Generalized frames for Gabor operators in seismic imaging

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Summary

Gabor methods deal with nonstationarity in seismic signals by decomposing a signal into roughly stationary parts, processing each part separately, and assembling the results into a global final. In numerical wavefield propagation, a complex geological region is broken up into small regions of nearly constant velocity, and the wavefield is propagated through each region separately. In frequency slicing, a signal is broken up into separate frequency bands, each processed separately, and the results assembled for the final, complete result

We discuss this decomposition/reassembling as a mathematical windowing procedure that is accurately described by the theory of generalized frames. With frame theory, it is shown that a collection of local wavefield propagators, combined via a suitable partition of unity, remains a stable propagator — a highly desirable property in numerical simulations. These results apply more generally to combinations of linear operators that are useful for many nonstationary filtering operations.

Introduction

In seismic data processing, we are often in a situation where there are good theories and algorithms for numerically simulating a physical process in a homogeneous setting – say, for instance, propagating a wave in a uniform velocity field, or computing attenuation in a constant Q environment. When the setting becomes more complex – a heterogeneous velocity field, say – new theoretical methods and algorithms must be developed to deal with the added complexity.

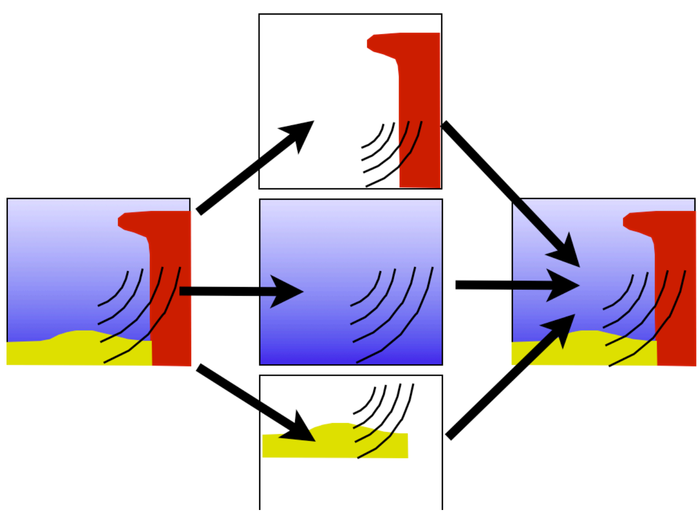

Figure 1: Propagating a wavefield in a complex region: decompose, propagate, re-assemble.

One way forward is a divide-and-conquer method, where a signal is decomposed into parts, each part processed as if in a homogeneous environment, and the individuals parts reassembled into one processed whole. This composition might be in different domains –

spatial, frequency, or other. Figure 1 shows an example of a spatial decomposition based on velocity profiles. A complex region (a salt dome) is broken up into three simpler regions, a wave propagates through each of the regions, and is reassembled into a final result.

Our research group has been examining the mathematics of this process, with the goal of formalizing some approaches that have been taken in Gabor techniques, including Gabor deconvolution and Gabor wavefield propagation. Key to Gabor methods is the notion of windowing a signal into various parts, processing each part, and reassembling. Our view is to make a precise mathematical description of these algorithms, leading to a numerical calculus that can give useful information about the stability, accuracy and efficiency of these methods.

Theory and Method

The mathematical theory is built on linear operators acting on a Hilbert space: roughly speaking, the Hilbert space is the collection of signals that we wish to study (seismic data, numerical wavefields, and the like) while the linear operators are the operations we do to the signal (smooth them, de-noise them, propagate them, migrate them). More basically, the Hilbert space is made up of vectors, and the linear operators are matrices acting on those vectors.

The windowing operation takes a signal and localizes it (in space, in frequency, etc) by multiplying the signal with a fixed collection of window functions – Gaussian, boxcars, etc. The windows are typically chosen so the corresponding operators P_i have the following property:

Definition: A sequence $P_1, P_2, P_3 ...$ of linear operators on a Hilbert space is called a generalized frame if there exist two positive constants a, b with

$$aI \leq \Sigma_i P_i^\star P_i \leq bI$$

where I is the identity operator. Generalized frames were introduced by Sun (2006) and are an important extension of the notion of frames, pseudo-frames, fusion frames, and other generalizations of the idea of basis in linear theory. A special case of a generalized frame occurs when the above sum equals the identity operator I, in which case the windows form a partition of unity (POU).

From the frame, one defines an analysis operator V as

$$Vf = (P_1 f, P_2 f, P_3 f, ...)$$

its adjoint the synthesis operator $V^\star$

$$V^\star (f_1, f_2, f_3 ...) = \Sigma_i P_i^\star f_i$$

and the frame operator S as

$$S = V^\star V.$$

The process of decomposing a signal into parts, applying an operator to each part, and reassembling, is described succinctly by the formula

$$A_{fin} = \Sigma_i P_i^\star A_i P_i$$

where the A_i are the operations on each individual piece, and A_{fin} is the final result obtained by putting all the pieces back together. A deep result (Stinespring's theorem) in linear operator theory shows that the result is nicely bounded when the P_i's form a generalized frame:

Theorem: Suppose $P_1, P_2, P_3, ...$ form a generalized frame, and $A_{fin} = \Sigma_i P_i^\star A_i P_i$ is the operator obtained by decomposition/reassembling. Then the norm of A_{fin} is bounded by the inequality

$$\|A_{fin}\| \leq b \max \|A_i\|,$$

where b is the constant in the frame definition.

Consequently, for wavefield propagations, using a partition of unity ($b = 1$), and stable propagators on each region ($\|A_i\| = 1$), we can guarantee that the final propagator A_{fin} is also stable. Similarly for frequency slicing, a good choice in partitioning ensures algorithm stability.

It is also useful to consider non-symmetric partition schemes, where the analysis operators P_i may be different from synthesis operators Q_i say. In this case, we have operators of the form

$$A_{fin} = \Sigma_i Q_i^\star A_i P_i.$$

Less is known about the stability of such operators, although they are heavily used in practice.

Now that we have this formalism in a mathematical form, our next steps are to create a numerical calculus demonstrating how such operators combine, giving estimates for how accurately they solve the underlying partial differential equations of the physics involved.

Examples

As an example of the decomposition technique, we create a synthetic example of wavefield propagation, consisting of a single pulse generating a wavefront that travels through a heterogeneous medium. The medium consists of a uniform fast region surrounding a circular slow region in the center (Figure 2). Two windows are used, a circular one to isolate the region in the center, and its complement to isolate the surrounding region. The wavefront is propagated using a time-stepping method, where at each time step, the wavefield is separated into the two components (fast region and slow regions), each component is forward propagated by one step at the appropriate velocity using a Fourier-based phase propagator, and the two components reassembled before the next time step occurs.

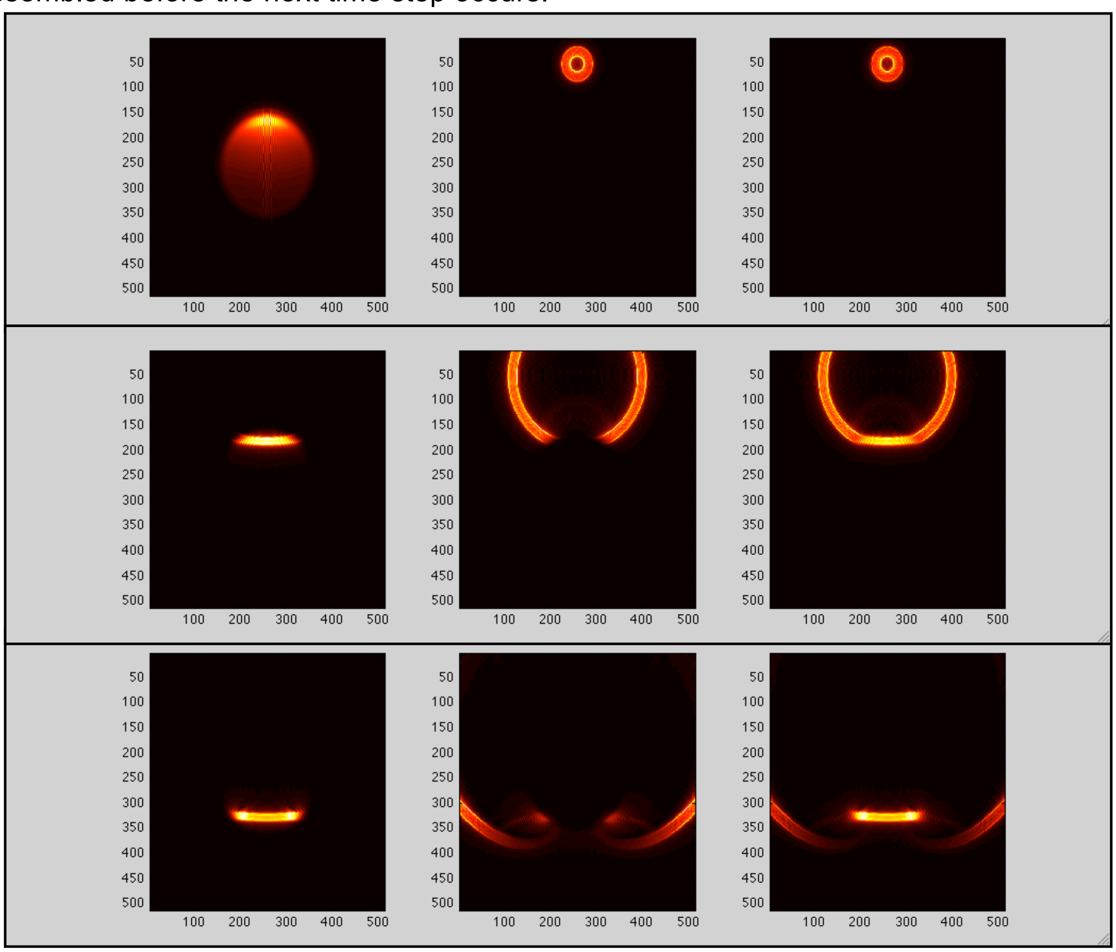

Figure 2: Three panels of a wavefield propagation experiment.

Figure 2 shows some snapshots from the simulation.
In the first panel, we see three square regions: the left square shows the signal in the slow region (normalized to be visible in the display, but overall very low amplitude), the middle square shows the initial wavefront in the fast region, and the right square shows the combined signal (note no contribution from the left square, confirming it was very low amplitude).
In the second panel, we see the wavefront hit the slow circular region. In the left square, we see the flattening and concentration of the wavefront as it crosses into the slow region. In the combined region on the right, we see the region is no longer circular.
In the third panel, in the combined region on the right, we see the slow part of the wavefront has fallen far behind the fast part, as expected, with some interesting phenomena linking the two.

This is a simple example to illustrate the concept of windowing and generalized frames applied to numerical wavefield propagation. More realistic examples are discussed in the papers by Wards (2008).

Conclusions

Nonstationary data processing is a valuable technique for representing physical phenomena in heterogeneous environments. The decomposition/process/re-assemble method is accurately described by generalized frame theory, a mathematical research area of great interest in scientific computation. This approach gives a mathematical formulation of a commonly used numerical technique, which in particular is a useful description of nonstationary Gabor methods. From this approach, we can demonstration numerical stability of the Gabor wavefield propagators and have shown examples of successful modeling of propagation in heterogeneous environments. This approach is a first step towards a good mathematical model or numerical calculus for solving differential equations using these nonstationary techniques.

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References

Asgari, M., and Khosravi, A., 2005, Frames and bases of subspaces in Hilbert spaces: J. Math. Anal. Appl., 308, 541–553.
Casazza, P., and Kutyniok, G., 2004, Frames of subspaces: (Wavelets, Frames and Op. Thy) Contemp. Math., 345, 87–113.
Christensen, O., 2003, An Introduction to Frames and Riesz Bases: Birkhäuser.
Eldar, Y., 2003, Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors: J. Fourier Anal. Appl., 9, No. 77–96.
Fornasier, M., 2003, Decompositions of Hilbert spaces: Local construction of global frames, in Bojanov, B., Ed., Proc. Int. Conf. on Constructive Function Theory Varna, DARBA, Sofia, 271–281.
Fornasier, M., 2004, Quasi-orthogonal decompositions of structured frames: J. Math. Anal. Appl., 289, 180–199.
Lamoureux, M.P., Margrave, G.F., and Ismail, S., 2008, Solving physics PDEs using Gabor multipliers, CREWES Ann. Report.
Li, S., and Ogawa, H., 2004, Pseudoframes for subspaces with applications: J. Fourier Anal. Appl., 10, 409–431.
Sun, W., 2006, G-frames and G-Riesz bases: J. Math. Anal. Appl., 322, 437–452..
Wards, B.D., Margrave, G.F., Lamoureux, M.P., 2008, Phase-shift time-stepping for reverse-time migration: The Marmousi data experience, CREWES Ann. Report.
Wards, B.D., Margrave, G.F., Lamoureux, M.P., 2008, Fast phase-shift time-stepping for reverse-time migration, CREWES Ann. Report.