

The computation of gridded traveltimes when assuming circular wavefronts

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Summary

Traveltimes computations are an integral part of modelling and imaging seismic data by providing efficient kinematic information on the location of propagated energy. The traveltimes may be computed analytically using simplifying assumptions, or may be estimated on a complex geological structure using raytracing or gridded traveltimes. A basic requirement for the propagation of gridded traveltimes is the estimation of one point on a corner of a square given the traveltimes on the other three corners. A number of solutions are available to solve for the unknown time and are based on either a plane-wave assumption, a finite difference solution to the Eikonal equation, or an assumption that the wavefront at the square is curved. A solution for a curved wavefront assumption requires estimating the center of curvature, and requires solving a quartic equation. An alternate method is presented to estimate the center of curvature for a curved wavefront that uses an iterative procedure and does not require solving the quartic equation.

Introduction

Accurate modelling and Kirchhoff depth migrations compute traveltimes by estimating the time along a ray path or on a grid. These times may represent the propagation of energy from a scatterpoint to the surface, defining a diffraction shape for a poststack migration. They may also define the traveltimes between a source and scatterpoint, or scatterpoint to a receiver for a prestack migration.

The gridded method places a grid on the velocity field, then, assuming a given starting or source point, computes the traveltimes on the grid surrounding the point. The traveltimes on the adjacent grid points are computed and the process repeated to expand the area of known traveltimes. This region expands away from the source point until the desired objective (such as traveltimes on the surface, or the arrival at a desired point) is met, as illustrated in Figure 1a. This figure shows a partial grid and the corresponding traveltime contour for a constant velocity. The traveltimes at the surface are then mapped to define a diffraction on a time section in (b).

The velocity in each square formed by the grid is assumed to be locally constant, and in structured areas will vary from square to square. The inclusion of anisotropy parameters with the velocity information enables state-of-the-art anisotropic prestack depth migration (Perez, 2004).

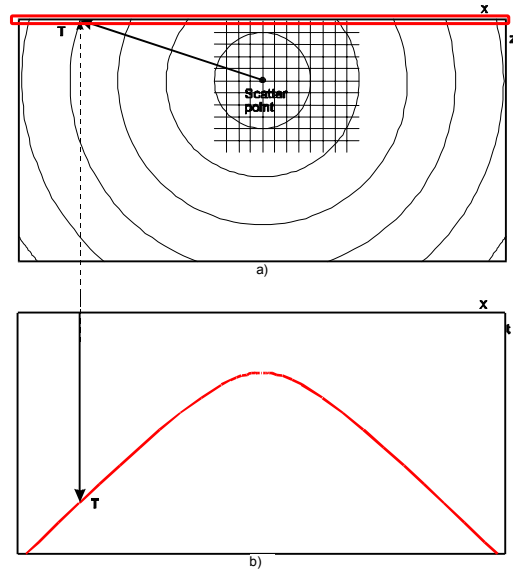


Figure 1: Part of a grid is shown on a) that also contains constant velocity traveltime contours from a scatterpoint. Times at the surface are mapped to the time section in b) to illustrate a zero-offset diffraction.

The spreading of times on the grid may be accomplished by a number of techniques that involve estimating the time on a corner of a square when the times on the other three corners are known. The geometry for estimating this time is illustrated in Figure 2, which shows one square taken from the grid with time t_1 at the origin, and times t_2 and t_3 on the adjacent corners. The unknown time t_4 that we are estimating is on the corner opposite t_1 . The square has a local velocity v , and each side has a distance h .

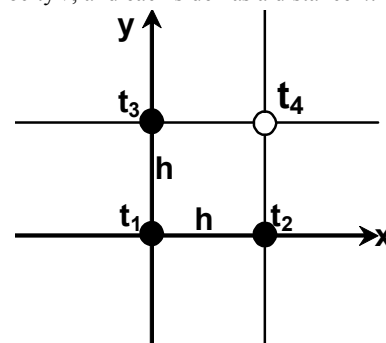


Figure 2: Geometry for one square for gridded traveltime estimations. Times t_1 , t_2 , and t_3 are known, t_4 is unknown.

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A solution to the problem is illustrated in Figure 3 that assumes the energy is propagating through the square as a plane-wave. The traveltime t_3 is interpolated on the x axis using the two known times t_1 and t_2 . Connecting the corner time t_3 with the interpolated time t_3 defines the angle of a plane-wave. The construction of a parallel plane-wave that passes through the corner point of t_4 will intersect the x axis at time t_4 .

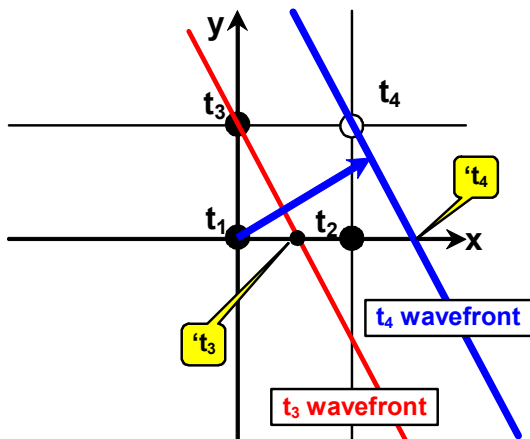


Figure 3: Plane wave assumption

Note the distance on the x axis between t_3 and t_1 is equal to the distance between t_4 and t_2 , giving a simple estimate for the time on the wavefront t_4 , i.e.,

$$t_4 = t_2 + t_3 - t_1. \quad (1)$$

Vidale (1988) introduced a method for computing t_4 that was based on a finite difference solution to the eikonal equation, i.e.,

$$t_4 = t_1 + \sqrt{\frac{2h^2}{v^2} - (t_2 - t_3)^2} \quad (2)$$

where h is the dimension one a side of the square and v is the local velocity of the square. This method still assumes a plane-wave, but chooses a more appropriate angle for the wavefront.

Vidale also mentions in his paper that a curved wavefront may be assumed that is locally circular at the square. This geometry is illustrated in Figure 4 and assumes the velocity is now constant over the region that contains the virtual center of curvature at a point (x_0, z_0) . The time at the center of curvature is designated t_0 , which is not assumed to be zero, as the wavefronts typically have varying curvature. Once the time and location of the center of curvature is known, then t_4 can be computed from

$$t_4 = \frac{1}{v} \sqrt{(x_0 + h)^2 + (z_0 + h)^2}. \quad (3)$$

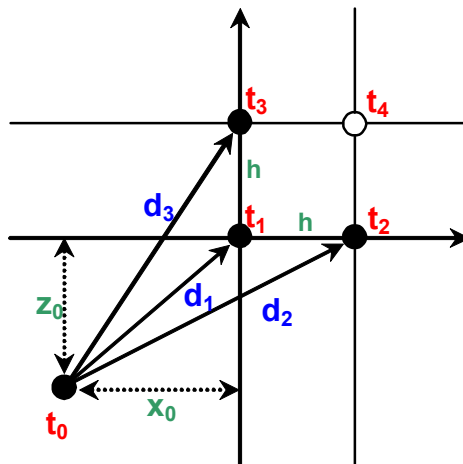


Figure 4: Geometry for curved wavefront assumption where the center of curvature is (x_0, z_0) .

The initial objective is to estimate the location of the center of curvature (x_0, z_0) , and to define the time t_0 at this virtual source. Vidale points out that the three equations (See Eq. (4)) that define the times of t_1 , t_2 , and t_3 from the center of curvature may be combined into a quartic equation that can be solved to get four possible solutions for x_0 . This task is straight forward, but not trivial, and is the reason for the iterative approach.

This problem is a simplified form of the Loran C navigation method in which the differences in traveltimes from radiating antenna are identified as hyperbolic contours on maps. In our application the traveltime differences $(t_2 - t_1)$ and $(t_3 - t_1)$ define the hyperbolae as illustrated in Figure 5. This figure shows four hyperbolae that intersect at four locations, corresponding to the four solutions of the quartic equation. The relative amplitudes of t_2 to t_1 and t_3 to t_1 provide additional information that eliminate one side of the hyperbolic pair, reducing the number of possible solutions to two. Logic must then be used to decide which solution is chosen. The significant effort to solve this problem is illustrated by the size of the equations which are too large to be included. The quartic equation alone contains over 200 characters of the hyperbolic parameters, and one solution of a quartic equation using MATHEMATICA contains over 1000 characters and occupies nearly a page of text.

The method that follows has simpler equations to solve for the center of curvature, but does involve iterating to an acceptable accuracy.

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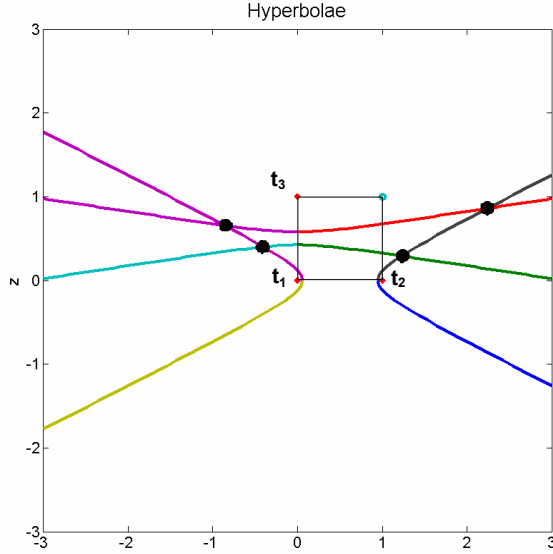


Figure 5: Two sets of hyperbolae that result from two time differences, $t_2 - t_1$ and $t_3 - t_1$.

The iterative solution

From the geometry of Figure 4 we can write three equations for the d 's as

$$\begin{aligned} d_1 &= v(t_1 - t_0) = \sqrt{x_0^2 + z_0^2} \\ d_2 &= v(t_2 - t_0) = \sqrt{(h - x_0)^2 + z_0^2} \\ d_3 &= v(t_3 - t_0) = \sqrt{x_0^2 + (h - z_0)^2} \end{aligned} \quad (4)$$

We can rewrite these equations as

$$\begin{aligned} x_0^2 + z_0^2 &= v^2(t_1^2 - 2t_1t_0 + t_0^2) \\ x_0^2 - 2x_0h + h^2 + z_0^2 &= v^2(t_2^2 - 2t_2t_0 + t_0^2) \\ x_0^2 + z_0^2 - 2z_0h + h^2 &= v^2(t_3^2 - 2t_3t_0 + t_0^2) \end{aligned} \quad (5)$$

Subtracting the first equation from the second and third we obtain

$$\begin{aligned} x_0 &= \frac{v^2}{2h} [2t_0(t_2 - t_1) - t_2^2 + t_1^2] + \frac{h}{2} \\ z_0 &= \frac{v^2}{2h} [2t_0(t_3 - t_1) - t_3^2 + t_1^2] + \frac{h}{2} \end{aligned} \quad (6)$$

These new equations are only dependent on t_0 ; all the other terms are constant. They are then substituted into

$$t_0 = t_1 - \frac{1}{v} \sqrt{x_0^2 + z_0^2}, \quad (7)$$

giving one equation where the only variable is t_0 , i.e.,

$$t_0 = t_1 - \frac{1}{v} \sqrt{\left\{ \frac{v^2}{2h} [2t_0(t_2 - t_1) - t_2^2 + t_1^2] + \frac{h}{2} \right\}^2 + \left\{ \frac{v^2}{2h} [2t_0(t_3 - t_1) - t_3^2 + t_1^2] + \frac{h}{2} \right\}^2} \quad (8)$$

This equation could be simplified by squaring, but that introduces additional solutions that must be identified and eliminated. In its present form equation (8) can be written as a function $f(t_0) = 0$ that is suitable for a Newton-Raphson (NR) iterative solution. The function is differentiable, i.e.,

$$\begin{aligned} \frac{df(t_0)}{dt_0} &= 1 + \frac{1}{v} \left[\left\{ \frac{v^2(t_2 - t_1)}{2h} [2t_0 - t_2 - t_1] + \frac{h}{2} \right\}^2 + \left\{ \frac{v^2(t_3 - t_1)}{2h} [2t_0 - t_3 - t_1] + \frac{h}{2} \right\}^2 \right]^{-1/2} \cdot \\ &\left[2 \left\{ \frac{v^2(t_2 - t_1)}{2h} [2t_0 - t_2 - t_1] + \frac{h}{2} \right\} \frac{v^2(t_2 - t_1)}{h} + 2 \left\{ \frac{v^2(t_3 - t_1)}{2h} [2t_0 - t_3 - t_1] + \frac{h}{2} \right\} \frac{v^2(t_3 - t_1)}{h} \right] \end{aligned} \quad (9)$$

Given a starting point at $t_{0,1}$, the NR method uses the derivative to predict an improved solution $t_{0,2}$.

Two solutions for $f(t_0) = 0$ are possible and depend on the starting point of the iterative solution. Starting with $t_0 = 0$ is a good choice and is suitable for all points in the third quadrant where both x_0 and z_0 are less than zero.

Testing and evaluating the methods

The accuracy of the different methods were evaluated using an array of points that represented many different centers of curvature. They were defined in the third quadrant with 100 by 100 points and with a range of $5h$ by $5h$ from the axis. At each center point, the traveltimes t_1 , t_2 , and t_3 were computed, and then, using only those times, the traveltimes t_4 was estimated.

The number of iterations required to reach a minimum threshold was recorded and displayed in Figure 6. A threshold of 10^{-8} for the normalized error in t_0 was used, and an average iteration number appears to be less than 5. The number of iterations outside this quadrant may be higher, especially on the x and y axis.

The relative errors in estimating t_4 in the third quadrant are compared in Figure 7 using (a) the plane-wave assumption, (b) the Vidale finite difference method, and (c) the iterative method. The plots in (a) and (b) have a maximum error of one percent, while (c) has a maximum error of $1.5E-12$. In this region that was limited to five square dimensions ($5h$), the plane wave solution has the largest error, while the eikonal method also showed some significant error.

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However it should be noted that their computational requirements are significantly less than the iterative method.

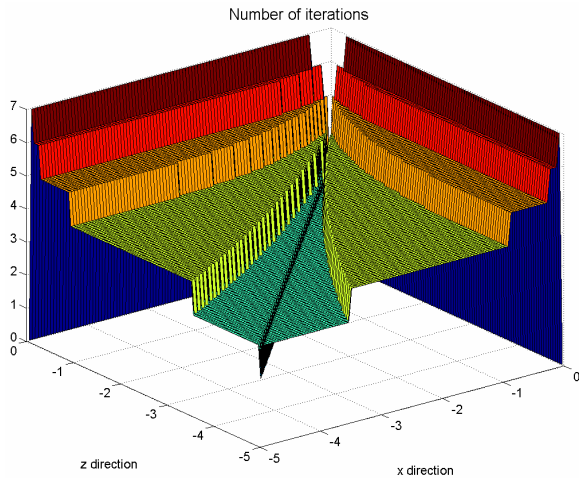


Figure 6: Iterations required to get an error in the third quadrant that is less than 10^{-8} .

Conclusions and comments

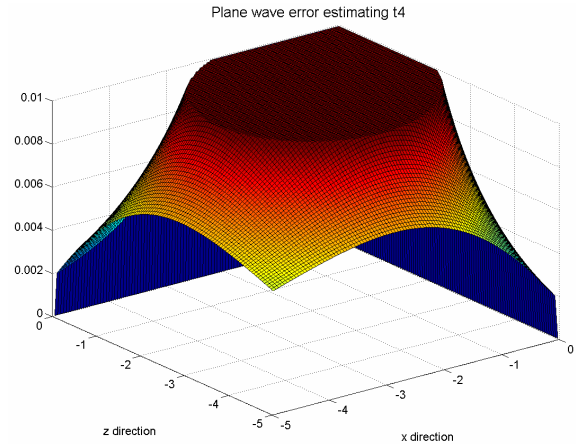
An iterative method was presented to compute a gridded traveltimes estimate that assumes curved wavefronts. The method is simpler than the quartic solution. The number of iterations required for very accurate estimates is typically five or less. Its accuracy was compared with that of a plane-wave assumption and with Vidale's finite difference method.

When the center of curvature is relatively close to the dimensions of the grid, i.e. less than $5h$, then it may be more appropriate to use the curved wavefront assumption than Vidale's method.

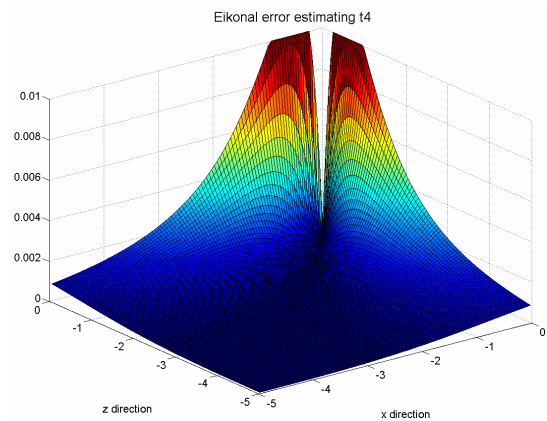
References

Perez, M. 2004, *Traveltimes Tomography in Transversely Isotropic Media*, PhD Dissertation, University of Calgary.

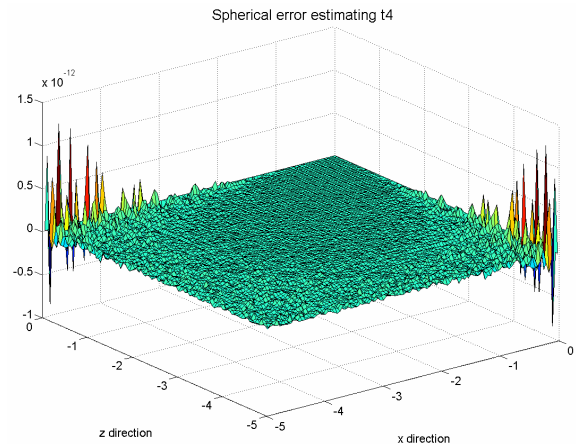
Vidale, J., 1988, *Finite-difference Calculation of traveltimes*, *Bulletin of the Seismological Society of America*, Vol. 78, No. 6, pp. 2062-2076.



(a)



(b)



(c)

Figure 7: Comparison of the errors in estimating t_4 , with a) the plane wave method, b) the Vidale method, and c) the iterative method.