Snell's law in transversely isotropic media using linearized group velocities and related quantities

P.F. Daley

ABSTRACT

Using a linearized approximation for the quasi-compressional phase velocity, $v_{qP}\left(n_k\right)$ in a transversely isotropic (TI) medium, which is a subset of the related quasi-compressional (qP) wave propagation in an orthorhombic medium, a linearized compressional group velocity, $V_{qP}\left(N_k\right)$, is derived as a function of group angles only. In addition, analytic expressions for the components of the slowness vector in terms of group velocities and angles are also obtained. These expressions are used here to set up the generally nonlinear equations that are required to be solved for the reflected and transmitted rays due to an incident ray, at a plane interface between two transversely isotropic media, when the axes of anisotropy, in both media, are, in general, not aligned with the interface. The total medium is assumed to be composed of finite elements, specifically Delauney triangles, which are used to account for vertical and lateral inhomogeneities as well as anisotropy with arbitrary orientation.

INTRODUCTION

In the geophysical literature related to wave propagation in anisotropic media, specifically quasi-compressional *(qP)* waves in a medium displaying orthorhombic symmetry, a linearized approach to get an approximate phase velocity expression for quasi-compressional *(qP)* wave propagation has been presented in Backus (1965). This and other methods, such a perturbation theory, have been employed to extract expressions for the qP case as well as for the two shear wave modes, qS_1 and qS_2, for the general 21 parameter anisotropic medium (Every, 1980, Every and Sachse., 1992, Jech. and Pšenčík, 1989, Pšenčík and Gajewski, 1998, Song, Every, and Wright, 2001, Pšenčík. and Farra, 2005). Once phase velocity approximations have been obtained, eikonals with similar accuracy may be written. From these, using the method of characteristics, (Courant and Hilbert, 1962) the formulae for the vector components of group velocity may be derived

Employing other methods of approximation, Daley and Krebes (2005) derived an expression for the qP group velocity obtained in terms of group angles. This formula was originally presented in Song and Every (2000) where the results were"***…*** *not established … by rigorous derivation but we were lead to [them] by plausibility arguments that are backed up by the numerical results …*" .

It is shown in Daley and Krebes (2005) that, for a weakly anellipsoidal anisotropic medium, the average deviation of this approximation from the exact expression for the qP group velocity, in medium with orthorhombic symmetry, over a range of polar angles $\left(0 \le \Theta < 2\pi\right)$ for a number of azimuths, Φ, was of the order of 1%. For this reason it was chosen for use in a number of procedures where speed was essential in ray tracing in two and three dimensions. Specifically, Born-Kirchhoff type migration, in a tessellated

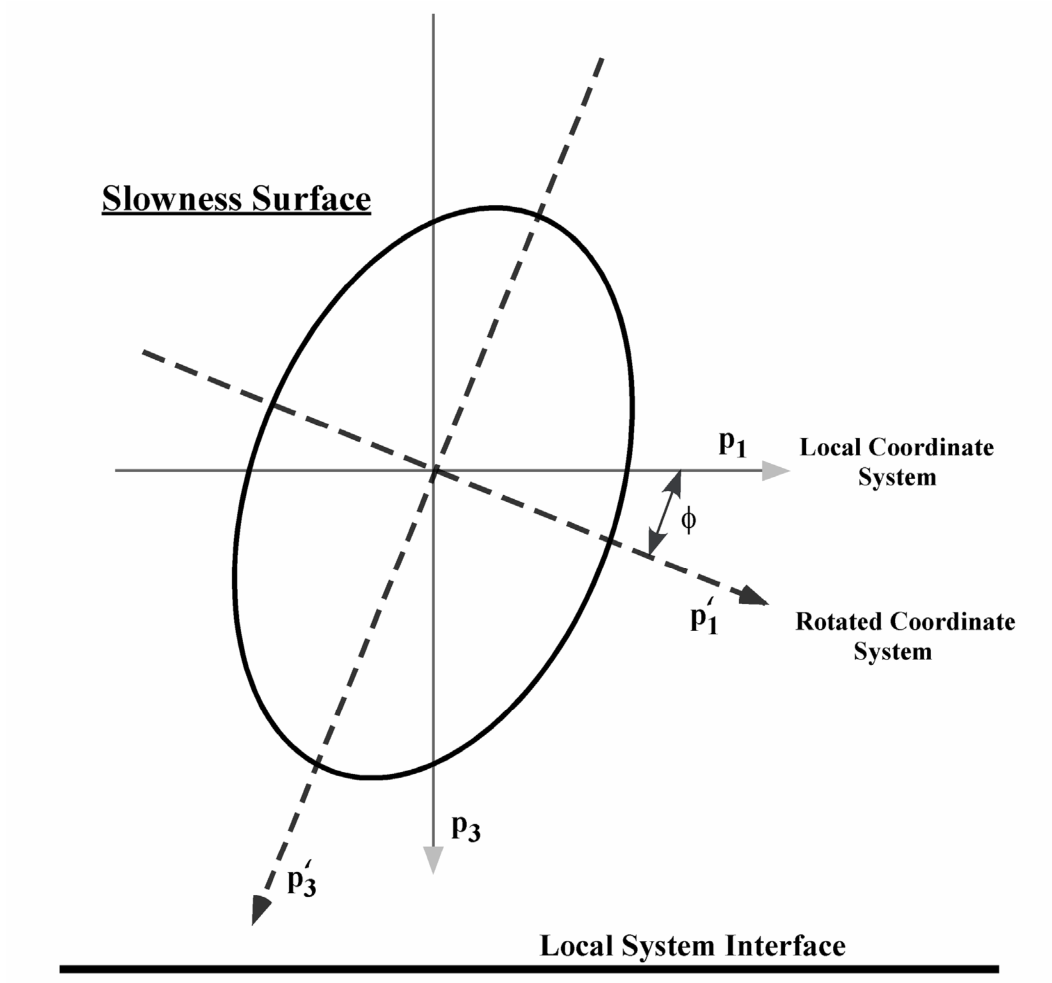

FIG. 1. Relationship between local and rotated coordinate systems. The angle ϕ is measured positive clockwise from the local coordinate system.

2D medium, was of interest. The anisotropic parameters and orientation of anisotropy are assumed constant within any triangle. What is required to be determined is an analogue of Snell's Law at the plane interface between two adjacent triangles. This will be treated in what follows for a transversely isotropic (TI) type medium as an initial step in the more complex 3D problem, where there is yet to appear an algorithm of any sophistication to produce a *quality* tetrahedral discretization.

Before proceeding further, a mathematically informal explanation of the duality of the ray or group velocity space and slowness space will be given. The informality is attributable to the weakly anelliptic assumption used here, specifically, both surfaces are assumed to be convex, which implies that there are no triplication points on the ray surface. Thus, given a point on the ray (group velocity) surface, the vector starting at the origin of this surface and normal to the tangent at the point at which the ray touches the ray surface is equal in direction to the corresponding vector in slowness space and its magnitude in slowness space is equal to the inverse of its magnitude in ray space. The dual of this is that in slowness space, for some arbitrary slowness vector, the vector beginning at the origin of the slowness surface that is normal to the tangent to the point at which the slowness vector impinges on the slowness surface, is equal in direction to the corresponding ray vector and has a magnitude that is the inverse of the length of tfining the vector in slowness space. In what follows, all figures will reference slowness space only, with Figure 1 defining the local and rotated coordinate systems in this space.

A SIMPLE EXAMPLE

It will be assumed that the horizontal component of the slowness vector, p_1, is known in the incident (upper) medium, and it is required to determine the ray angle and magnitude for both the reflected ray in the upper medium and the transmitted ray in the lower medium, in the special instance where the axes of anisotropy in both media are aligned with the intervening interface, i.e., the rotated and model axes are aligned, with the incident ray angle known. (Figures 2 and 3.)

The linearized quasi-compressional phase velocity, $v_{qP}\left(n_k\right)$, in a TI medium may be written as

$$v_{qP}^2\left(n_k\right) = A_{11}n_1^2 + A_{33}n_3^2 + E_{13}n_1^2n_3^2 \tag{1}$$

with the anelliptic term defined as

$$E_{13} = 2\left(A_{13} + 2A_{55}\right) - \left(A_{11} + A_{33}\right). \tag{2}$$

This expression should be compared to that given for a *mildly* anisotropic medium as presented in Gassmann (1964) as an indication of how the linearization simplifies the phase velocity expression.

The *2D* phase velocity propagation vector direction is defined as the unit vector

$$\mathbf{n} = (n_1, n_3) = (\sin\theta, \cos\theta). \tag{3}$$

Thus the slowness vector has the form

$$\mathbf{p} = (p_1, p_3) = \left(\frac{n_1}{v_{qP}(n_k)}, \frac{n_2}{v_{qP}(n_k)}\right). \tag{4}$$

It was shown by Daley and Hron (1979) that the slowness vector for the *qP* elliptical (degenerate) case may be written exactly in terms of the group velocity quantities, angle, Θ, and velocity, $V_{qP}\left(N_k\right)$, as

$$\mathbf{p} = (p_1, p_3) = \left(\frac{\left[N_1 V_{qP}(N_k)\right]_{ellip.}}{A_{11}}, \frac{\left[N_3 V_{qP}(N_k)\right]_{ellip.}}{A_{33}}\right) \tag{5}$$

where

$$\mathbf{N} = (N_1, N_3) = (\sin\Theta, \cos\Theta) \tag{6}$$

and the group velocity for the elliptical case is given by

$$\left[\frac{1}{V_{qP}^2\left(N_k\right)}\right]_{ellip.} = \frac{N_1^2}{A_{11}} + \frac{N_3^2}{A_{33}} = \frac{\sin^2\Theta}{A_{11}} + \frac{\cos^2\Theta}{A_{33}}. \tag{7}$$

The general group velocity expression is obtained by using the elliptical case as a trial solution with the additional assumption that the polarization vector components, g_k, may be approximated as $g_1 g_3 \approx N_1 N_3$. This is not an unreasonable assumption, as in the original linearization process the polarization vector components were approximated by n_k, i.e., $g_1 g_3 \approx n_1 n_3$. The resultant group velocity expression for a *qP* ray in a TI medium is

$$\frac{1}{V_{qP}^2\left(N_k\right)} = \frac{N_1^2}{A_{11}} + \frac{N_3^2}{A_{33}} - \frac{E_{13}N_1^2N_3^2}{A_{11}A_{33}} = \frac{\sin^2\Theta}{A_{11}} + \frac{\cos^2\Theta}{A_{33}} - \frac{E_{13}\sin^2\Theta\cos^2\Theta}{A_{11}A_{33}}. \tag{8}$$

As it has been assumed that the axes of anisotropy, in both the incident (upper) and transmitted (lower) layer, were aligned with the intervening interface, the reflected ray would have the same magnitude and acute angle with the normal at the point of incidence as the incident ray. It is to be remembered that the approximate expressions employed here are for *weakly anelliptic* anisotropy, and it will be assumed that a user will not seriously violate this constraint. This is shown graphically in Figure 2 where the reflected ray vector is equal in magnitude to the incident ray vector, and both vectors have the same acute angle with respect to the normal to the interface as do the corresponding slowness vectors.

The sine of the group angle in the medium of transmission $\left(x = \sin\Theta\right)$ will be used as the parameter to be determined using Newton's Method. In the elliptical case it may be determined analytically as

$$x = \frac{p_1 A_{11}}{\left[A_{33} + \left(A_{11} - A_{33}\right)p_1^2\right]^{1/2}} \tag{9}$$

Since Snell's Law in the elliptical case requires the equivalence of the horizontal components of the slowness vector,

$$p_1 = \frac{N_1 V_{qP}\left(N_k\right)}{A_{11}} = \frac{\sin\Theta V_{qP}\left(\Theta\right)}{A_{11}} = \frac{x V_{qP}\left(x\right)}{A_{11}}. \tag{10}$$

The equation to be solved numerically in an iterative manner is

$$F\left(x\right) = p_1 A_{11} - x V_{qP}\left(x\right) = 0. \tag{11}$$

The derivative of $F\left(x\right)$ with respect to *x* is

$$\frac{dF\left(x\right)}{dx} = -V_{qP}\left(x\right) - x\frac{dV_{qP}\left(x\right)}{dx} \tag{12}$$

where

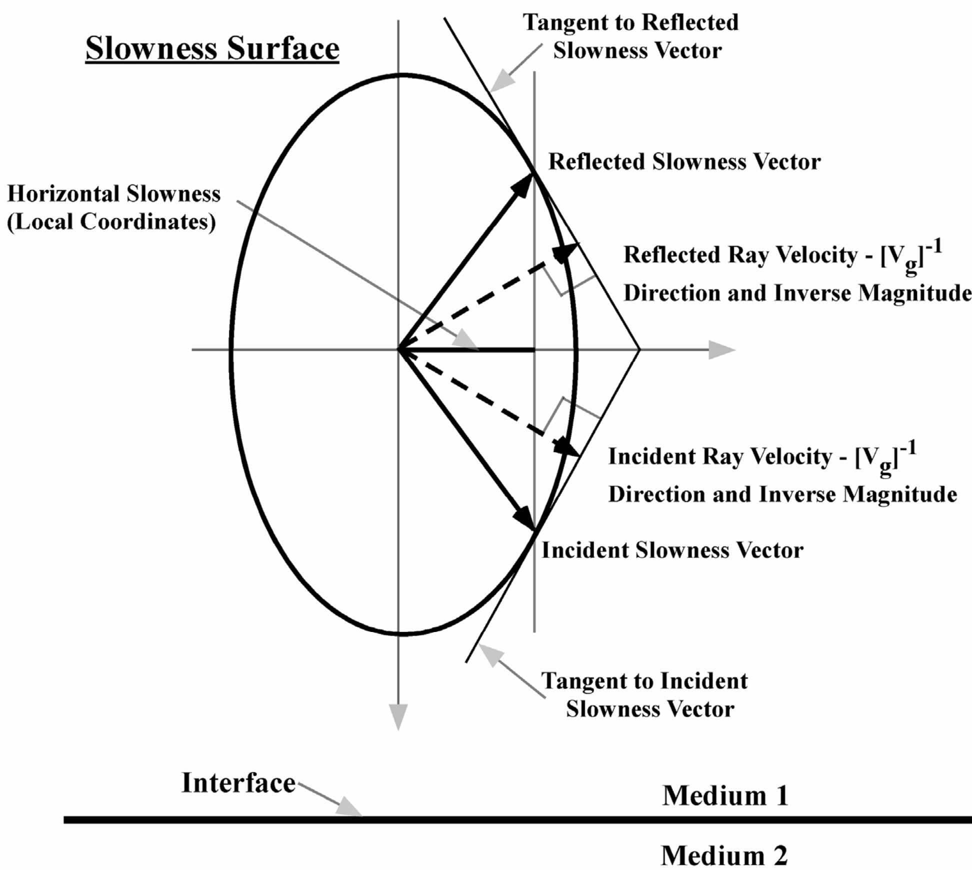

FIG. 2. Incident and reflected slowness vectors and inverse group (ray) velocity vectors for the case where the rotated and local coordinate systems align.

$$\frac{dV_{qP}(x)}{dx} = x\left[\frac{1}{A_{11}} - \frac{1}{A_{33}} - \frac{(1-2x^2)E_{13}}{A_{11}A_{33}}\right], \tag{13}$$

so that with x_{k+1}, being a refinement of the iterated solution, x_k, Newton's Method has the standard formulation

$$x_{k+1} = x_k - \frac{F(x_k)}{dF(x_k)/dx} \tag{14}$$

with x_0 given by equation (8). Once $x = \sin\Theta$ has been determined, the group velocity, and as a result, the components of the slowness vector in the lower medium, may be determined employing a generalization of equation (5).

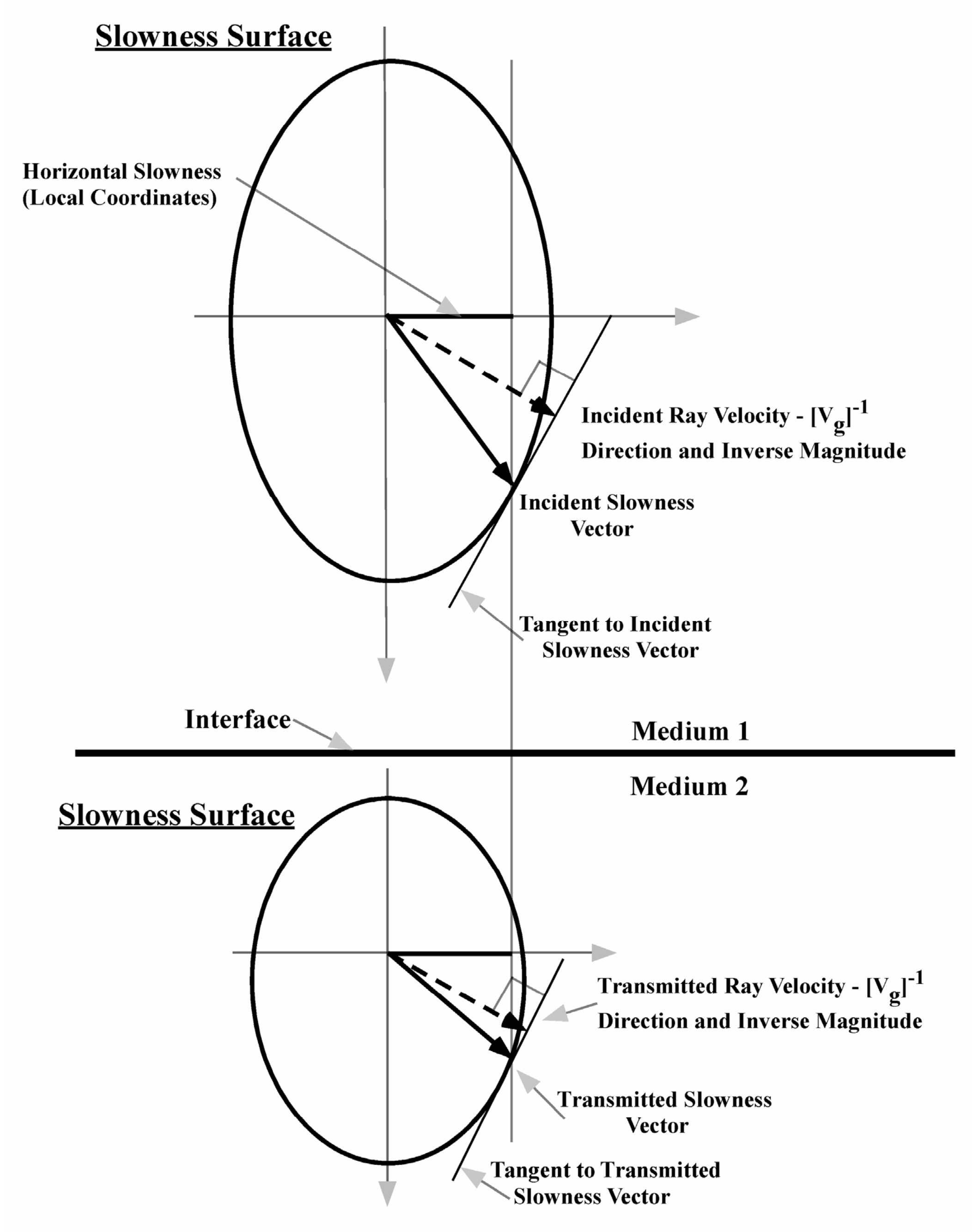

FIG. 3. Incident and transmitted slowness vectors and inverse group (ray) velocity vectors for the case where the rotated and local coordinate systems align.

REFLECTION AT A PLANE INTERFACE WITH THE AXES OF ANISOTROPY NOT ALIGNED WITH THE INTERFACE IN THE INCIDENT MEDUM

For this problem, it is assumed that the incident group velocity is known at the interface, in the rotated coordinate system. That is, its magnitude is known in both the rotated and local Cartesian system, but the incident angle is known only in the rotated system. This coordinate system is oriented at an acute angle ϕ, measured positive clockwise from the normal to the interface, into the incident medium, defining the local coordinate system. If $\phi = 0$, as would be expected, the magnitudes of the incident and reflected qP group velocities are equal as are the acute angles Θ_i and Θ_r, measured from the normal to the interface, the subscripts "i" and "r" referring to incident and reflected, respectively.

Given that ϕ is known, the angle $\Theta_i' = \Theta_i - \phi$ is easily determined, the primed angle indicating that it is measured in the rotated coordinate system. What is required to be solved for is the reflected angle and magnitude of the qP group velocity in this system. The horizontal component of the slowness vector **p** in the primed system may be obtained, with $x' = \sin(\Theta - \phi)$

$$F(p_1') = p_1' A_{11} - x' V_{qP}(x') = 0 \tag{15}$$

$$p_1' = \frac{x' V_{qP}(x')}{A_{11}} \tag{16}$$

In a similar manner, the vertical component of the slowness vector in the rotated system is obtained from

$$G(p_3') = p_3' A_{33} - \left[1 - (x')^2\right]^{1/2} V_{qP}(x') = 0 \tag{17}$$

$$p_3' = \frac{\left[1 - (x')^2\right]^{1/2} V_{qP}(x')}{A_{33}} \tag{18}$$

As the rotation of the primed system relative to the local system is orthonormal, the slowness vector p, in local coordinates, may be obtained as

$$\begin{bmatrix} p_1 \\ p_3 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p_1' \\ p_3' \end{bmatrix} \tag{19}$$

Using the methods of the previous section the reflected group velocity and angle may be obtained. The orientation of the local vertical axis in both slowness space and ray space must be taken in the proper sense as a reflected and transmitted ray and slowness vertical component of each individual vector have different signs.

TRANSMISSION AT A PLANE INTERFACE WITH THE AXES OF ANISOTROPY NOT ALIGNED WITH THE INTERFACE

In the previous section showed how to determine the horizontal component of slowness in the incident medium if the axis of anisotropy was not aligned within that medium. Here it will be assumed that this quantity is known, and what is required to be established are the slowness vector and group velocity magnitude, and angle with respect to the vertical component of the slowness vector which is known as the angle of rotation ϕ of the rotated (primed) system in the transmitted medium in both the local and rotated coordinate system. This case is depicted in Figure 4

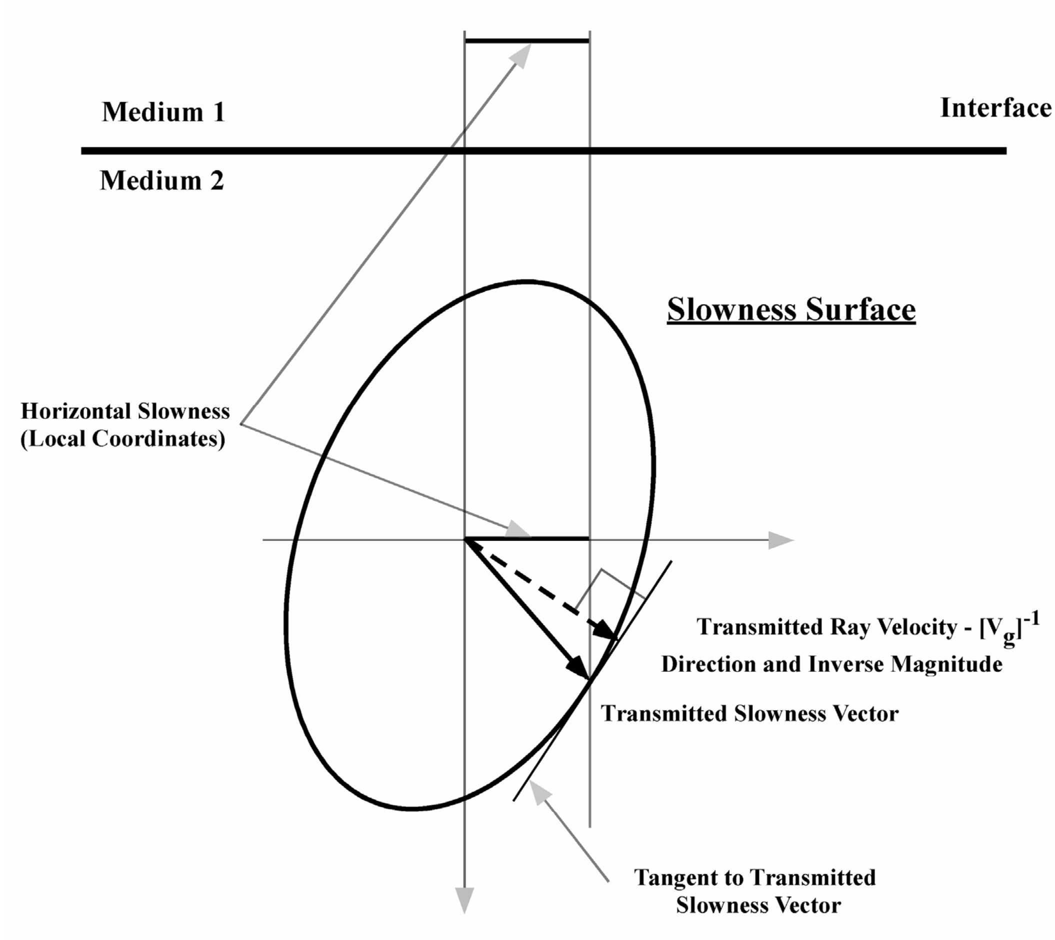

FIG. 4. Transmission at an interface where the local and rotated coordinate systems are not aligned.

In a similar manner as in the previous section where $\mathbf{p} = (\, p_1, p_3\,)$ was obtained in local coordinates, the vector $\mathbf{p}' = (\, p_1', p_3'\,)$ in the rotated system is obtained in the lower medium through the orthonormal transformation

$$\begin{bmatrix} p_1' \\ p_3' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p_1 \\ p_3 \end{bmatrix}. \tag{20}$$

As in the first section of this report with $x = \sin$ being replaced by $x' = \sin(\Theta - \phi)$ the following equation, which also must be solved numerically, is obtained

$$F(\,x'\,) = p_1' A_{11} - x' V_{qP}(\,x'\,) = 0. \tag{21}$$

The ancillary formulae for accomplishing this are given in equations (9) – (14). The vertical component of displacement, p_3' is obtained from equation (18), and the slowness vector may be put in terms of the local coordinate system using equation (20), while the group velocity magnitude, $V(\Theta)$, is obtained from the sequence $\Theta' = \sin^{-1} x'$, $\Theta = \Theta' - \phi$, $V(\Theta) = V(\Theta' - \phi)$.

As the value of ϕ in all probability takes on a different value, say $\hat{\phi}$, the value of $\mathbf{p} = (p_1, p_3)$ in this new local coordinate system is obtained from the results of solving equation (21), so that

$$\begin{bmatrix} p_1 \\ p_3 \end{bmatrix} = \begin{bmatrix} \cos\hat{\phi} & -\sin\hat{\phi} \\ \sin\hat{\phi} & \cos\hat{\phi} \end{bmatrix} \begin{bmatrix} p_1' \\ p_3' \end{bmatrix} \tag{22}$$

This updated specification of the slowness vector components is used as input, either for a reflected or transmitted ray at the edge between the current triangle of propagation and the next adjacent triangle.

CONCLUSIONS AND FUTURE WORK

The basic formulae and solution method for tracing rays in a tessellated transversely isotropic medium have been presented. Linearized forms of the phase and group velocities were used in the development. Numerical experimentation with a more complex media type, an orthorhombic medium, for arbitrary azimuthal angles has been shown to provide acceptable results using these weakly anelliptic approximations for phase and group velocities. In a symmetry plane, which amounts to a transversely isotropic medium, more accurate results are expected than in some arbitrary azimuthal plane. It is to be remembered that the formulae used are for weakly anelliptic media and contravention of this constraint will lead to a degradation of results. For a medium that satisfies this constraint, the average deviation from the exact solution over the polar angle range of $0 \le \Theta < 2\pi$ was of the order of 0.05%. In a medium, with the alternate specification of anisotropic parameters of $\varepsilon \approx 0.2$, $\delta \approx 0.7$, the average deviation, when compared to the exact solution over the total polar range, was of the order of 1.5%.

What remains to be done is to determine how to best specify the anisotropic parameters within a triangle, using either their centroid values or a weighted average on each element edge, or a combination. Further, it is required to determine if some type of smoothing needs to be done. This may not be required if an appropriate manner of anisotropic parameterization is employed.

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