Generalized Frames for Gabor Operators in Imaging

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POTSI PSEUDODIFFERENTIAL OPERATOR THEORY & SEISMIC IMAGING

CREWES Consortium for Research in Elastic Wave Exploration Seismology

Outline

Wavefield propagation in a heterogeneous medium

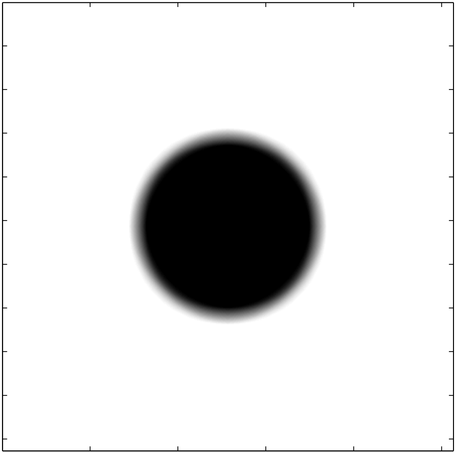

Figure: Numerical simulation of seismic propagation.

Wavefield propagation in a heterogeneous medium

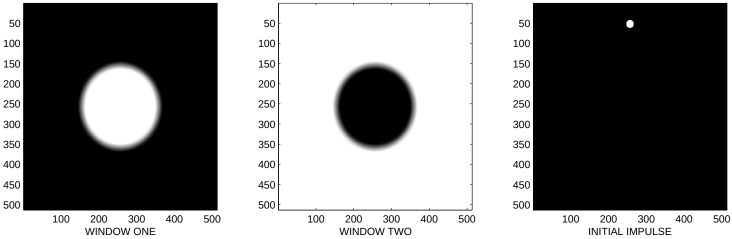

Figure: Numerical simulation, two parts summing to the whole.

Complex medium

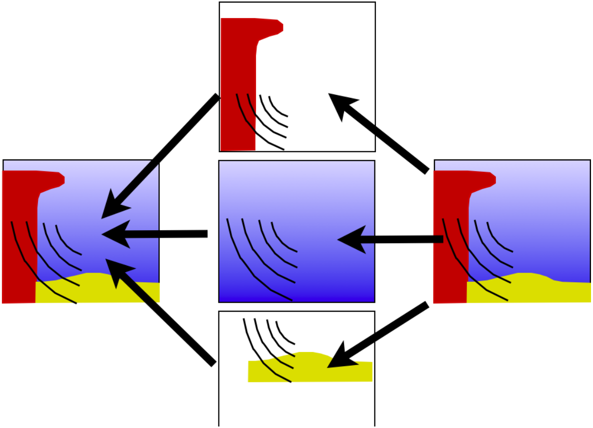

Figure: Three or more regions, each could be complex.

Data flow

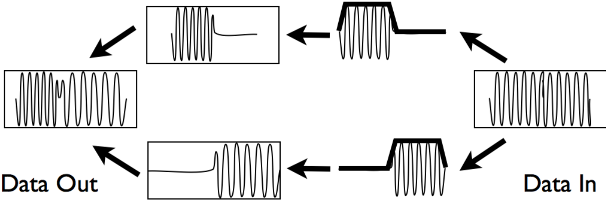

Figure: Two windows for data, process, recombine

Data flow

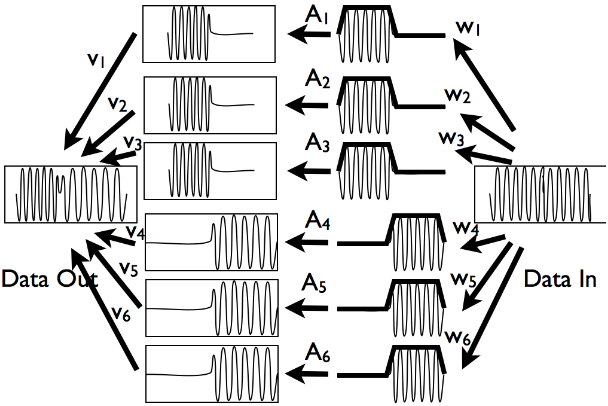

Figure: Many windows for data, process, recombine

Data flow - as mathematical operators

Represent the transformations as block matrices

$$g = \begin{bmatrix} V_1^* & V_2^* & \cdots & V_n^* \end{bmatrix} \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} f$$

In operator notation, we write

$$g = [V^* A W] f$$

In summation notation, we write

$$g = \sum_k V_k^* A_k W_k f$$

Generalized frame

Here is a generalization of frame theory:

Definition

A set of operators $\{W_1, W_2, \ldots, W_n\}$ forms a generalized frame if there are constants $a, b > 0$ with

$$c \cdot \mathbb{I} \leq \sum_k W_k^* W_k \leq b \cdot \mathbb{I}.$$

When the W_k operators are multiplication by functions $w_k(x)$, this definition means

$$a \leq \sum_k |w_k(x)|^2 \leq b, \text{ for all } x.$$

The W_k could give localization in space, in time, or in frequency.

Frame theory

The difference between a basis and a frame

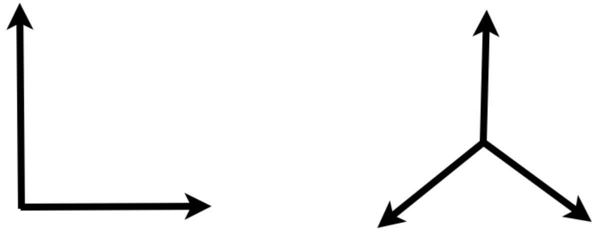

Figure: A basis, and a MB frame in 2D

– a frame is a set of vectors that spans a linear space, but a bit more redundant than a basis.
– wavelet, Gabor, curvelet, ridgelet frames ...
– frame theory gives algorithms that treat redundancy efficiently.

Frame theory

- Analysis operator $W = [W_1, W_2, \ldots, W_n]^t$, Synthesis op. W^*
- Frame operator $S = W^* W$ (positive, invertible)
- normalized frame $\widetilde{W_k} = W_k S^{-1/2}$
- partition of unity (POU) condition

$$\sum_k \widetilde{W_k}^* \widetilde{W_k} = \mathbb{I}$$

Theorem

If the generalized frame $\{W_1, W_2, \ldots, W_n\}$ form a POU, then the operator norms on the windowed operator satisfies

$$||\sum_k W_k^* A_k W_k|| \leq \max ||A_k||.$$

In particular, if the A_k are each stable wavefield propagators, then the combined windowed propagator is stable.

Frame theory

– POU condition on the W_k is important for the functional calculus
– can use pre, post-windows $\{W_1, W_2 \ldots\}$ and $\{V_1, V_2 \ldots\}$ with

$$\sum_k V_k^* W_k = \mathbb{I}.$$

– however, the norm of the combined operator may grow with the number of windows

$$||\sum_k V_k^* A_k W_k|| \approx \sqrt{n} \max ||A_k||$$

which can cause numerical instability.

An unstable propagator

By picking windows and shifts just right, get unstable propagator.

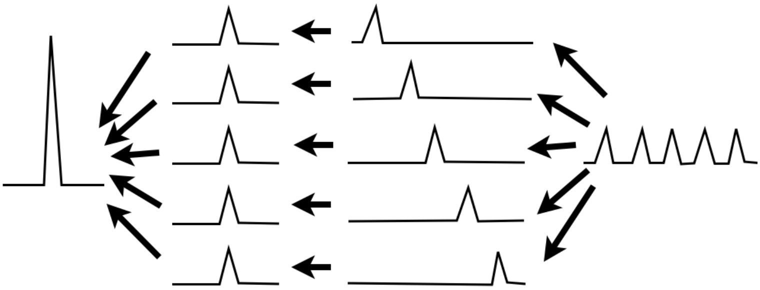

Figure: Breaking up a waveform into five, shift just right

Input waveform has energy $\sqrt{1^2+1^2+1^2+1^2+1^2} = \sqrt{5}$
Output waveform has energy $\sqrt{5^2} = 5$, increases!

Possible solutions

– shifting the waveform seems to be an issue (FIOs)
– can hope for better results using differential operators (PsDOs)
– controlling the step size may control instability (we control step size already for fidelity, could be why our current methods are stable anyway)

Preserving minimum phase signals

When designing these windowed operators, should preserve as much physics as possible.

Eg: dynamite blast is "minimum phase":
most of the energy is concentrated near the start. As it propagates, it remains minimum phase.

What kind of linear operators preserve minimum phase?

Minimum phase signals

Theorem

(Paley-Wiener) If a signal is causal, then its log amplitude spectrum is integrable. In particular, not too many zeros.

Any causal signal has a minimum phase equivalent $(f_0, f_1, f_2, \ldots)$, given by a complex function

$$
\begin{aligned}
F(z) &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log A(\theta)\,d\theta\right) \\
&= \sum_{n=0}^{\infty} f_n z^n
\end{aligned}
$$

which is analytic on the disk $|z| < 1$. No zeros or poles! (outer)

Paley rests in Banff

Figure: Fossil Mountain, north of Banff townsite, Paley perished 1933.

Decay operator

A nonstationary, min-phase preserving operator is given by the map

$$(f_0, f_1, f_2, f_3, \ldots) \mapsto (r^0 f_0, r^1 f_1, r^2 f_2, r^3 f_3 \ldots),$$

for any positive constant $r \leq 1$. This maps analytic functions as

$$F(z) \mapsto F(rz)$$

and it is easy to check this maps outer to outer.

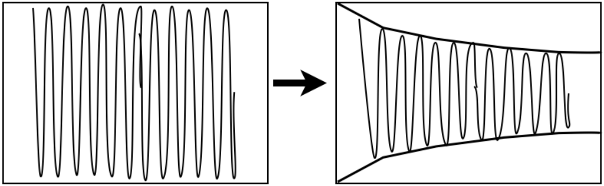

Figure: Constant decay

The only diagonal min-phase preserving operators we found.

General Q-decay operator

A more general min-phase preserving operator can be prescribed on analytic functions as

$$F(z) \mapsto G(z)F(z\phi(z)),$$

where $G(z), \phi(z)$ are both outer functions.
The $G(z)$ gives stationary convolution (min-phase preserving). The other part gives a type of Q-attenuation decay. The delta spike

$$\delta_n = (0, 0, 0, \ldots, 1, 0, 0, \ldots)$$

maps to the function $G(z)(z\phi(z))^n$ which has Fourier spectrum

$$H_n(\theta) = G(e^{i\theta})e^{nh(\theta)}$$

With increasing time shifts n, we get exponentially increasing decay, which is frequency dependent. $(Re(h) \leq 0)$

Is that all there is?

– Have not been able to find any other min-phase preserving operators. Only some partial results.

– Can show the image of the delta spikes show spectral decay:

$$|H_{n+1}(\theta)| \leq |H_n(\theta)| \text{ for all } n, \theta.$$

– There are convexity conditions as well.

– Still a gap to show the Q-decay operators cover everything.

Is that all there is?

The spectral decay condition can be visualized:

a_{00}	0	0	0	...
a_{10}	a_{11}	0	0	...
a_{20}	a_{21}	a_{22}	0	...
a_{30}	a_{31}	a_{32}	a_{33}	...
...	...	...	...	...

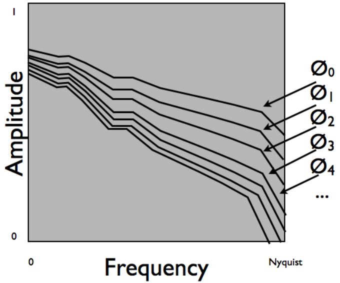

Figure: Each column of min-phase preserving operators in a min-phase signal signal. Plot of spectra shows a decreasing sequence of functions.

Pushing signal to min phase

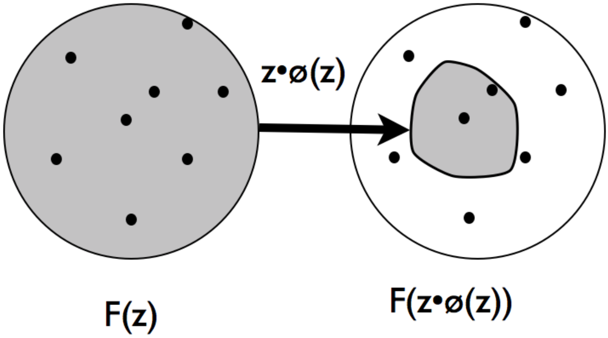

Figure: Map on the unit disk in complex plane

Applying a nonstationary min-phase preserving operator gets rid of zeros. Makes a signal look ''more like" min-phase.

Conclusions

– generalized frame theory gives a context for analyzing our window algorithms.
– covers windows in space, time, or frequency slices.
– POU condition with symmetric windows gives stable propagators.
– can construct counterexamples in the non-symmetric case.
– enforcing minimum phase properties on operators leads to analytic function theory, outer functions, maps of the unit disk.
– future work is to use this framework to better design operators for our imaging needs.

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