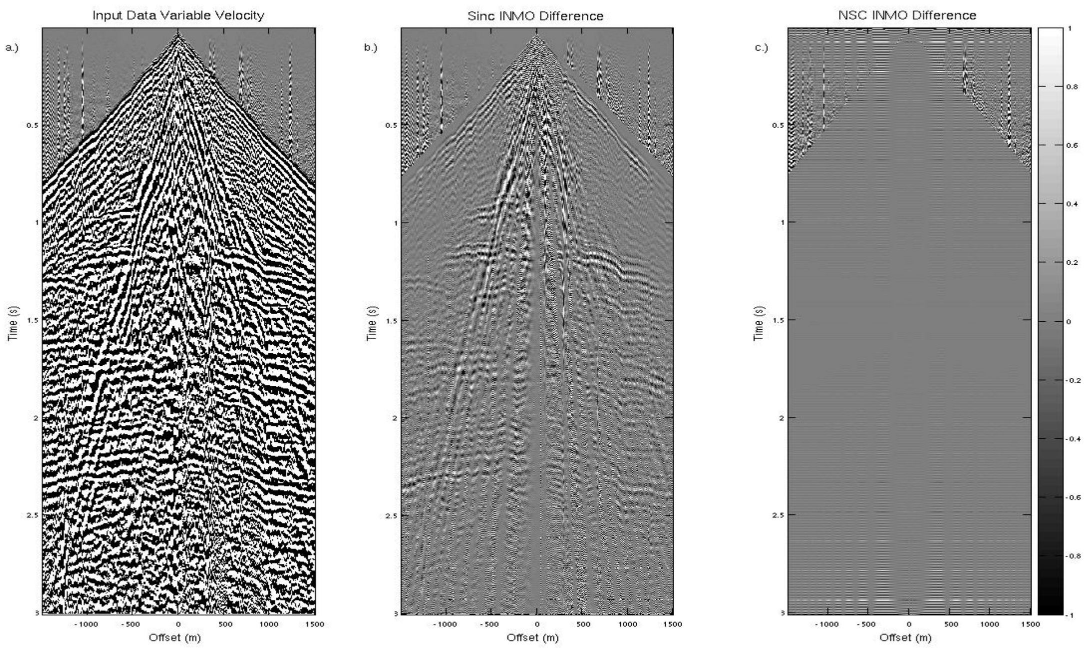

Figure 2. NMO is applied and then removed on the shot gather (2a.) using an 8-point sinc function algorithm and separately, using integral transform. The difference between the input data and the inverse sinc-function NMO is shown in 2b. In 2c, the same difference is shown, but using the results from the integral transforms by matrix-vector multiplication.