Stabilizing explicit frequency-space migration using local WKBJ operators

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Summary

We introduce a new operator for explicit wavefield extrapolation. We modify the commonly used locally homogeneous wavefield extrapolator to include a local vertical gradient, whose purpose is simply to enhance operator stability when spatially localized. The locally homogeneous operator assumes that wavefield extrapolation across a single depth step can be done with straight raypaths using the assumed constant velocity at the output point. Such operators can produce excellent seismic images but the straight ray assumption means that their spatial aperture is infinite, which leads to instability when the operator is localized by spatial windowing. Adjusting the operator to accommodate a suitably chosen positive vertical velocity gradient causes raypath curvature which naturally limits the operator within a finite aperture. The required modification to the locally homogeneous operator is essentially a WKBJ-style integrated phase. The resulting operator has a finite aperture and is sufficiently stable when localized to be used in an explicit depth migration scheme. We demonstrate operator fidelity with excellent images of the Marmousi model.

Introduction

Many current wavefield extrapolation migration algorithms (for example Hale, 1991; Wu, 1994) are space-frequency methods related to or derived from the phase-shift method introduced by Gazdag (1978). In this method, the wavefield is Fourier-transformed over time and the lateral spatial coordinates resulting in a plane-wave decomposition. Under the assumption of constant velocity (i.e. acoustic wavespeed) an appropriate phase-shift is applied to each plane wave to extrapolate the wavefield a single step in the vertical (z) direction, that is across a homogeneous layer. Later, Gazdag and Sguazzero (1984) extended the method to accommodate lateral inhomogeneity in the layer by interpolating a final result from a suite of *reference wavefields* each extrapolated with a well-chosen constant *reference velocity.* This extended method was called PSPI, an acronym for *Phase-Shift-Plus-Interpolation.* A wavefield can be marched though a highly heterogeneous medium by quantizing the velocity model in the vertical direction into a suite of heterogenous layers and recursively applying the PSPI extrapolator to step across each one. Applying a suitable imaging condition at each step results in a high-fidelity depth migration.

Generalized PSPI

Mathematically, 2D wavefield extrapolation from $z = 0$ to $z = \Delta z$ may be represented in abstract operator notation as

$$\Psi(x, z = \Delta z, \omega) = \mathbf{T}_{\alpha(\Delta Z)} \Psi(x, z = 0, \omega), \tag{1}$$

where ω is temporal frequency, x is the lateral spatial coordinate, z is depth, and $\mathbf{T}_{\alpha(\Delta Z)}$ is the wavefield extrapolation operator for a single step through a laterally variable medium. In our approach, $\mathbf{T}_\alpha$ is taken to be a Fourier integral operator (Duistermaat, 1996) where the subscript α is called the *operator symbol* and is a mathematical function of position, wavenumber, and frequency that describes the physics of the propagating waves. In principle, it is possible to find exact symbols for highly complex lateral velocity variations (Fishman et al., 1997), which describe all internal scattering as well as primary transmitted waves. Here we are concerned with more approximate expressions and in particular we begin with the explicit form for equation 1 known as the *locally homogeneous approximation* (Fishman et al., 1997)

$$\begin{aligned}\Psi(x, z = \Delta z, \omega) &\approx \Psi_{LH}(x, z = \Delta z, \omega) \\ &= \int_{\mathbb{R}} \phi(k_x, z = 0, \omega) \alpha\left(k\left(x\right), k_x, \omega, \Delta z\right) e^{ik_x x} dk_x,\end{aligned} \tag{2}$$

where k_x is the wavenumber dual to x, the symbol $\alpha\left(k\left(x\right), k_x, \omega, \Delta z\right)$ is

$$\alpha\left(k\left(x\right), k_x, \omega, \Delta x\right) = \left\{ \begin{array}{c} e^{i \Delta z k_z(x)}, |k_x| \leq \frac{\omega}{v(x)} \\ e^{-|\Delta z k_z(x)|}, |k_x| > \frac{\omega}{v(x)} \end{array} \right. , \tag{3}$$

where

$$\begin{aligned} k_z(x) &= \sqrt{k(x)^2 - k_x^2} \\ k(x) &= \tfrac{\omega}{v(x)}, \end{aligned} \tag{4}$$

and $\phi(k_x, z, \omega)$ is the Fourier transformation over lateral spatial coordinates of the input wavefield $\Psi(x, z, \omega)$. This is called "locally homogeneous" because the form of the symbol is mathematically the same as for the exact homogeneous case except that the actual velocity function $v(x)$ is substituted for the homogeneous velocity. In exploration seismology the locally homogeneous operator is often called *Generalized Phase-Shift Plus Interpolation* (GPSPI) (Margrave and Ferguson, 1999) because the formula can be derived by eliminating the explicit interpolation in PSPI by taking the limiting form when a unique reference velocity is used for each and every output point.

The GPSPI algorithm of equation 2 can be viewed as applying a nonstationary filter (Margrave, 1998), where the spatial form of the filter depends upon x, to the input data $\Psi(x, z = 0, \omega)$. In mathematical analysis, equation 2 is a Fourier integral operator, which is a generalization of pseudodifferential operators, written in the standard, or Kohn-Nirenberg, form.

The wavefield $\Psi(x, \Delta z, \omega)$ produced by equation 2 using symbol α from equation 3 at a specific output point $(x_0, \Delta z)$ is identical to the wavefield $\Psi_0(x, \Delta z, \omega)$, again at the specific point $(x_0, \Delta z)$, produced by using symbol α given by

$$\alpha\left(k\left(x\right), k_x, \omega, \Delta z\right) = \begin{cases} e^{i\Delta z k_{z0}}, |k_x| \le \frac{\omega}{v_0} \\ e^{-|\Delta z k_{z0}|}, |k_x| > \frac{\omega}{v_0} \end{cases}, \tag{5}$$

where

$$\begin{gathered} k_{z0} = \sqrt{k_0^2 - k_x^2} \\ k_0 = \frac{\omega}{v_0}, \end{gathered} \tag{6}$$

and $v_0 = v(x = x_0)$. A new operator is constructed for each required output point along the line using the local velocity at the output point $v(x_0, \Delta z)$.

Method

We will consider the case of local velocity that is not homogeneous but varies linearly in the z direction. In this case, the exponential terms in equation 5 may be replaced by integrations, so that

$$\begin{aligned} &\alpha\left(k\left(x\right), k_x, \omega, \Delta z\right) \\ &= \begin{cases} e^{\left(i\int_0^{\Delta z}\sqrt{\frac{\omega^2}{v(x,z')^2} - k_x^2}dz'\right)}, |k_x| \le \frac{\omega}{v_0} \\ e^{\left(-\left|\int_0^{\Delta z}\sqrt{\frac{\omega^2}{v(x,z')^2} - k_x^2}dz'\right|\right)}, |k_x| > \frac{\omega}{v_0} \end{cases}. \end{aligned} \tag{7}$$

This uses a local (in x) WKBJ approximation (Aki and Richards, 2002), and is a WKBJ extension of the locally homogeneous operator. Though equation 7 can accomodate other than vertical gradients, here we will only consider the vertical case.

Although velocity models of the earth may contain a vertical velocity gradient that can be approximated by a local constant gradient, this is not the reason for this approach. The advantage of this approach is that a velocity with positive gradient in z limits the aperture of the migration operator on the line $(x, 0)$ $x \in \mathbb{R}$. Consider a point source at depth in a medium with a positive vertical velocity gradient. The raypaths leaving away from the point source in the upper hemisphere will all intersect the surface within a finite radius from the source (Figure 1).

In an $\omega - x$ domain implementation of GPSPI, such as the FOCI algorithm (Margrave et al., 2005), the operator is designed to operate on the spatial extent of the data in x. Numerically, this may be truncated at some finite spatial distance. For computational speed, this truncation contains less than all x offsets available. The final operator width is frequently specified in terms of an odd number of "points", with the total width spanning the trace spacing multiplied by the number of points. The operator's radial extent is then calculated by subtracting one from the number of points, dividing by two, and multiplying by the trace spacing.

Recalling that the operator is a nonstationary filter, this truncation will result in the Gibbs phenomenon. This leads to instability in the propagation, and great pains must be taken to correct this problem (Margrave et al., 2005). The FOCI algorithm works by first designing a truncated operator for propagation, and then using Wiener match filtering to stabilize the operator. In addition to FOCI there are other methods for the stabilization of $\omega - x$ domain extrapolators (Hale, 1991, for example).

Due to its intrinsic natural self-truncation, this $v(z)$ approach allows direct spatial truncation by simple windowing with dramatically less instability. Other methods may also provide stability, but the main benefit of the $v(z)$ approach is its simplicity.

Choosing the v(z) parameters

Restricting attention to a specific lateral location at a specific Δz step, we drop the explicit x dependence from our velocity notation. Thus at every lateral position, we seek to replace the actual migration velocity, called here v_{loc}, with an equivalent linear $v(z)$ chosen to achieve a specific operator aperture (or radius) x_r. For the choice of $v(z) = v_0 + A\Delta z$, two constraints are required to uniquely determine the velocity function. Since the $v(z)$ velocity function is designed to stabilize the operator and not to represent any physical gradient, for our first constraint we require that the vertical traveltime across Δz through the $v(z)$ medium be $\Delta z / v_{loc}$. For our second constraint, we choose the gradient such that a ray with a takeoff angle of 90° at the output point is incident at the input depth at precisely the desired operator's aperture radius, x_r.

It follows from results in Slotnick (1959) that the second constraint expresses as

$$x_r = \frac{1}{pA}\left(\sqrt{1 - p^2 v_0^2} + \sqrt{1 - p^2(v_0 + A\Delta z)}\right) \tag{8}$$

for a given horizontal distance reached x, traversed depth Δz, ray parameter $p = (v_0 + A\Delta z)^{-1}$, vertical velocity gradient (accelerator) A, and initial velocity v_0. Solving this result for A,

$$A = \frac{2 v_0 \Delta z}{x_r^2 - \Delta z^2}. \tag{9}$$

The first constraint expresses as

$$\ln\left(1 + \frac{A\Delta z}{v_0}\right) = \frac{A\Delta z}{v_{loc}}. \tag{10}$$

From equation 9, x_r must not approach Δz too closely, or a singularity results.

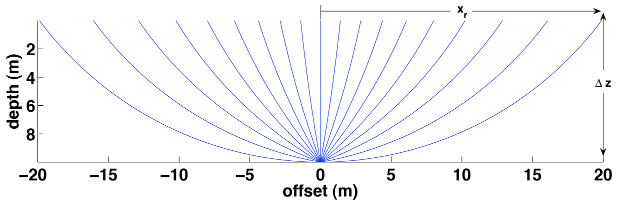

Fig. 1: Ray tracing through a $v(z)$ medium. Maximum horizontal distance was limited to 20 meters, and rays were traced through 10 meters depth. Takeoff angles from $-90°$ to $90°$ were calculated in ten degree increments.

Combining equation 9 and equation 10 and solving for v_0 yields equation 11,

$$v_0 = v_{loc} \ln \left(1 + \frac{2\Delta z^2}{x_r^2 - \Delta z^2} \right) \frac{(x_r^2 - \Delta z^2)}{2\Delta z^2}. \tag{11}$$

With equations 9 and 11, the required $v(z)$ is expressed purely in terms of v_{loc}, x_r and Δz. For each Δz step we can replace the given $v(x)$ function with a $v(x,z)$ which has the same vertical traveltime but yields a naturally finite operator aperture.

An example of ray tracing through a $v(z)$ medium defined using these equations is shown in Figure 1. This medium was described with parameters chosen to limit the maximum aperture radius to 20m through a depth step of 10m in a 2000ms^{-1} reference medium. In this case then, $v_{loc} = 2000$ms^{-1}, $\Delta z = 10$m, $x_r = 20$m, yielding $v_0 = 1533$ms^{-1} and $A = 102$s^{-1}. These parameters are chosen purely to demonstrate the raybending effect: this value for x_r is smaller than would be typically used in actual migration.

Marmousi migration testing

Prestack migrations of the Marmousi model were calculated using the FOCI method of Margrave et al. (2005) where we compared the FOCI operator to an $\omega - x$ operator that included the $v(z)$ symbol in the operator design but was otherwise identical in all operational parameters. Though the $v(z)$ operator shows more instability than the FOCI operator in the wavelike region, it produces high-quality images.

The Marmousi image using standard FOCI is shown in Figure 2, and the image using $v(z)$ is shown in Figure 3. Both procedures yield high-quality images.

Parameter choice and performance

The primary factor in choosing parameters for the operation of the $v(z)$ operator is time-to-compute vs. image quality. Simply put, more points mean a better image, but more points mean a slower calculation. We performed all calculations using MATLAB 7.1 on Linux-based PC computers with 3.06 GHz Intel Pentium 4 CPUs. The 31 point (187.5m) operator required approximately 4 minutes per shot record to migrate, resulting in a total

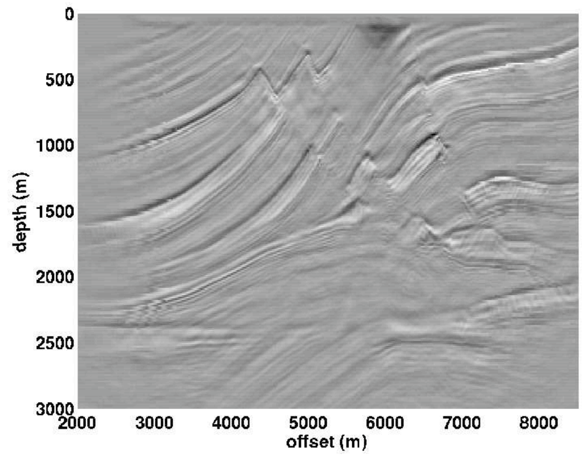

Fig. 2: Marmousi image, FOCI migration with full stabilization. (31 point forward operator, 41 point inverse operator, 31 point final window, 187.5m effective operator width.)

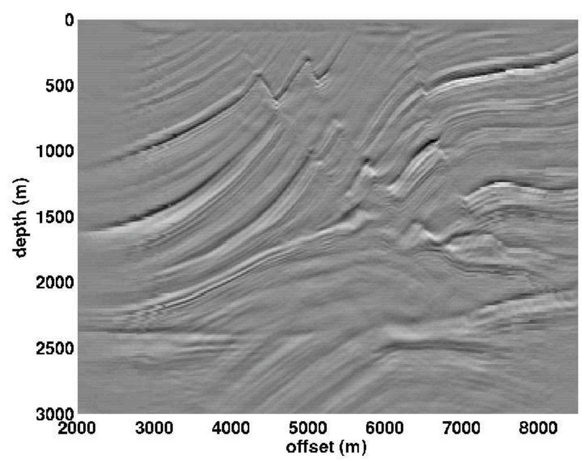

Fig. 3: Marmousi image, $v(z)$ migration. (31 point (187.5m) operator, 40m aperture radius.)

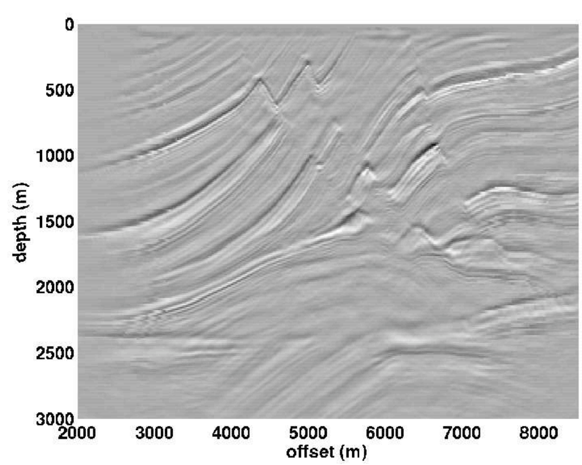

Fig. 4: Marmousi image, $v(z)$ migration. (101 point (625m) operator, 80m aperture radius.)

time of about 16 hours. For example, recalculation with a 101 point operator required approximately 4.6 minutes to calculate each shot record, with a total time of slightly more than 18 hours. Use of this operator does yield small improvements (Figure 4). These were gained at the cost of an extra 15% calculation time.

Aperture radius does not significantly affect calculation time. It does, however, affect the quality of the image if the radius is not significantly larger than the depth step of each extrapolation. Also, the stabilization of the operator is ineffective if the radius is not significantly smaller than the radial extent of the operator. In the Marmousi data set traces are 12.5m apart. Thus a 31 point operator has an effective radial extent of 15×12.5m= 187.5m. Therefore the choice of a 40m aperture radius is smaller than the radial extent of the operator (187.5m), but larger than the depth step of the extrapolation (12.5m).

Testing suggests that a useful aperture width measure is such that the ratio of operator radial extent to aperture radius is somewhat greater than the ratio of aperture radius to depth-step size. Dividing these two ratios and calling the quotient ρ,

$$\rho = \left(\frac{dx(OW-1)/2}{x_r}\right)\left(\frac{dz}{x_r}\right), \tag{12}$$

allows a numerical description of the aperture radius x_r where OW is the operator width in points, dx is trace spacing in meters, and dz is depth-step size in meters. Therefore, aperture radius x_r may be suggested as follows, given that useful values of this ratio are in the range $\rho = (1, 2.5]$. Solving equation 12 for aperture,

$$x_r = \sqrt{\frac{dxdz(OW-1)}{2\rho}}. \tag{13}$$

This aperture radius balances the requirement for effective natural truncation of the operator to avoid Gibbs phenomenon instability, while maximizing the quality of the final image. The stability factor ρ allows the processor to tune the image quality and stability as necessary for any given situation.

Conclusions

Replacing the usual GPSPI square-root symbol exponential as in equation 3 with an exponential that is an integral involving a linear velocity over depth as in equation 7 results in an operator symbol that may be used directly in any $\omega - x$ GPSPI-derived algorithm including the high-performance FOCI algorithm.

Though the operator may not be perfectly stable, it nonetheless results in stable and accurate images demonstrating that it is practically stable. The $v(z)$ operator naturally and effectively truncates the extrapolation operator in the $\omega - x$ domain, and therefore obviates the need for an additional explicit stabilization step. Although this does save a trivial amount of computation time, the true benefit is its simplicity.

Acknowledgments

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