# A non-linear, three-parameter AVO method that can be solved non-iteratively

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## ABSTRACT

The rationale is presented for a three-parameter AVO expression which includes a term quadratic in the shear velocity reflectivity. It is shown that this can be solved explicitly without recourse to iterative techniques. It is compared to other AVO methods in its ability to estimate various reflectivities from noisy synthetic data.


## INTRODUCTION

Two-parameter methods have been the staple of practical AVO inversion for many years. In a previous study (Ursenbach, 2004a,b) we presented new two-parameter inversion methods. One involved adding a term quadratic in the shear velocity or impedance, and this had the effect of removing what is usually the single largest component of theoretical error for most inversions. Furthermore, unlike most non-linear methods, this inversion could be completed in a single step, without recourse to iterative techniques.

In this study we extend this technique to three-parameter methods by augmenting the traditional Aki-Richards equation with a single quadratic term. The method of solution is slightly more complicated than in the two-parameter case, but can still be carried out non-iteratively.

Notation: The individual reflectivities are defined as one-half of the relative contrast of a property. Thus using $\alpha$ to represent the P-wave velocity, the $\alpha$-reflectivity is defined as

$$R_\alpha = \frac{1}{2}\frac{\Delta\alpha}{\alpha} = \frac{1}{2}\frac{\alpha_2-\alpha_1}{(\alpha_1+\alpha_2)/2} = \frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}$$

where the subscript 1 denotes the layer above the interface and subscript 2 the layer below. Similar definitions are used for reflectivities of the S-wave velocity ($\beta$), density ($\rho$), P-wave impedance ($I=\rho\alpha$), S-wave impedance ($J=\rho\beta$) and shear modulus ($\mu=\rho\beta^2$).

## THEORY

We have previously shown (Ursenbach, 2003a,b), using synthetic reflectivities derived from data on 110 interfaces, that errors in AVO results (except the P-impedance reflectivity) are strongly correlated with the S-wave velocity reflectivity (and with the S-impedance reflectivity and also the $\mu$ reflectivity, all of which are usually similar). This is illustrated in Figure 1, which displays the error in reflectivity estimates for the Aki-Richards methods. It is particularly noticeable in the shear-impedance reflectivity.

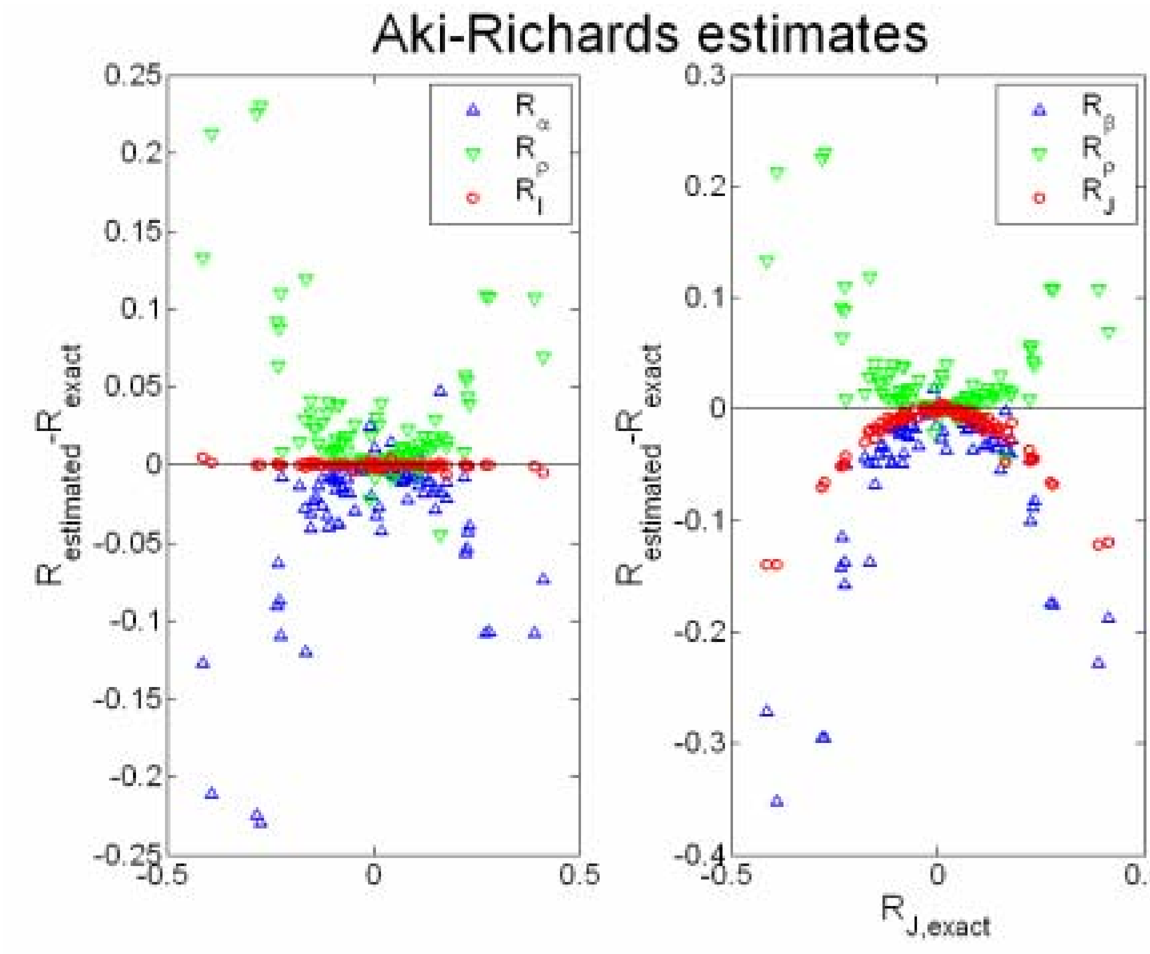


Figure 1: Errors in $R_\alpha$, $R_\beta$, $R_\rho$, $R_I$ (= $R_\alpha$+$R_\rho$), and $R_J$ (=$R_\beta$+$R_\rho$), as obtained by the Aki-Richards theory from AVO inversion of noise-free synthetic $R_{PP}$ amplitudes.

To correct this behavior we suggest an augmented Aki-Richards expression as follows:

$$R_{PP}(\theta) \approx \frac{1}{2\cos^2\theta}\frac{\Delta\alpha}{\alpha} - 4\gamma^2\sin^2\theta\frac{\Delta\beta}{\beta} + B_2\left(\frac{\Delta\beta}{\beta}\right)^2 + \left(\frac{1}{2} - 2\gamma^2\sin^2\theta\right)\frac{\Delta\rho}{\rho}. \quad (1)$$

Here $R_{PP}$ is the P-wave reflection coefficient, θ is the average of the P-wave reflection and transmission angles, and $\gamma \equiv \beta / \alpha = (\beta_1+\beta_2) / (\alpha_1+\alpha_2)$. The appropriate expression for $B_2$ can be derived and is given as

$$B_2 = \gamma \frac{\sin^2\theta - \cos^2\varphi}{\cos\theta\cos\varphi} B_1, \tag{2}$$

where φ is the average of the S-wave reflection and transmission angles, and $B_1$ is the coefficient of the linear $\Delta\beta/\beta$ term, $-4\gamma^2\sin^2\theta$. The quantity $\cos\varphi$ is well-approximated by $\sqrt{(1-\gamma^2\sin^2\theta)}$.

Seeking a least-squares solution to Eq. (1) yields three equations which must be solved simultaneously. Two are of the form (see Appendix for more details)

$$c_1 R_\alpha + c_2 R_\rho = c_3 + c_4 R_\beta + c_5 R_\beta^2, \tag{3}$$

i.e., they are linear in $R_\alpha$ and $R_\rho$ and have linear and quadratic terms in $R_\beta$. These two equations are solved for $R_\alpha$ and $R_\rho$, which gives each of them as a quadratic function of $R_\beta$. The third equation is of the form

$$0 = c_1 R_\beta^3 + c_2 R_\beta^2 + \left(c_{3a} + c_{3b} R_\alpha + c_{3c} R_\rho\right) R_\beta + \left(c_{4a} + c_{4b} R_\alpha + c_{4c} R_\rho\right), \tag{4}$$

which, when substitution is made for $R_\alpha$ and $R_\rho$, becomes a cubic polynomial in $R_\beta$. This is solved, with $R_\beta$ being equal to the real root having the smallest magnitude. $R_\beta$ is then substituted back into the expressions for $R_\alpha$ and $R_\rho$.

## RESULTS

We use data published by Castagna and Smith (1994) to generate synthetic reflection data for 110 interfaces as described in earlier publications (Ursenbach, 2003a,b). [The earlier studies employed 125 interfaces, but some of these have been deleted because of concerns over physicality.] On a scale where the maximum $R_{PP}$ is ~1, Gaussian noise is then added which has a magnitude of ~0.1. For each interface we perform four AVO inversions with four methods: Aki-Richards (1980), Smith-Gidlow (1987), Fatti et al. (1994), and Eq. (1) of this study. The results are displayed by subtracting the exact value from the predicted value for the various reflectivities, and then plotting these against $R_J = R_\beta + R_\rho$.

In Figure 2 we compare impedance reflectivity predictions from Eq. (1) against results from the Aki-Richards and Fatti et al. methods. As expected all methods give very good results for $R_I$, which, in the presence of zero-offset data, is very stably predicted. In the case of $R_J$ we see that Eq. (1) has somewhat less scatter than Aki-Richards, and is very similar to the result of Fatti et al. For comparison, estimates of $R_J$ are also given for noise-free data.

In Figure 3 we compare velocity and density reflectivity predictions from Eq. (1) against results from the Aki-Richards and Smith-Gidlow methods. Eq. (1) appears to deal with noise far better than the Aki-Richards method. In the absence of noise (not shown) the two methods give much less disparate results. Comparison to the Smith-Gidlow method however shows that the two-parameter method is even more stable for noisy data.

Even when the density reflectivity is crudely predicted by Gardner's relation (1974) from the Smith-Gidlow $R_\alpha$, the result is still better than Eq. (1).

## CONCLUSIONS

For noisy synthetic data Eq. (1) gives estimates similar to the Fatti method for impedance reflectivities, but more scattered than the Smith-Gidlow method for velocity reflectivities and even for a Gardner-density reflectivity. However it gives results which improve very significantly on the Aki-Richards method.

As long-offset data becomes more frequently available, interest in the density reflectivity will continue to increase. Methods designed to tame the Aki-Richards method, and to extract density information without the aid of empirical relations, might profitably be applied to Eq. (1).

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## ACKNOWLEDGMENTS

The author wishes to thank Bill Goodway for insights into the dataset employed.

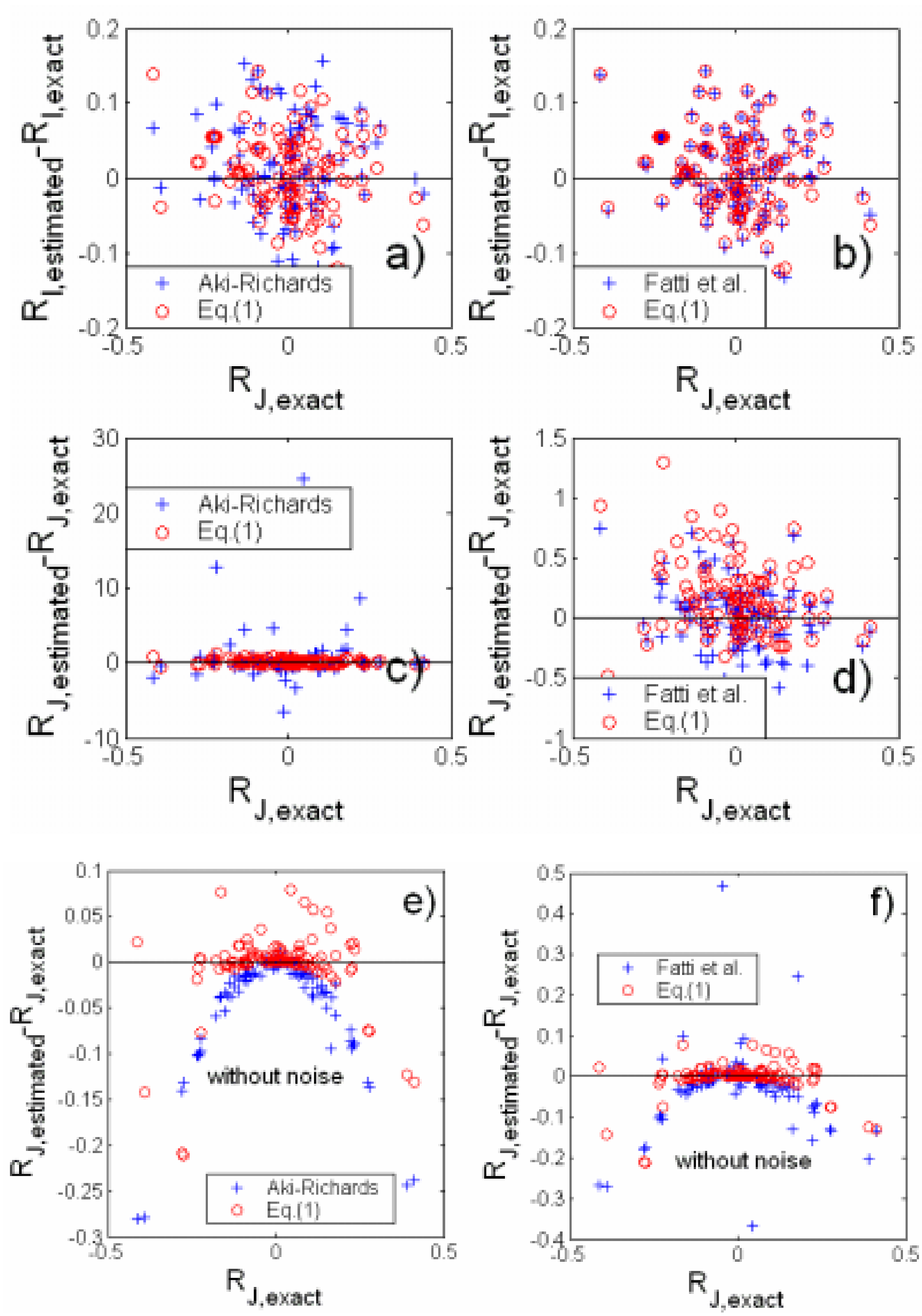


FIG. 2. This figure illustrates the error in various impedance reflectivity estimates. Note the differences in *y*-axis scales. Parts a) and b) illustrate that the P-impedance reflectivity $R_I$ is predicted equally well by all methods. Parts c) and d) show that estimates of the S-impedance reflectivity $R_J$ are similar for Eq.(1) and the Fatti method (perhaps slightly better for the latter) but that Aki-Richards results are strongly corrupted by noise in the data. For comparison, results for $R_J$ from noise-free data are shown in e) and f).

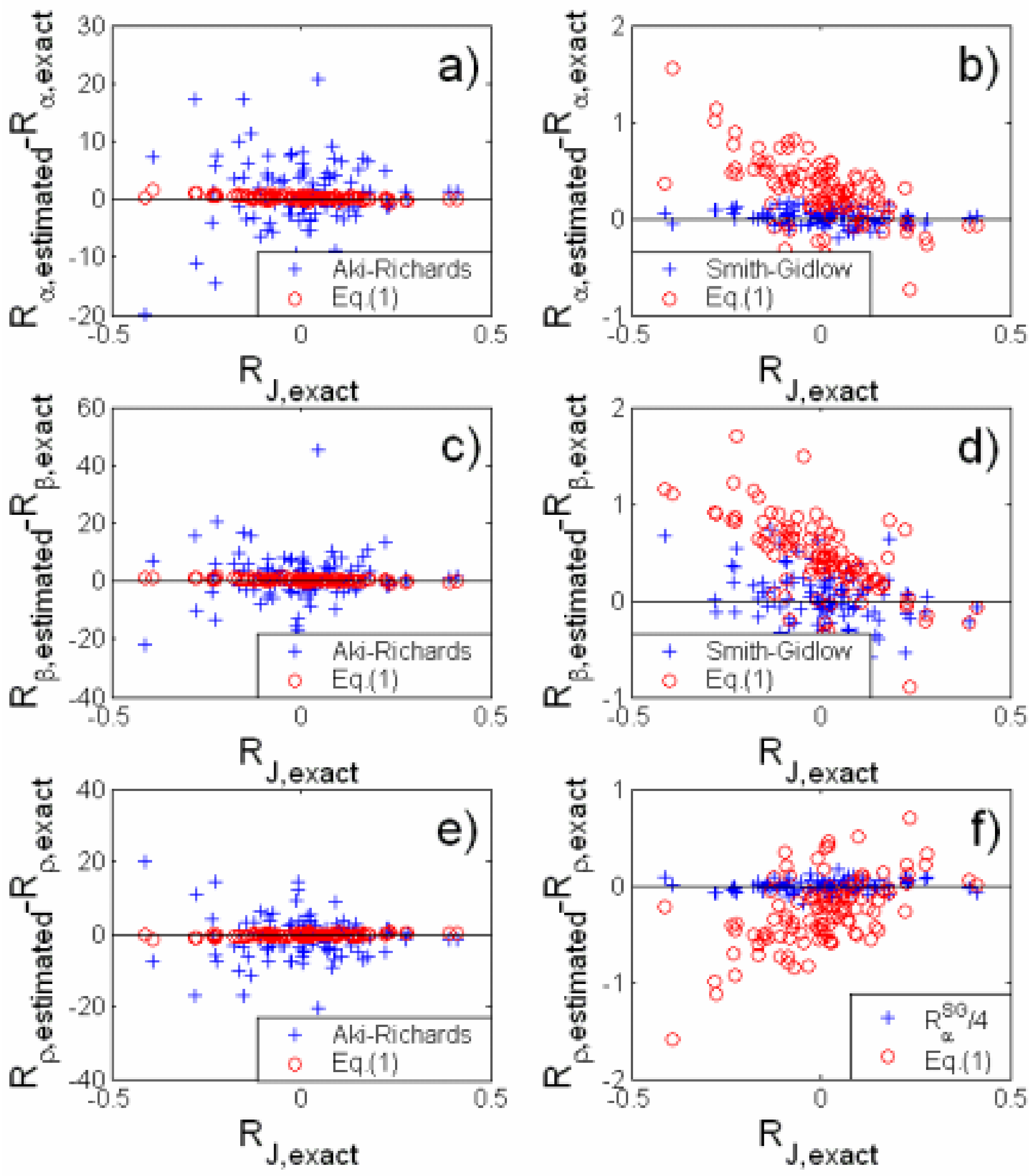


FIG. 3. This figure illustrates the error in velocity and density reflectivity estimates. Note the differences in *y*-axis scales. Parts a), c) and e) illustrate that, when using noisy data, Eq. (1) gives a far more stable estimate of $R_\alpha$, $R_\beta$ and $R_\rho$ than does the Aki-Richards method. Parts b), d) and f) show that Smith-Gidlow estimates are in turn superior to those of Eq.(1), even when $R_\rho$ is estimated simply as $R_\alpha/4$, as per the Gardner relation. In the case of $R_\beta$, however, the Smith-Gidlow method is not as strongly superior to Eq. (1).

## APPENDIX

If Eq. (1) is written as

$$R_{PP}(\theta) = A\frac{\Delta\alpha}{\alpha} + B_1\frac{\Delta\beta}{\beta} + B_2\left(\frac{\Delta\beta}{\beta}\right)^2 + C\frac{\Delta\rho}{\rho}, \tag{A1}$$

then to find the least squares solution we first construct the quantity

$$S = \sum_i\Big[R_{\mathrm{PP}}(\theta_i) - \left\{A(\theta_i)R_\alpha + B_1(\theta_i)R_\beta + B_2(\theta_i)R_\beta^2 + C(\theta_i)R_\rho\right\}\Big]^2. \tag{A2}$$

Next we find the stationary point of S with respect to the three parameters:

$$\frac{\partial S}{\partial R_\alpha} = 0, \quad \frac{\partial S}{\partial R_\beta} = 0, \quad \frac{\partial S}{\partial R_\rho} = 0 \tag{A3}$$

The equations in A3 can be written explicitly as

$$\begin{aligned}&\left[\sum A_i^2\right]\frac{\Delta\alpha}{\alpha} + \left[\sum A_iC_i\right]\frac{\Delta\rho}{\rho} \\ &\quad = \left[\sum A_iR(\theta_i)\right] - \left[\sum A_iB_{1,i}\right]\frac{\Delta\beta}{\beta} - \left[\sum A_iB_{2,i}\right]\left(\frac{\Delta\beta}{\beta}\right)^2,\end{aligned} \tag{A4}$$

$$\begin{aligned}&\left[\sum A_iC_i\right]\frac{\Delta\alpha}{\alpha} + \left[\sum C_i^2\right]\frac{\Delta\rho}{\rho} \\ &\quad = \left[\sum C_iR(\theta_i)\right] - \left[\sum C_iB_{1,i}\right]\frac{\Delta\beta}{\beta} - \left[\sum C_iB_{2,i}\right]\left(\frac{\Delta\beta}{\beta}\right)^2,\end{aligned} \tag{A5}$$

and

$$\begin{aligned}&\left[\sum A_iB_{1,i}\right]\frac{\Delta\alpha}{\alpha} + 2\left[\sum A_iB_{2,i}\right]\frac{\Delta\alpha}{\alpha}\frac{\Delta\beta}{\beta} \\ &\qquad + \left[\sum C_iB_{1,i}\right]\frac{\Delta\rho}{\rho} + 2\left[\sum C_iB_{2,i}\right]\frac{\Delta\rho}{\rho}\frac{\Delta\beta}{\beta} \\ &= \left[\sum B_{1,i}R(\theta_i)\right] + \left(2\left[\sum B_{2,i}R(\theta_i)\right] - \left[\sum B_{1,i}^2\right]\right)\frac{\Delta\beta}{\beta} \\ &\qquad - 3\left[\sum B_{1,i}B_{2,i}\right]\left(\frac{\Delta\beta}{\beta}\right)^2 - 2\left[\sum B_{2,i}^2\right]\left(\frac{\Delta\beta}{\beta}\right)^3.\end{aligned} \tag{A6}$$

Eqs (A4) and (A5) are of the form of Eq. (3). Solving these two equations gives $R_\alpha$ and $R_\rho$ as quadratic functions of $R_\beta$. Substituting for $R_\alpha$ and $R_\rho$ in Eq. (A6) results in an equation of the form of Eq. (4).

One may just as easily carry out this derivation from the three-term Fatti approximation (in terms of I, J and $\rho$) rather than from the Aki-Richards expression (in terms of $\alpha$, $\beta$ and $\rho$). The expression for $B_2$ is the same in both cases.