Derivation of generalized reflections from point sources in a two-layer VTI medium

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ABSTRACT

Detailed expressions are derived for generalized reflections from point sources in a two-layer medium possessing vertical transverse isotropy. First, the displacements resulting from P-wave and S-wave point sources in a homogeneous medium are derived. It is shown that these are given by integrals over the corresponding plane-wave displacements. The generalized reflection coefficients are then derived and are seen to be integrals over plane-wave reflection coefficients.

INTRODUCTION

In a previous series of investigations, the generalized reflection coefficients from point sources in two-layer isotropic media were studied to clarify their potential implications for AVO (amplitude variation with offset). The influence of spherical-wave effects on different AVO classes was investigated for conventional and converted-wave reflections in both elastic media (Haase & Ursenbach, 2004a) and anelastic media (Haase & Ursenbach, 2004b). A highly accurate approximation was also presented to allow more rapid calculation in the case of P-waves in elastic media (Ursenbach & Haase, 2004).

Although the same theory of generalized reflections is in principle known for VTI (vertical transverse isotropy) media (e.g., Rommel, 1992; Tsvankin, 2001), the explicit expressions available for isotropic media (Aki & Richards, 1980) are not published for the anisotropic case. This paper will begin with the theoretical result given by Tsvankin (2001) and from these derive explicit expressions for generalized reflections in a two-layer VTI medium. Thus this study provides the theoretical basis for calculations carried out on VTI media (Haase & Ursenbach, 2005a,b).

The first step follows Tsvankin's derivation of the Weyl integral for a homogeneous VTI medium. Then we obtain detailed versions of this for P-waves (equation 10) and for S-waves (equation 15). Next we demonstrate that these can be expressed in terms of plane-wave displacement parameters (equations 16 and 17), with notation due to Graebner (1992). From these we obtain the central result of this paper, the generalized reflection coefficients for a two-layer medium (equations 34-37).

THEORY

Weyl integral for point forces in a homogeneous, VTI medium

Tsvankin (2001) shows that the frequency spectrum of the displacement is given by (see his equation 2.25):

$$\mathbf{S}(\omega,\mathbf{x}) = \frac{1}{(2\pi)^3\,\omega^2}\int_{-\infty}^{\infty}\mathbf{H}^{-1}\mathbf{F}e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k} \tag{1}$$

where ω is the frequency, $\mathbf{x}$ is the spatial Cartesian coordinate, and $\mathbf{k}$ is the wave vector. $\mathbf{F}$ is the direction and magnitude of a point force, $\mathbf{F}\delta(\mathbf{x})h(t)$, where $\delta(\mathbf{x})$ is a 3-D delta function, $h(t)$ represents the source pulse, and, if the force is oriented along the i^{th} axis, then $\mathbf{F}$ can be written as $\mathbf{F} = B(\delta_{i1},\delta_{i2},\delta_{i3})$. $\mathbf{H}$ is related to the Christoffel matrix and is given by [cf. Tsvankin (2001), p. 14]

$$\mathbf{H} = \begin{bmatrix} c_{11}p_1^2 + c_{66}p_2^2 + c_{55}p_3^2 - \rho & (c_{11}-c_{66})\,p_1p_2 & (c_{13}+c_{55})\,p_1p_3 \\ (c_{11}-c_{66})\,p_1p_2 & c_{66}p_1^2 + c_{11}p_2^2 + c_{55}p_3^2 - \rho & (c_{13}+c_{55})\,p_2p_3 \\ (c_{13}+c_{55})\,p_1p_3 & (c_{13}+c_{55})\,p_2p_3 & c_{33}p_3^2 + c_{55}\left(p_1^2+p_2^2\right) - \rho \end{bmatrix}.$$

Where the c_{ij} are elastic constants, the p_i are slowness components ($\mathbf{p} = \mathbf{k}/\omega$), and ρ is the density of the medium. Equation 1 is converted to a Weyl integral by integrating over k_3 to yield (cf. equations 2.26 and 2.27 of Tsvankin (2001))

$$\mathbf{S}(\omega,\mathbf{x}) = -\frac{i\omega}{4\pi^2}\sum_{\nu=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{U}^{(\nu)}(p_1,p_2)e^{-i\omega(p_1x_1+p_2x_2+p_3^{(\nu)}x_3)}dp_1dp_2 \tag{2}$$

where ν is an index for the three eigenvalues of $\mathbf{H}$, so that $p_3^{(\nu)}$ are poles of the kernel in equation 1, and $\mathbf{U}(\nu)$ is given by the residue of $\mathbf{H}^{-1}\mathbf{F}$ at the poles:

$$\mathbf{U}^{(\nu)}(p_1,p_2) = \mathrm{Res}\left[\frac{\mathrm{adj}(\mathbf{H})}{\det(\mathbf{H})}\mathbf{F}\right]_{p_3^{(\nu)}} = \left[\frac{p_3 - p_3^{(\nu)}}{\det(\mathbf{H})}\mathbf{H}^{\mathrm{adj}}\mathbf{F}\right]_{p_3=p_3^{(\nu)}}. \tag{3}$$

Recalling that $\mathbf{F}$ is oriented along the i^{th} axis, we can obtain slightly more explicit expressions using the $p_3^{(\mathrm{P})} \equiv \xi$ and $p_3^{(\mathrm{SV})} \equiv \eta$ poles, which are of interest to us:

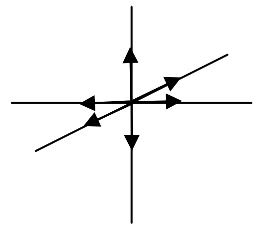
$$\mathbf{U}^{(\mathrm{P},i)}(p_1,p_2) = \frac{B\left(H_{1i}^{\mathrm{adj}},H_{2i}^{\mathrm{adj}},H_{3i}^{\mathrm{adj}}\right)}{2c_{33}c_{55}\xi\left(\xi^2-\eta^2\right)\left(\rho - c_{55}\left[p_1^2+p_2^2\right] - c_{66}\xi^2\right)} \tag{4}$$

$$\mathbf{U}^{(\mathrm{SV},i)}(p_1,p_2) = \frac{B\left(H_{1i}^{\mathrm{adj}},H_{2i}^{\mathrm{adj}},H_{3i}^{\mathrm{adj}}\right)}{2c_{33}c_{55}\eta\left(\eta^2-\xi^2\right)\left(\rho - c_{55}\left[p_1^2+p_2^2\right] - c_{66}\eta^2\right)} \tag{5}$$

The adjoint matrix elements are complicated, and we will defer evaluating them as they must first be manipulated further.

Point-sources for P-waves

The point source for a P-wave is represented by six equal-magnitude point forces oriented in the six axial directions away from the origin, as indicated in the diagram below:



According to the method of moments (Krebes, course notes), the effect of two oppositely directed forces is given by applying a derivative to equation 2 along the direction of the two forces. If we restrict consideration to the P-wave emanating from such a point source ($\nu = \mathrm{P}$), but sum over the three Cartesian coordinate directions, equation 2 becomes

$$
\begin{aligned}
\mathbf{S}^{(\mathrm{P})}(\omega,\mathbf{x}) &= \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\mathbf{S}^{(\nu=\mathrm{P},i)}(\omega,\mathbf{x}) \\
&= -\frac{i\omega}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{i=1}^{3}\mathbf{U}^{(\mathrm{P},i)}(p_1,p_2)\left[\frac{\partial}{\partial x_i}e^{-i\omega(p_1x_1+p_2x_2+\xi x_3)}\right]dp_1dp_2 \\
&= -\frac{\omega^2}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sum_{i=1}^{3}p_i\mathbf{U}^{(\mathrm{P},i)}(p_1,p_2)\right]e^{-i\omega(p_1x_1+p_2x_2+\xi x_3)}dp_1dp_2 \\
&= -\frac{\omega^2}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{B\left(\sum p_iH_{1i}^{\mathrm{ad}},\sum p_iH_{2i}^{\mathrm{ad}},\sum p_iH_{3i}^{\mathrm{ad}}\right)}{2c_{33}c_{55}\xi\left(\xi^2-\eta^2\right)\left(\rho-c_{55}\left[p_1^2+p_2^2\right]-c_{66}\xi^2\right)}\right]e^{-i\omega(p_1x_1+p_2x_2+\xi x_3)}dp_1dp_2
\end{aligned}
\tag{6}
$$

We now define $p_0=\sqrt{p_1^2+p_2^2}$. In particular we define $p_1 = p_0\cos\phi$ and $p_2 = p_0\sin\phi$ to produce (cf. Tsvankin, Eq. (2.28))

$$\begin{aligned}\mathbf{S}^{(\mathrm{P})}(\omega,\mathbf{x}) &= -\frac{\omega^2}{4\pi^2}\int_0^{2\pi}\int_0^{\infty}\left[\sum_{i=1}^{3} p_i \mathbf{U}^{(\mathrm{P},i)}(p_0\cos\phi, p_0\sin\phi)\right]e^{-i\omega(p_0 r\cos(\phi-\alpha)+\xi z)}p_0 dp_0 d\phi \\ &= -\frac{\omega^2}{4\pi^2}\int_0^{2\pi}\int_0^{\infty}\left[\frac{-B\left(p_0T_r^{\mathrm{P}}\cos\phi, p_0T_r^{\mathrm{P}}\sin\phi, \xi T_z^{\mathrm{P}}\right)}{2c_{33}c_{55}\xi\left(\xi^2-\eta^2\right)}\right]e^{-i\omega(p_0 r\cos(\phi-\alpha)+\xi z)}p_0 dp_0 d\phi \\ &= \frac{B\omega^2}{8\pi^2 c_{33}c_{55}}\int_0^{2\pi}\int_0^{\infty}\left(p_0T_r^{\mathrm{P}}\cos\phi, p_0T_r^{\mathrm{P}}\sin\phi, \xi T_z^{\mathrm{P}}\right)\frac{e^{-i\omega(p_0 r\cos(\phi-\alpha)+\xi z)}}{\left(\xi^2-\eta^2\right)}\frac{p_0}{\xi}dp_0 d\phi\end{aligned}$$

(7)

We have also substituted $x_1 = r\cos\alpha$, $x_2 = r\sin\alpha$ and $x_3 = z$, and note that ξ and η depend on p_0 but not on ϕ, so all ϕ dependence is given explicitly in the above expressions. The quantities T_r^{P} and T_z^{P} result from summing and simplifying the adjoint matrix elements in equation 6. They can be expressed as

$$T_r^{\mathrm{P}} = c_{55}p_0^2 + (c_{33} - [c_{13}+c_{55}])\xi^2 - \rho \tag{8}$$

$$T_z^{\mathrm{P}} = (c_{11} - [c_{13}+c_{55}])p_0^2 + c_{55}\xi^2 - \rho \tag{9}$$

We will now focus on the ϕ dependence of equation 7. From the last line we can isolate three integrals. First,

$$\begin{aligned}&\int_0^{2\pi}\cos\phi e^{-i\omega p_0 r\cos(\phi-\alpha)}d\phi \\ &= \int_\alpha^{2\pi+\alpha}\cos(\phi+\alpha)e^{-i\omega p_0 r\cos\phi}d(\phi+\alpha) \\ &= \int_0^{2\pi}\cos(\phi+\alpha)e^{-i\omega p_0 r\cos\phi}d\phi \\ &= \int_0^{2\pi}(\cos\phi\cos\alpha - \sin\phi\sin\alpha)e^{-i\omega p_0 r\cos\phi}d\phi \\ &= \cos\alpha\int_0^{2\pi}\cos\phi e^{-i\omega p_0 r\cos\phi}d\phi - \sin\alpha\int_0^{2\pi}\sin\phi e^{-i\omega p_0 r\cos\phi}d\phi \\ &= \cos\alpha(-2\pi i)J_1(\omega p_0 r) - \sin\alpha\cdot 0\end{aligned}$$

Second,

$$\int_0^{2\pi} \sin\phi e^{-i\omega p_0 r\cos(\phi-\alpha)} d\phi,$$
$$=\int_\alpha^{2\pi+\alpha} \sin(\phi+\alpha) e^{-i\omega p_0 r\cos\phi} d(\phi+\alpha),$$
$$=\int_0^{2\pi} \sin(\phi+\alpha) e^{-i\omega p_0 r\cos\phi} d\phi,$$
$$=\int_0^{2\pi} (\sin\phi\cos\alpha+\sin\alpha\cos\phi) e^{-i\omega p_0 r\cos\phi} d\phi,$$
$$=\sin\alpha\int_0^{2\pi} \cos\phi e^{-i\omega p_0 r\cos\phi} d\phi+\cos\alpha\int_0^{2\pi} \sin\phi e^{-i\omega p_0 r\cos\phi} d\phi$$
$$=\sin\alpha(-2\pi i)J_1(\omega p_0 r)-\cos\alpha\cdot 0$$

Thirdly,

$$\int_0^{2\pi} e^{-i\omega p_0 r\cos(\phi-\alpha)} d\phi$$
$$=\int_\alpha^{2\pi+\alpha} e^{-i\omega p_0 r\cos\phi} d(\phi+\alpha)$$
$$=\int_0^{2\pi} e^{-i\omega p_0 r\cos\phi} d\phi$$
$$=2\pi J_0(\omega p_0 r)$$

Here J_0 and J_1 are zeroth- and first-order Bessel functions, obtained using standard identities. Substituting these results back into equation 7 yields the final expressions for P-waves emanating from a point source in a homogeneous VTI medium:

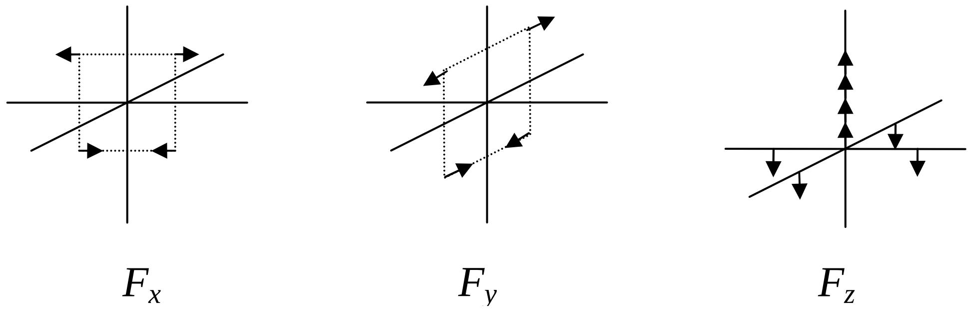
$$\mathbf{S}^{(\mathrm{P})}(\omega,\mathbf{x})=\frac{B\omega^2}{4\pi c_{33}c_{55}}\int_0^\infty\left[-ip_0T_r^{\mathrm{P}}J_1(\omega p_0 r)\cos\alpha,-ip_0T_r^{\mathrm{P}}J_1(\omega p_0 r)\sin\alpha,\xi T_z^{\mathrm{P}}J_0(\omega p_0 r)\right]\frac{e^{-i\omega\xi z}}{\left(\xi^2-\eta^2\right)}\frac{p_0}{\xi}dp_0$$

Transforming from (S_x, S_y, S_z) to (S_r, S_α, S_z) further yields

$$\mathbf{S}^{(\mathrm{P})}(\omega,\mathbf{x})=\frac{B\omega^2}{4\pi c_{33}c_{55}}\int_0^\infty\left[-ip_0T_r^{\mathrm{P}}J_1(\omega p_0 r),0,\xi T_z^{\mathrm{P}}J_0(\omega p_0 r)\right]\frac{e^{-i\omega\xi z}}{\left(\xi^2-\eta^2\right)}\frac{p_0}{\xi}dp_0. \qquad (10)$$

Point-sources for SV-waves

The point source for an SV-wave is somewhat more complicated than for a P-wave. The required toroidal motion can be produced by a combination of sixteen equal-magnitude point forces oriented about the origin as indicated in the diagram below (cf. Figure 6.11 in Aki & Richards (1980).):



By a procedure analogous to the method of moments, the effect of these combinations of point forces can be generated by applying particular 2nd-order derivatives to equation 2. The choice of 2nd-derivative can be deduced from inspection of the above diagrams. The first set, denoted F_x, are equivalent to applying $\partial^2/\partial x\partial z$ to the solution for the point force (B,0,0). The second set, denoted F_y, are equivalent to applying $\partial^2/\partial y\partial z$ to the solution for the point force (0,B,0). The third set, denoted F_z, are equivalent to applying $-\partial^2/\partial x^2 - \partial^2/\partial y^2$ to the solution for the point force (0,0,B). We note that these operators are exactly the components of the double curl applied to an SV-wave potential for isotropic media in order to obtain SV-wave displacements (see Aki and Richards (1980), p. 217). If we now restrict consideration to the SV-wave emanating from such a point source (ν = SV), but sum over the three Cartesian coordinate directions, equation 2 becomes

$$\begin{aligned}
\mathbf{S}^{(\mathrm{SV})}(\omega,\mathbf{x}) &= \frac{\partial^2}{\partial x\partial z}\mathbf{S}^{(\nu=\mathrm{SV},1)}(\omega,\mathbf{x}) + \frac{\partial^2}{\partial y\partial z}\mathbf{S}^{(\nu=\mathrm{SV},2)}(\omega,\mathbf{x}) - \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\mathbf{S}^{(\nu=\mathrm{SV},3)}(\omega,\mathbf{x}) \\
&= -\frac{i\omega}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\mathbf{U}^{(\mathrm{SV},1)}(p_1,p_2)\frac{\partial^2}{\partial x\partial z}e^{-i\omega(p_1x_1+p_2x_2+\eta x_3)} + \dots\right]dp_1dp_2 \\
&= \frac{i\omega^3}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[p_1\eta\mathbf{U}^{(\mathrm{SV},1)} + p_2\eta\mathbf{U}^{(\mathrm{SV},2)} - \left(p_1^2+p_2^2\right)\mathbf{U}^{(\mathrm{SV},3)}\right]e^{-i\omega(p_1x_1+p_2x_2+\eta x_3)}dp_1dp_2 \qquad (11)
\end{aligned}$$

where

$$\begin{aligned}
&p_1\eta\vec{U}_1^{(\mathrm{SV})} + p_2\eta\vec{U}_2^{(\mathrm{SV})} - (p_1^2+p_2^2)\vec{U}_3^{(\mathrm{SV})} \\
&= \frac{B\left(p_1\eta H_{11}^{\mathrm{ad}} + p_2\eta H_{12}^{\mathrm{ad}} - (p_1^2+p_2^2)H_{13}^{\mathrm{ad}},\, p_1\eta H_{21}^{\mathrm{ad}} + p_2\eta H_{22}^{\mathrm{ad}} - (p_1^2+p_2^2)H_{23}^{\mathrm{ad}},\, p_1\eta H_{31}^{\mathrm{ad}} + p_2\eta H_{32}^{\mathrm{ad}} - (p_1^2+p_2^2)H_{33}^{\mathrm{ad}}\right)}{2c_{33}c_{55}\eta\left(\eta^2-\xi^2\right)\left(\rho - c_{55}\left[p_1^2+p_2^2\right] - c_{66}\eta^2\right)}
\end{aligned}$$

As in the P-wave case we define $p_0 = \sqrt{p_1^2+p_2^2}$, $p_1 = p_0\cos\phi$, and $p_2 = p_0\sin\phi$. We further evaluate the $\mathbf{H}_{ij}^{\mathrm{adj}}$ to produce (cf. Tsvankin, Eq. (2.28))

$$\begin{aligned}\mathbf{S}^{(\mathrm{SV})}(\omega,\mathbf{x}) &= \frac{i\omega^3}{4\pi^2}\int_0^{2\pi}\int_0^{\infty}\left[p_0\eta(\mathbf{U}^{(\mathrm{SV},1)}\cos\phi+\mathbf{U}^{(\mathrm{SV},2)}\sin\phi)-p_0^2\mathbf{U}^{(\mathrm{SV},3)}\right]e^{-i\omega(p_0 r\cos(\phi-\alpha)+\eta z)}p_0dp_0d\phi \\ &= \frac{i\omega^3}{4\pi^2}\int_0^{2\pi}\int_0^{\infty}\left[\frac{-Bp_0\left(\eta T_r^{\mathrm{SV}}\cos\phi,\eta T_r^{\mathrm{SV}}\sin\phi,p_0T_z^{\mathrm{SV}}\right)}{2c_{33}c_{55}\eta\left(\eta^2-\xi^2\right)}\right]e^{-i\omega(p_0 r\cos(\phi-\alpha)+\eta z)}p_0dp_0d\phi \\ &= \frac{iB\omega^3}{8\pi^2c_{33}c_{55}}\int_0^{2\pi}\int_0^{\infty}\left(\eta T_r^{\mathrm{SV}}\cos\phi,\eta T_r^{\mathrm{SV}}\sin\phi,p_0T_z^{\mathrm{SV}}\right)\frac{e^{-i\omega(p_0 r\cos(\phi-\alpha)+\eta z)}}{\left(\xi^2-\eta^2\right)}\frac{p_0^2}{\eta}dp_0d\phi \qquad (12)\end{aligned}$$

We have also once again substituted $x_1 = r\cos\alpha$, $x_2 = r\sin\alpha$ and $x_3 = z$, and note that ξ and η depend on p_0 but not on ϕ, so all ϕ dependence is given explicitly in the above expressions. The quantities T_r^{SV} and T_z^{SV} result from summing and simplifying the adjoint matrix elements in equation 6 (MAPLE was used to assist in this procedure). They can be expressed as

$$T_r^{\mathrm{SV}} = c_{33}\eta^2+(c_{13}+2c_{55})p_0^2-\rho \qquad (13)$$

$$T_z^{\mathrm{SV}} = c_{11}p_0^2+\left(c_{13}+2c_{55}\right)\eta^2-\rho \qquad (14)$$

The ϕ dependence of equation 12 is dealt with in the identical fashion to that in equation 7. The procedure applied to equation 12 yields the final expressions for SV-waves emanating from an SV-point source in a homogeneous VTI medium:

$$\mathbf{S}^{(\mathrm{SV})}(\omega,\mathbf{x}) = \frac{iB\omega^3}{4\pi c_{33}c_{55}}\int_0^{\infty}\left(-i\eta T_r^{\mathrm{SV}}J_1(\omega p_0 r)\cos\alpha,-i\eta T_r^{\mathrm{SV}}J_1(\omega p_0 r)\sin\alpha,p_0T_z^{\mathrm{SV}}J_0(\omega p_0 r)\right)\frac{e^{-i\omega\xi z}}{\left(\xi^2-\eta^2\right)}\frac{p_0^2}{\eta}dp_0$$

Converting from (S_x, S_y, S_z) to (S_r, S_α, S_z) further yields

$$\mathbf{S}^{(\mathrm{SV})}(\omega,\mathbf{x}) = \frac{iB\omega^3}{4\pi c_{33}c_{55}}\int_0^{\infty}\left(-i\eta T_r^{\mathrm{SV}}J_1(\omega p_0 r),0,p_0T_z^{\mathrm{SV}}J_0(\omega p_0 r)\right)\frac{e^{-i\omega\xi z}}{\left(\xi^2-\eta^2\right)}\frac{p_0^2}{\eta}dp_0 . (15)$$

Relationship to plane-wave solutions

Graebner (1992) has shown that plane waves in a homogeneous VTI medium are eigenvectors of the equations

$$\begin{pmatrix} c_{11}p^2+c_{55}q^2-\rho & \left(c_{13}+c_{55}\right)pq \\ \left(c_{13}+c_{55}\right)pq & c_{55}p^2+c_{33}q^2-\rho \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $(u,v) = (l_\alpha,m_\alpha)$ for the eigenvalue $q = \xi$, and where $(u,v) = (m_\beta,-l_\beta)$ for $q = \eta$. Here p signifies the same as p_0 above. Graebner obtains explicit expressions for l_i and m_i by multiplying the first equation by u, the second by v, and then subtracting one from the other. His equations 5 and 6 follow directly. There are, of course, other ways the equations can be manipulated. In particular, if, for $q = \xi$, we multiply the first equation by ξ, the second by p, and then subtract we obtain

$$\frac{l_\alpha}{m_\alpha} = \frac{pT_r^{\mathrm{P}}}{\xi T_z^{\mathrm{P}}}.$$

Similarly, if, for $q = \eta$, we multiply the first equation by p, the second by η, and then add we obtain

$$\frac{m_\beta}{l_\beta} = \frac{\eta T_r^{\mathrm{SV}}}{pT_z^{\mathrm{SV}}}.$$

Thus, within an arbitrary scaling factor, we can rewrite equations 10 and 15 as

$$\mathbf{S}^{(\mathrm{P})}(\omega,\mathbf{x}) = \omega^2 \int_0^\infty \left(-il_\alpha J_1(\omega p_0 r), 0, m_\alpha J_0(\omega p_0 r)\right) \frac{e^{-i\omega\xi z}}{\left(\xi^2 - \eta^2\right)} \frac{p_0}{\xi} dp_0 . \qquad (16)$$

$$\mathbf{S}^{(\mathrm{SV})}(\omega,\mathbf{x}) = i\omega^3 \int_0^\infty \left(-im_\beta J_1(\omega p_0 r), 0, l_\beta J_0(\omega p_0 r)\right) \frac{e^{-i\omega\xi z}}{\left(\xi^2 - \eta^2\right)} \frac{p_0^2}{\eta} dp_0 . \qquad (17)$$

Derivation of displacement reflection coefficients

Above we considered waves originating from a point source at the coordinate system origin in an infinite homogeneous medium. Now we consider a P-wave in a two-layer VTI medium beginning at a point source at $(0,0,h)$, where $h < 0$. The interface passes through the origin, so the point source is above the interface. The wave can be detected at a receiver above the interface ($z < 0$) as 1) a direct wave, 2) a reflected P-wave, or 3) a reflected SV-wave. It can also be detected at a receiver below the interface ($z > 0$) by 1) a transmitted P-wave, or 2) a transmitted SV-wave. As the initial wave contacts the interface, varying amounts of energy go into the four reflection/transmission modes, and these amounts are determined by enforcing four boundary conditions, namely, continuity of various components of displacement and traction across the welded interface.

Our objective is to see how to incorporate plane-wave reflection and transmission coefficients into point-source expressions. We consider the expressions due to Graebner (1992). The essential plane-wave form is given in his equations 9 and 10. These differ from the expressions here in the sign of the z-term of the exponential factor for the downgoing incident wave. To be consistent with this we can modify equations 16 and 17 of this paper by replacing $(-i)$ with i. We then proceed to write down (cf. Graebner's equations 11 and 12) the r and z components of (the frequency spectrum of) the displacement above and below the interface:

$$S_r^+(\omega,r,z\le 0)=i\int_0^\infty J_1(\omega p_0 r)\left(Al_{\alpha,1}e^{i\omega\xi_1|z-h|}+Bl_{\alpha,1}e^{-i\omega\xi_1 z}+(-i\omega p_0)Cm_{\beta,1}e^{-i\omega\eta_1 z}\right)dp_0 \quad (18)$$

$$S_z^+(\omega,r,z\le 0)=\int_0^\infty J_0(\omega p_0 r)\left(Am_{\alpha,1}e^{i\omega\xi_1|z-h|}-Bm_{\alpha,1}e^{-i\omega\xi_1 z}+(-i\omega p_0)Cl_{\beta,1}e^{-i\omega\eta_1 z}\right)dp_0 \quad (19)$$

$$S_r^-(\omega,r,z\ge 0)=i\int_0^\infty J_1(\omega p_0 r)\left(Dl_{\alpha,2}e^{i\omega\xi_2 z}+(-i\omega p_0)Em_{\beta,2}e^{i\omega\eta_2 z}\right)dp_0 \quad (20)$$

$$S_z^-(\omega,r,z\ge 0)=\int_0^\infty J_0(\omega p_0 r)\left(Dm_{\alpha,2}e^{i\omega\xi_2 z}-(-i\omega p_0)El_{\beta,2}e^{i\omega\eta_2 z}\right)dp_0 \quad (21)$$

By comparison with equation 16 we can see that $A\propto \omega^2 p_0\Big/\left[\left({\xi_1}^2-{\eta_1}^2\right)\xi_1\right]$. Applying the boundary conditions will then yield the ratios $R_{PP}=B/A$, $R_{PS}=C/A$, $T_{PP}=D/A$, $T_{PS}=E/A$. These require the following (frequency spectra of) traction components as well:

$$\begin{aligned}\tau_{rz}^+(\omega,r,z\le 0)&=c_{55}^+\left[\frac{\partial S_r^+}{\partial z}+\frac{\partial S_z^+}{\partial z}\right]\\ &=\omega c_{55}^+\int_0^\infty J_1(\omega p_0 r)\Big(-A\left(\xi_1 l_{\alpha,1}+p_0 m_{\alpha,1}\right)e^{i\omega\xi_1|z-h|}+B\left(\xi_1 l_{\alpha,1}+p_0 m_{\alpha,1}\right)e^{-i\omega\xi_1 z}\\ &\qquad+(-i\omega p_0)C\left(\eta_1 m_{\beta,1}-p_0 l_{\beta,1}\right)e^{-i\omega\eta_1 z}\Big)dp_0\end{aligned} \quad (22)$$

$$\begin{aligned}\tau_{zz}^+(\omega,r,z\le 0)&=c_{13}^+\frac{1}{r}\frac{\partial\left(rS_r^+\right)}{\partial r}+c_{33}^+\frac{\partial S_z^+}{\partial z}\\ &=i\omega\int_0^\infty J_0(\omega p_0 r)\Big(A\left(c_{13}^+p_0 l_{\alpha,1}+c_{33}^+\xi_1 m_{\alpha,1}\right)e^{i\omega\xi_1|z-h|}+B\left(c_{13}^+p_0 l_{\alpha,1}+c_{33}^+\xi_1 m_{\alpha,1}\right)e^{-i\omega\xi_1 z}\\ &\qquad+(-i\omega p_0)C\left(c_{13}^+p_0 m_{\beta,1}-c_{33}^+\eta_1 l_{\beta,1}\right)e^{-i\omega\eta_1 z}\Big)dp_0\end{aligned} \quad (23)$$

and similarly for τ_{rz}^- and τ_{zz}^-. The boundary conditions then are

$$S_r^+(\omega,r,z=0)=S_r^-(\omega,r,z=0) \quad (24)$$

$$S_z^+(\omega,r,z=0)=S_z^-(\omega,r,z=0) \quad (25)$$

$$\tau_{rz}^+(\omega,r,z=0)=\tau_{rz}^-(\omega,r,z=0) \quad (26)$$

$$\tau_{zz}^+(\omega,r,z=0)=\tau_{zz}^-(\omega,r,z=0) \quad (27)$$

The first of these conditions yields the expression

$$0=i\int_0^\infty J_1(\omega p_0 r)\left(Ae^{i\omega\xi_1 h}l_{\alpha,1}+Bl_{\alpha,1}+(-i\omega p_0)Cm_{\beta,1}-Dl_{\alpha,2}-(-i\omega p_0)Em_{\beta,2}\right)dp_0\,. \quad (28)$$

In order for this to be satisfied for all values of the elastic parameters, the integrand must vanish for each value of p_0. For this and the other three conditions we therefore obtain the simpler expressions

$$-Ae^{i\omega\xi_1 h}l_{\alpha,1} = Bl_{\alpha,1} + (-i\omega p_0)Cm_{\beta,1} - Dl_{\alpha,2} - (-i\omega p_0)Em_{\beta,2} \tag{29}$$

$$Ae^{i\omega\xi_1 h}m_{\alpha,1} = Bm_{\alpha,1} - (-i\omega p_0)Cl_{\beta,1} + Dm_{\alpha,2} - (-i\omega p_0)El_{\beta,2} \tag{30}$$

$$\begin{aligned} c_{55}^{+}Ae^{i\omega\xi_1 h}\left(\xi_1 l_{\alpha,1} + p_0 m_{\alpha,1}\right) = c_{55}^{+}\left[B\left(\xi_1 l_{\alpha,1} + p_0 m_{\alpha,1}\right) + (-i\omega p_0)C\left(\eta_1 m_{\beta,1} - p_0 l_{\beta,1}\right)\right] \\ + c_{55}^{-}\left[D\left(\xi_2 l_{\alpha,2} + p_0 m_{\alpha,2}\right) + (-i\omega p_0)E\left(\eta_2 m_{\beta,2} - p_0 l_{\beta,2}\right)\right] \end{aligned} \tag{31}$$

$$\begin{aligned} -Ae^{i\omega\xi_1 h}\left(c_{33}^{+}\xi_1 m_{\alpha,1} + c_{13}^{+}p_0 l_{\alpha,1}\right) = B\left(c_{33}^{+}\xi_1 m_{\alpha,1} + c_{13}^{+}p_0 l_{\alpha,1}\right) - (-i\omega p_0)C\left(c_{33}^{+}\eta_1 l_{\beta,1} - c_{13}^{+}p_0 m_{\beta,1}\right) \\ - D\left(c_{33}^{-}\xi_2 m_{\alpha,2} + c_{13}^{-}p_0 l_{\alpha,2}\right) + (-i\omega p_0)C\left(c_{33}^{-}\eta_2 l_{\beta,2} - c_{13}^{-}p_0 m_{\beta,2}\right) \end{aligned} \tag{32}$$

We can write this conveniently in matrix notation:

$$\begin{aligned} &\begin{bmatrix} l_{\alpha,1} & m_{\beta,1} & -l_{\alpha,2} & -m_{\beta,2} \\ m_{\alpha,1} & -l_{\beta,1} & m_{\alpha,2} & -l_{\beta,2} \\ c_{55}^{+}\left(\xi_1 l_{\alpha,1} + p_0 m_{\alpha,1}\right) & c_{55}^{+}\left(\eta_1 m_{\beta,1} - p_0 l_{\beta,1}\right) & c_{55}^{-}\left(\xi_2 l_{\alpha,2} + p_0 m_{\alpha,2}\right) & c_{55}^{-}\left(\eta_2 m_{\beta,2} - p_0 l_{\beta,2}\right) \\ c_{33}^{+}\xi_1 m_{\alpha,1} + c_{13}^{+}p_0 l_{\alpha,1} & -c_{33}^{+}\eta_1 l_{\beta,1} + c_{13}^{+}p_0 m_{\beta,1} & -c_{33}^{-}\xi_2 m_{\alpha,2} - c_{13}^{-}p_0 l_{\alpha,2} & c_{33}^{-}\eta_2 l_{\beta,2} - c_{13}^{-}p_0 m_{\beta,2} \end{bmatrix} \times \\ &\qquad \begin{bmatrix} e^{-i\omega\xi_1 h}B/A \\ e^{-i\omega\xi_1 h}(-i\omega p_0)C/A \\ e^{-i\omega\xi_1 h}D/A \\ e^{-i\omega\xi_1 h}(-i\omega p_0)E/A \end{bmatrix} = \begin{bmatrix} -l_{\alpha,1} \\ m_{\alpha,1} \\ c_{55}^{+}\left(\xi_1 l_{\alpha,1} + p_0 m_{\alpha,1}\right) \\ -c_{33}^{+}\xi_1 m_{\alpha,1} - c_{13}^{+}p_0 l_{\alpha,1} \end{bmatrix} \end{aligned} \tag{33}$$

Comparing this to equations 13-16 of Graebner we see that $B/A = R_{\mathrm{PP}}^{\text{plane-wave}}e^{i\omega\xi_1 h}$, $C/A = R_{\mathrm{PS}}^{\text{plane-wave}}e^{i\omega\xi_1 h}/(-i\omega p_0)$, $D/A = T_{\mathrm{PP}}^{\text{plane-wave}}e^{i\omega\xi_1 h}$, and $E/A = T_{\mathrm{PS}}^{\text{plane-wave}}e^{i\omega\xi_1 h}/(-i\omega p_0)$.

We can now write down the complete integrals for calculation of the generalized reflection coefficients:

$$
\begin{aligned}
S_r^{\mathrm{PP}}(\omega, r, z \le 0) &= i\int_0^\infty J_1(\omega p_0 r) A\big(B/A\big) l_{\alpha,1} e^{i\omega\xi_1(h-z)} dp_0 \\
&= i\omega^2 \int_0^\infty R_{\mathrm{PP}}^{\text{plane-wave}}(p_0) J_1(\omega p_0 r) \frac{l_{\alpha,1}}{\xi_1^2 - \eta_1^2} e^{i\omega\xi_1(h-z)} \frac{p_0}{\xi_1} dp_0
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
S_z^{\mathrm{PP}}(\omega, r, z \le 0) &= \int_0^\infty J_0(\omega p_0 r) A\big(-B/A\big) m_{\alpha,1} e^{i\omega\xi_1(h-z)} dp_0 \\
&= -\omega^2 \int_0^\infty R_{\mathrm{PP}}^{\text{plane-wave}}(p_0) J_0(\omega p_0 r) \frac{m_{\alpha,1}}{\xi_1^2 - \eta_1^2} e^{i\omega\xi_1(h-z)} \frac{p_0}{\xi_1} dp_0
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
S_r^{\mathrm{PS}}(\omega, r, z \le 0) &= i\int_0^\infty J_1(\omega p_0 r) A\big(C/A\big)(-i\omega p_0) m_{\beta,1} e^{i\omega(\xi_1 h - \eta_1 z)} dp_0 \\
&= i\omega^2 \int_0^\infty R_{\mathrm{PS}}^{\text{plane-wave}}(p_0) J_1(\omega p_0 r) \frac{m_{\beta,1}}{\xi_1^2 - \eta_1^2} e^{i\omega(\xi_1 h - \eta_1 z)} \frac{p_0}{\xi_1} dp_0
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
S_z^{\mathrm{PS}}(\omega, r, z \le 0) &= \int_0^\infty J_0(\omega p_0 r) A\big(C/A\big)(-i\omega p_0) l_{\beta,1} e^{i\omega(\xi_1 h - \eta_1 z)} dp_0 \\
&= -\omega^2 \int_0^\infty R_{\mathrm{PS}}^{\text{plane-wave}}(p_0) J_0(\omega p_0 r) \frac{l_{\beta,1}}{\xi_1^2 - \eta_1^2} e^{i\omega(\xi_1 h - \eta_1 z)} \frac{p_0}{\xi_1} dp_0
\end{aligned} \tag{37}
$$

Some comments are in order regarding evaluation of these integrals. As p_0 approaches $1/\alpha$, p_0/ξ_1 will diverge. Therefore it is more convenient to replace $(p_0/\xi_1)\mathrm{d}p_0$ with $-d\xi_1$, even though it yields slightly more complicated integration limits. Next consider whether

$$
\xi_1^2 - \eta_1^2 = \left(\frac{1}{\alpha_1^2} - p_0^2\right) - \left(\frac{1}{\beta_1^2} - p_0^2\right) = \frac{1}{\alpha_1^2} - \frac{1}{\beta_1^2} \tag{38}
$$

will vanish at any point. Although the velocities α_1 and β_1 vary with direction, they will not be equal for any ray parameter, so this term will not produce a singularity.

SUMMARY

The results of equations 34-37 provide the methodology for calculating reflection coefficients for waves emanating from point sources in VTI media. They are explicitly given as weighted integrals over ray-parameter of the plane-wave reflection coefficients. They are defined for specific frequencies, and further manipulation is therefore necessary to obtain the actual reflection coefficients exhibited in time traces. Such manipulations have been carried out in order to present information relevant to AVO studies (Haase & Ursenbach, 2005a,b).

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