Frequency dependent reflection coefficients for a velocity ramp in 2D

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Abstract

We solve the 2D elastic wave equation using a velocity ramp. We require the density and modulus satisfy the relation established in (Lamoureux et al., 2012) and (Lamoureux et al., 2013) using some parameter α. Extending these velocities to the two dimensions, we compute the analytic solutions to the 2D elastic wave equation and find the reflection coefficients for a plane wave hitting the transition zone of the ramp at normal incidence. Finally, we conclude with a discussion of the case where the plane wave hits the transition zone of the 2D velocity ramp at non-normal incidence given varying density.

Problem

Recall the 2D elastic wave equation

$$\rho(x,z)\frac{\partial^2 u}{\partial t^2} = \nabla(\mathrm{K(x,z)} \cdot \nabla \mathrm{u})$$

where ρ represents the density and K represents the bulk modulus. As in (Lamoureux et al., 2012), we define $\rho(x,z) = c(x,z)^{\alpha-2}$ and $K(x,z) = c(x,z)^{\alpha}$ for some parameter α. This relation preserves the ratio

$$\frac{\rho(x,z)}{K(x,z)} = c(x,z)^2.$$

We are specifically interested in the velocity field which has constant velocity prior to a linear increasing ramp and constant after the ramp.

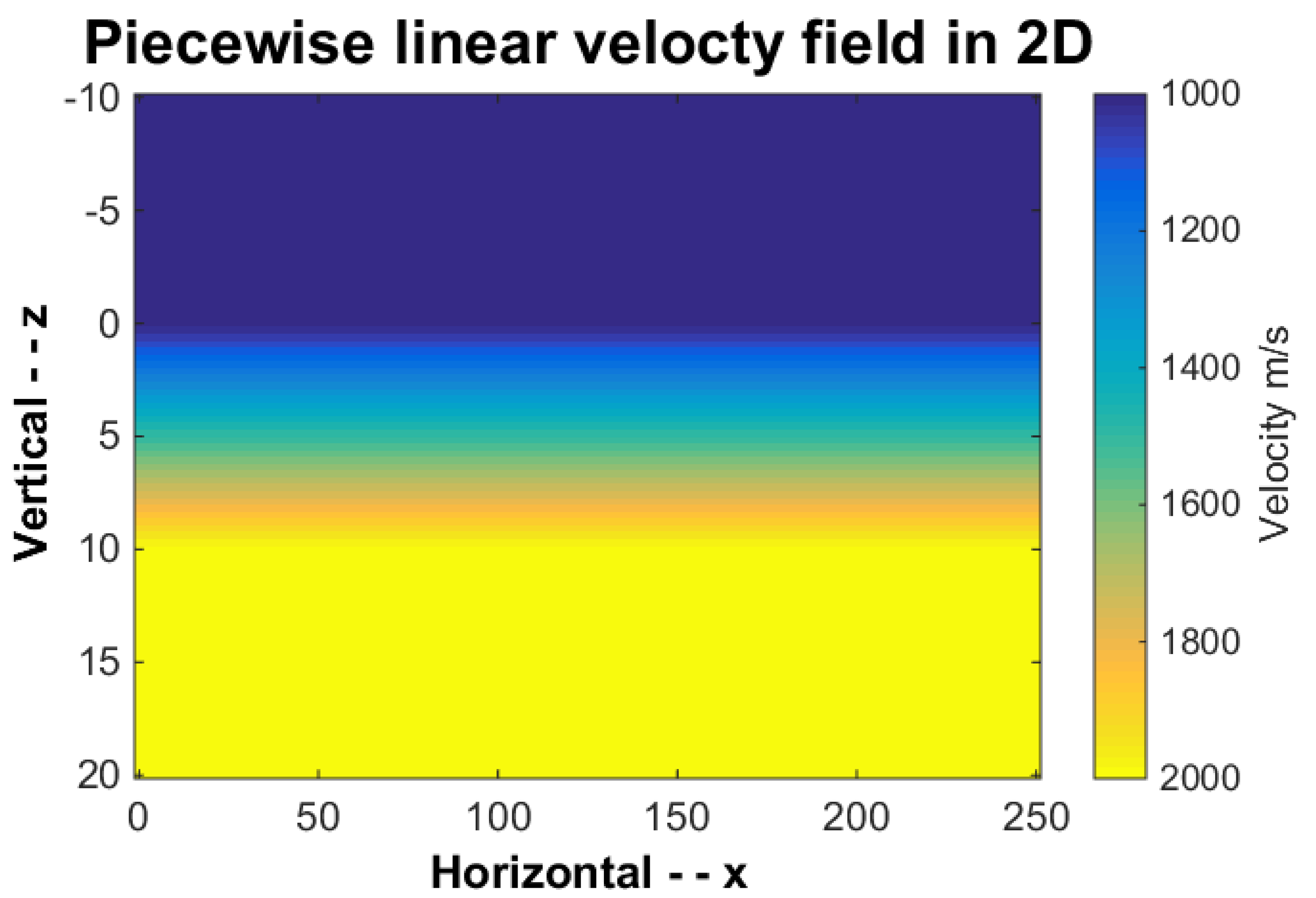

Figure 1: A velocity field which has a ramp moving from a constant velocity of 1000 m/s to 2000 m/s over 10 m.

Normal Incidence

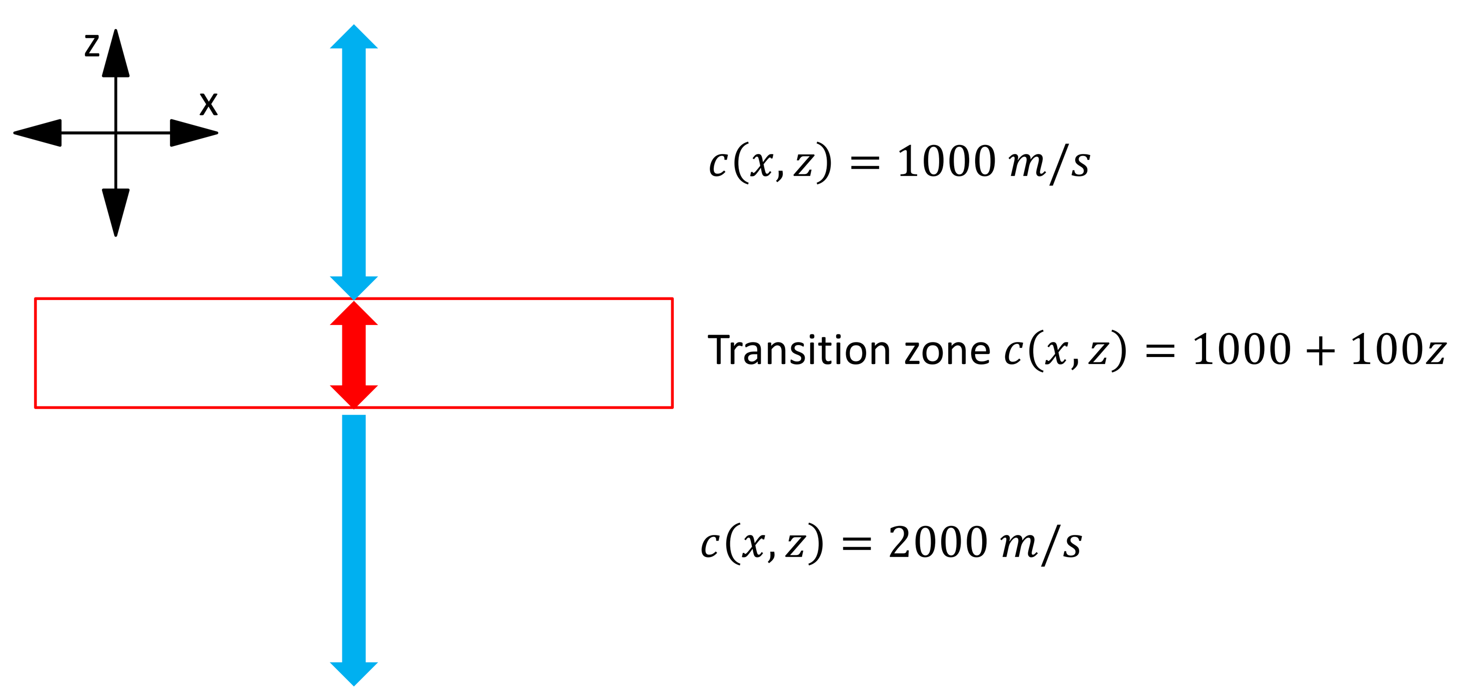

Figure 2: The plane wave hits the transition zone orthogonally. Some of the wave travels through the transition zone ($z = 0$ to $z = 10$) and is transmitted on the other side while some is reflected back.

In the normal incidence case, the plane wave is orthogonal to the transition zone of the velocity field. This means that the incident angle $\theta_1 = 0°$. We model Fig. 2 with the regional solutions

$$u_{\text{top}}(x,z,t) = e^{i\omega(z/1000-t)} + Re^{i\omega(-z/1000-t)}$$
$$u_{\text{trans}}(x,z,t) = A(1000+100z)^{n_1}e^{-i\omega t} + B(1000+100z)^{n_2}e^{-i\omega t}$$
$$u_{\text{bottom}}(x,z,t) = Te^{i\omega(z/2000-t)}$$

where $n_1 = (1-\alpha)/2 + \sqrt{(1-\alpha)/4-\omega^2}$ and $n_2 = (1-\alpha)/2 - \sqrt{(1-\alpha)/4-\omega^2}$. Note that R and T are the reflection and transmission coefficients respectively. Next, we impose the following continuity conditions to preserve displacement continuity and continuity of force:

$$u_{\text{top}} = u_{\text{trans}} \quad \text{at } z = 0,$$
$$u_{\text{trans}} = u_{\text{bottom}} \quad \text{at } z = 10,$$
$$K_{\text{top}}\nabla(u_{\text{top}}) = K_{\text{trans}}\nabla(u_{\text{trans}}) \quad \text{at } z = 0$$
$$K_{\text{trans}}\nabla(u_{\text{trans}}) = K_{\text{bottom}}\nabla(u_{\text{bottom}}) \quad \text{at } z = 10$$

Applying these conditions to the regional solutions above, we get the a matrix equation which solving gives the reflection coefficient

$$R(\omega) = \frac{(\omega + imn_1)(\omega + imn_2)(c_1^{n_1}c_2^{n_2} - c_1^{n_2}c_2^{n_1})}{(\omega^2 + m^2 n_1 n_2)(c_1^{n_1}c_2^{n_2} - c_1^{n_2}c_2^{n_1}) + (imn_2 - imn_1\omega)(c_1^{n_1}c_2^{n_2} + c_1^{n_2}c_2^{n_1})}$$

where m is the slope of the ramp, $c_1 = 1000\ m/s$, and $c_2 = 2000\ m/s$.

Non-Normal Incidence

In the non-normal incidence case, the plane wave hits the transition zone at an angle θ_1. In the normal incidence case, the wave was constant in the x-direction; however, that is not true for this case. As such, we will focus on the case when only density varies for the 2D elastic wave equation in the rest of this work.

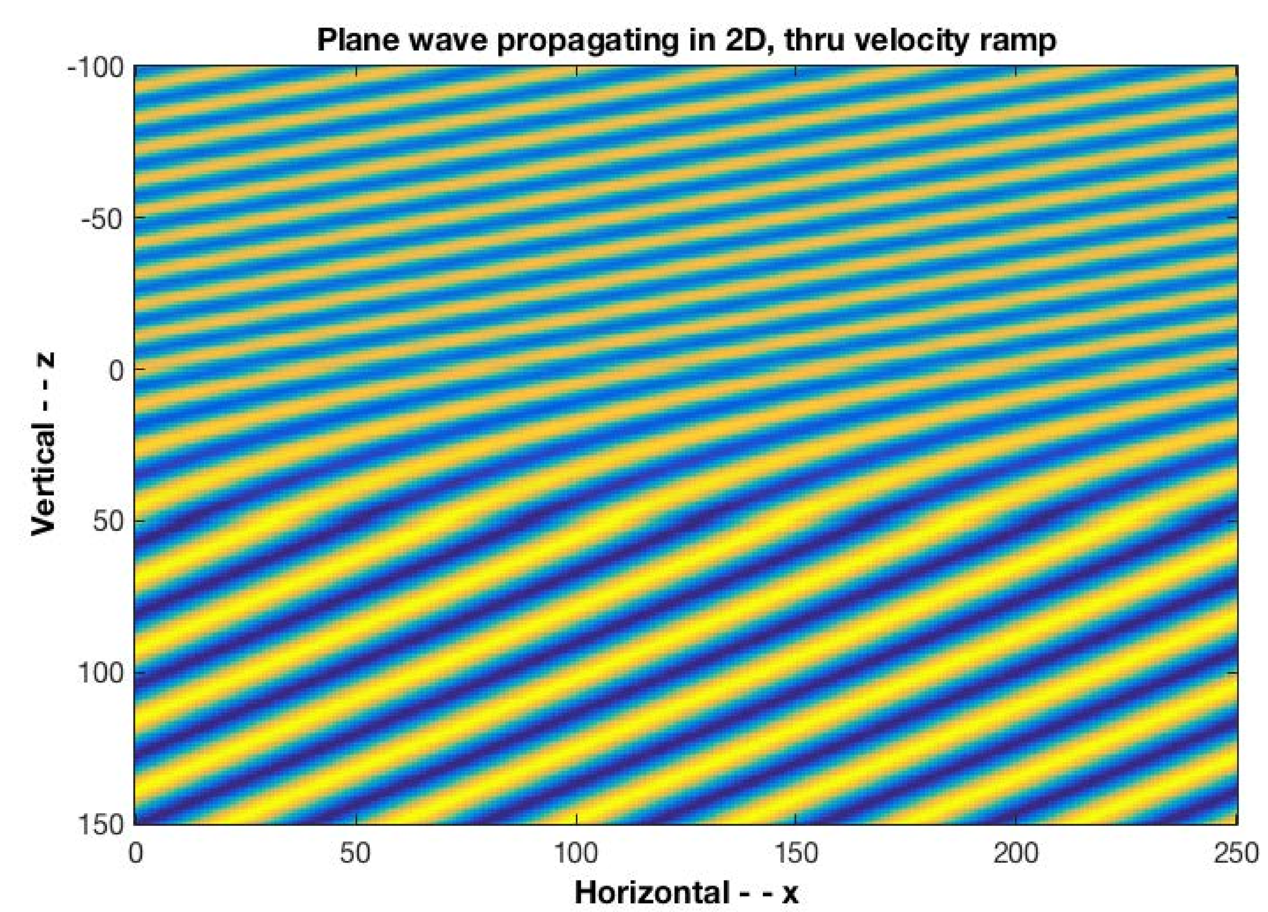

Figure 3: The plane wave hits the transition zone at an angle in this case. A portion of the wave travels through the transition zone from $z = 0$ to $z = 50$ and is transmitted on the other side while the rest is reflected.

To model Fig. 3, we use the following regional solutions:

$$u_{\text{top}} = e^{i(k_x x + k_z z - \omega t)} + Re^{i(k_x x - k_z z - \omega t)}$$
$$u_{\text{trans}} = e^{i(k_x x - \omega t)}\left(AZ_1(z) + BZ_2(z)\right)$$
$$u_{\text{bottom}} = Te^{i(k_x x + k_z' z - \omega t)}$$

where Z_1 and Z_2 are Whittaker functions. We also require that the parameters k_x, k_z, and k_z' satisfy the dispersion relation:

$$k_x^2 + k_z^2 = \frac{\omega^2}{c_1^2}$$

where $c_1 = 1000$ m/s.

Applying the continuity conditions from the previous case, we get a matrix equation which we solve to find the reflection coefficient:

$$R(\omega) = -(Z_1'(0)Z_2'(10) - Z_2'(0)Z_1'(10) - ik_z(Z_1(0)Z_2'(10) - Z_2(0)Z_1'(10)) - ik_z'(Z_1'(0)Z_2(10) - Z_2'(0)Z_1(10)) - k_z k_z'(Z_1(0)Z_2(10) - Z_2(0)Z_1(10)))/N$$

where

$$N = Z_1'(0)Z_2'(10) - Z_2'(0)Z_1'(10) + ik_z(Z_1(0)Z_2'(10) - Z_2(0)Z_1'(10)) - ik_z'(Z_1'(0)Z_2(10) - Z_2'(0)Z_1(10)) + k_z k_z'(Z_1(0)Z_2(10) - Z_2(0)Z_1(10)).$$

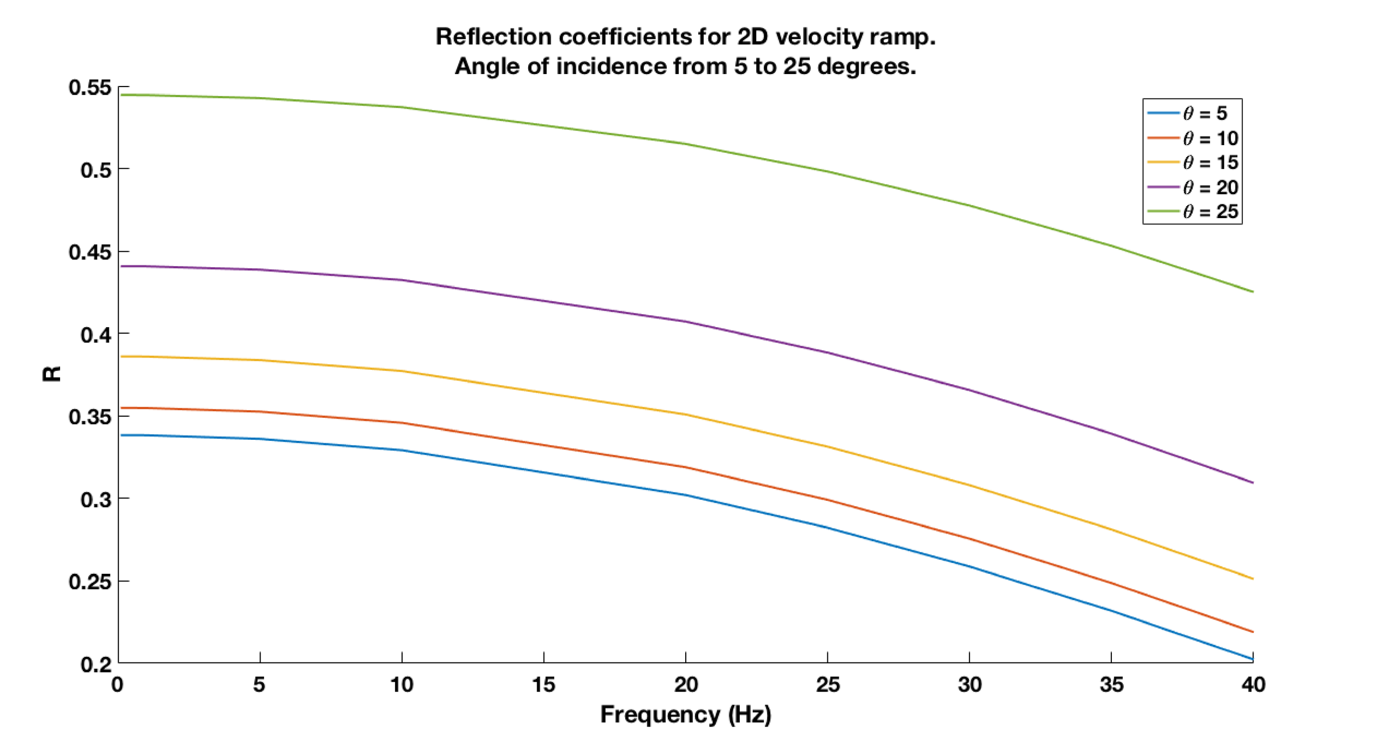

Figure 4: Reflection coefficients for non-normal incident case for the 2D velocity ramp. Each line represents the reflection coefficient for a different incidence angle θ. Specifically, $\theta = 5°, 10°, 15°, 20°, 25°$.

Future Work

Our next step will be to consider the case when the modulus varies for the non-normal incidence case of the 2D velocity ramp. From there, we would find the exact solution of the reflection coefficients for general α. It would also be interesting to extend this work to 3D.

Conclusions

We have extended the work of the authors in (Lamoureux et al.,2012) and (Lamoureux et al.,2013) to two dimensions for a velocity ramp. We found the reflection coefficient solutions for a 2D velocity ramp when the plane wave hits at normal incidence. For the non-normal incidence case, we looked at the case when only the density varied for the 2D velocity ramp and found an equation for the reflection coefficient with respect to k_x, k_z, ω, and k_z. Finally, we compared the reflection coefficients in this case for different incidence angles θ.

Acknowledgements

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