Quasi-compressional group velocity approximation in a weakly general 21 parameter anisotropic medium

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ABSTRACT

Using a linearized approximation of the phase velocity related to quasi-compressional *(qP)* wave propagation in a weakly general 21 parameter anisotropic medium, an approximate eikonal equation is constructed. The corresponding expression for the related group velocity is then derived. The degenerate (ellipsoidal) case of *(qP)* wave propagation in an anisotropic medium is explored and an exact group velocity expression obtained, together with the exact expressions for the slowness vector components, for this reduced case. This ellipsoidal group velocity is taken as the reference or background velocity surface. Slowness vector components are in terms of the group velocity vector angles. This result is employed as a trial solution in the approximate eikonal equation, where the related group velocity surface is taken to be a perturbed ellipsoid. The group velocity expressions, both approximate and exact, are numerically compared for an anisotropic model that may be classified as weakly anisotropic or, possibly more accurately, weakly anellipsoidal, as the background group velocity surface used is an ellipsoid.

INTRODUCTION

In the recent literature on wave propagation in anisotropic media a number of approximate techniques, usually based on perturbation theory, have been used to advance the understanding of wave propagation in these complex anisotropic structures. The motivation for this is that the exact analytical expressions for quantities such as eikonal equations, phase and group velocities and polarization vectors are so complex that they usually reveal inadequate information when attempting to determine their significance. The general linearized anisotropic problem is considered in Jech and Pšenčík (1989), Pšenčík and Gajewski (1998) and Every and Sachse (1992) and other cited references.

Explicit expressions for *qP* ray tracing, yielding linearized group velocity approximations, in the general, as well as subset media types, may be found in Pšenčík and Farra (2005). In that paper, an isotropic background medium is assumed. However, as the exact solution for a reduced linearized problem may be determined for an ellipsoid, it is this that will be used initially as a reference velocity surface or background medium. This follows from a statement in Mensch and Farra (1999): ”Examples obtained in homogeneous orthorhombic medium show that a reference media with ellipsoidal anisotropy is a better choice to develop the perturbation approach than an isotropic reference medium.” The extension to a more general anisotropic medium follows from this, as a sphere is point wise topologically equivalent to an ellipsoid, it is that surface type that will be used here as a reference surface. What is presented is an extension of the theory of a previous work (Daley and Krebes, 2006) where a similar problem for an orthorhombic medium was investigated.

To establish the accuracy of the approximations, characteristic theory is used to obtain the exact expressions for the group velocity vector components in a general orthorhombic medium, employing the exact *qP* phase velocity expression and hence eikonal (Every, 1980 and Schoenberg and Helbig, 1997). These exact formulae are not included, as they are cumbersome, and freely available software such as the series of programs, the latest being ANRAY95, (Gajewski and Psencik, 1989) may be used for these computations. For simplicity, but without much loss of generality, the medium of propagation is assumed to be homogeneous, i.e., the anisotropic elastic parameters are independent of the spatial coordinates.

THEORETICAL PRELIMINARIES

The linearization process presented by Backus (1965) results in an equation for the phase velocity in a general 21 parameter anisotropic medium (see also, Every and Sache, 1992). This approach is a result of approximating the components of the unit quasi – compressional, *qP*, polarization vector, $\mathbf{g}_{qP}$, by those of the unit phase vector, $\mathbf{n} = (n_1, n_2, n_3)$, to obtain a linearization *qP* phase velocity of the form

$$\begin{aligned} v_{qP}^2(n_k, x_k) = \{ & A_{11}n_1^4 + A_{22}n_2^4 + A_{33}n_3^4 + 2(A_{12} + A_{66})n_1^2 n_2^2 + \\ & 2(A_{13} + A_{66})n_1^2 n_3^2 + 2(A_{23} + A_{44})n_2^2 n_3^2 \} + \\ & 4\{ (A_{16}n_1 n_2 + A_{15}n_1 n_3)n_1^2 + (A_{24}n_2 n_3 + A_{26}n_1 n_2)n_2^2 + \\ & (A_{35}n_1 n_3 + A_{34}n_2 n_3)n_3^2 + (A_{14} + 2A_{56})n_1^2 n_2 n_3 + \\ & (A_{25} + 2A_{46})n_1 n_2^2 n_3 + (A_{36} + 2A_{45})n_1 n_2 n_3^2 \} \end{aligned} \tag{1}$$

with

$$\mathbf{n} = (n_1, n_2, n_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2}$$

where θ is the polar angle measured from the positive x_3 (vertical) axis $(0 \le \theta < \pi)$ and ϕ the azimuthal angle measured in a positive sense from the x_1 axis $(0 \le \phi < 2\pi)$.

To put equation (1) in a form that has been found to be more useful and instructive, add to and subtract from it the quantity (Daley and Krebes, 2006)

$$n_1^2 n_2^2 (A_{11} + A_{22}) + n_1^2 n_3^2 (A_{11} + A_{33}) + n_2^2 n_3^2 (A_{22} + A_{33}) . \tag{3}$$

The following formula results from equation (2) results, after some reorganization of the second and third group of terms, for the linearized *qP* phase velocity in a general anisotropic medium as

$$v_{qP}^2(x_k,n_k) = \{A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2\} + \{E_{12}n_1^2n_2^2 + E_{13}n_1^2n_3^2 + E_{23}n_2^2n_3^2\} + 4\{H_1(x_k,n_k)n_1^2 + H_2(x_k,n_k)n_2^2 + H_3(x_k,n_k)n_3^2\} \tag{5}$$

This equation is shown to be composed of three groups of terms: those specifying an ellipsoid, anellipsoidal deviation terms, and another term composed of the remaining 12 possible anisotropic parameters that describes an anisotropic medium for qP wave propagation The anellipsoidal deviation parameters, $E_{ij}(x_k)$, and the parameters, $H_k(x_k,n_k)$ are defined by the formulae

$$E_{12}(x_k) = 2(A_{12} + 2A_{66}) - (A_{11} + A_{22}) \tag{6}$$

$$E_{13}(x_k) = 2(A_{13} + 2A_{55}) - (A_{11} + A_{33}) \tag{7}$$

$$E_{23}(x_k) = 2(A_{23} + 2A_{44}) - (A_{22} + A_{33}). \tag{8}$$

$$H_1(x_k,n_k) = (A_{14} + 2A_{56})n_2n_3 + (A_{16}n_2 + A_{15}n_3)n_1 \tag{9}$$

$$H_2(x_k,n_k) = (A_{25} + 2A_{46})n_1n_3 + (A_{26}n_1 + A_{24}n_3)n_2 \tag{10}$$

$$H_3(x_k,n_k) = (A_{36} + 2A_{45})n_1n_2 + (A_{35}n_1 + A_{34}n_2)n_3 \tag{11}$$

It should be noted that the expressions $H_k(x_k,n_k)$ could have been written in a number of other ways. However, numerical experimentation with the formulae obtained in a later section indicates that the above arrangement produces the best results when compared to the exact solution. Also the arrangement of the terms within the $H_j(x_k,n_k)$ expressions are consistent with those of Every and Sache (1992).

The components of the slowness vector are defined in terms of the qP wave front normal vector and phase velocity as

$$\mathbf{p} = (p_1,p_2,p_3) = [v_{qP}(n_k)]^{-1}(n_1,n_2,n_3) \tag{12}$$

so that the specification of a pseudo eikonal equation is given by

$$G_{qP}(x_k, n_k, p_k) = 1 = \{A_{11}p_1^2 + A_{22}p_2^2 + A_{33}p_3^2\} + \\ \{E_{12}p_1p_2n_1n_2 + E_{13}p_1p_3n_1n_3 + E_{23}p_2p_3n_2n_3\} + \\ 4\{H_1(x_k, p_k)n_1^2 + H_2(x_k, p_k)n_2^2 + H_3(x_k, p_k)n_3^2\} \tag{13}$$

In the above equation the $H_k(x_k, n_k)$ may be inferred from the definitions, equations (9)-(11), for the $H_k(x_k, n_k)$ and the equal signs have to be taken within the context that an approximation is being considered. The above equation can be put in a form that is only a function of (x_k, p_k) with the introduction of the identities, $n_1^2 + n_2^2 + n_3^2 = 1$ and $v_{qP}^2(x_k, n_k)/v_{qP}^2(x_k, n_k) = 1$. The resulting linearized qP eikonal equation, obtained after minor rearrangement is

$$G_{qP}(p_k, x_k) = \{A_{11}p_1^2 + A_{22}p_2^2 + A_{33}p_3^2\} + \\ \{E_{12}p_1^2p_2^2 + E_{13}p_1^2p_3^2 + E_{23}p_2^2p_3^2\}/(p_kp_k) + \\ 4\{[(A_{14} + 2A_{56})p_2p_3 + (A_{16}p_1p_2 + A_{15}p_1p_3)]p_1^2 + \\ [(A_{25} + 2A_{46})p_1p_3 + (A_{24}p_2p_3 + A_{26}p_1p_2)]p_2^2 + \\ [(A_{36} + 2A_{45})p_1p_2 + (A_{35}p_1p_3 + A_{34}p_2p_3)]p_3^2\}/(p_kp_k) \tag{14}$$

The method of characteristics (Courant and Hilbert, 1962 and its equivalent, Červený, 2001) is often used to determine the rays, along which the energy traverses between one point in the medium and another. For a formulae equivalent to equation (14), but using an alternative notation and a spherical background velocity, Pšenčík and Farra (2006) derived formulae for the linearized vector components of the qP ray velocity in terms of phase vector components. They have named this method First Order Ray Tracing (FORT). Their derivation will not be repeated here, using equation (14), as the intent of this work is to obtain a scalar equation for the qP group velocity in terms of group angles.

In the degenerate ellipsoidal case, where the 3 symmetry plane correction coefficients $E_{ij}(x_k)$ as well as the functions are $H_k(x_k, n_k)$ identically zero the qP eikonal becomes

$$G_{qP}(x_k, p_k) = 1 = A_{11}p_1^2 + A_{22}p_2^2 + A_{33}p_3^2 . \tag{15}$$

The ray (group) velocity vector and corresponding slowness vector components are given generally in terms of some eikonal equation, $G(x_k, p_k)$, by

$$\frac{dx_i}{dt} = \frac{1}{2}\frac{\partial G(x_k, p_k)}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{1}{2}\frac{\partial G(x_k, p_k)}{\partial x_i} \tag{16}$$

An initial value problem is fully specified, given some initial conditions $\mathbf{x_0} = \mathbf{x}(t_0)$ and $\mathbf{p_0} = \mathbf{p}(t_0)$ at a reference time t_0. The progression of the ray in 3D Cartesian space as well as the magnitude and direction of the slowness vector at these points may be determined. In what follows the elastic anisotropic parameters are assumed to be spatially independent, so that, $dp_i/dt = 0$, and the initial conditions on $\mathbf{p}$ require that $\mathbf{p_0} = \mathbf{p}(t_0) = \mathbf{p}(t)$ equals some constant for all t. The group velocity in terms of its components may then be given as

$$\frac{d\mathbf{x}}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\right) = \left(A_{11}p_1, A_{22}p_{21}, A_{33}p_3\right) \tag{17}$$

with magnitude

$$\left|\frac{d\mathbf{x}}{dt}\right| = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2\right]^{1/2} = \left[A_{11}^2 p_1^2 + A_{22}^2 p_2^2 + A_{33}^2 p_3^2\right]^{1/2} . \tag{18}$$

It is convenient to introduce the group velocity angles, that is, the azimuthal and polar angles at which the ray propagates. The group azimuthal angle, Φ, $0 \le \Phi < 2\pi$ may be determined from

$$\tan\Phi = \left[\frac{dx_2}{dx_1}\right] = \left[\frac{dx_2/dt}{dx_1/dt}\right] = \frac{A_{22}p_2}{A_{11}p_1} = \frac{A_{22}}{A_{11}}\tan\phi . \tag{19}$$

Defining the projection of the 3D group velocity vector onto the (x_1, x_2) plane as

$$\frac{dr}{dt} = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2\right]^{1/2} = \left[A_{11}^2 p_1^2 + A_{22}^2 p_2^2\right]^{1/2} \tag{20}$$

the group polar angle, Θ $(0 \le \Theta \le \pi)$ is given by

$$\tan\Theta = \left[\frac{dr}{dx_3}\right] = \left[\frac{dr/dt}{dx_3/dt}\right] = \frac{\left[A_{11}^2 p_1^2 + A_{22}^2 p_2^2\right]^{1/2}}{A_{33} p_3}$$
$$= \frac{A_{11}\tan\theta\cos\phi\left[1+\left(A_{22}/A_{11}\right)^2\tan^2\phi\right]^{1/2}}{A_{33}} \quad . \tag{21}$$

After a moderate amount of algebra involving basic trigonometric manipulations, first solving equation (19) for Φ, and then substituting the result into equation (21) to obtain Θ, expressions for the components of the slowness vector, **p**, may be obtained in terms of the group rather than the phase angles and velocity. Defining a unit vector in the direction of ray propagation as

$$\mathbf{N} = (N_1, N_2, N_2) = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta), \tag{22}$$

the magnitude of the *qP* group velocity for the reduced (ellipsoidal) medium is

$$\frac{1}{V_{qP}^2(\Theta,\Phi)} = \frac{N_1^2}{A_{11}} + \frac{N_2^2}{A_{22}} + \frac{N_3^2}{A_{33}} . \tag{23}$$

From the above relations the phase slowness vector in this special case $\left(V_{qP} = V_e\right)$ may be written completely in terms of group velocity and angles as

$$\mathbf{p} = (p_1, p_2, p_3) = \left(\frac{V_e(\Theta,\Phi)N_1}{A_{11}}, \frac{V_e(\Theta,\Phi)N_2}{A_{22}}, \frac{V_e(\Theta,\Phi)N_3}{A_{33}}\right) . \tag{24}$$

This solution will be used as an initial approximation, or trial solution, in the approximate eikonal, equation (13), to determine the quasi- compressional *(qP)* group velocity, in terms of group angles, for the more general case of a general weakly anellipsoidal anisotropic medium

GROUP VELOCITY APPROXIMATION

In this section a *qP* group velocity estimate using a linearized approximation to the exact eikonal equation will be derived using the result obtained in the previous section as a trial solution. The ellipsoidal phase velocity is

$$\left[v_{qP}^2(x_k, p_k)\right]_e = A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 \tag{25}$$

and the general linearized phase velocity may be recovered from equation (13) as

$$G_{qP}(x_k,n_k,p_k) = v_{qP}^2(x_k,n_k) = \left[v_{qP}^2(x_k,n_k)\right]_e [1+[1+E_{12}[p_1p_2][n_1n_2]+ \\ E_{13}[p_1p_3][n_1n_3]+E_{23}[p_2p_3][n_2n_3]]_e + \\ 4\{[(A_{14}+2A_{56})p_2p_3+(A_{16}p_2+A_{15}p_3)p_1]n_1^2+ \\ [(A_{25}+2A_{46})p_1p_3+(A_{24}p_3+A_{26}p_1)p_2]n_2^2+ \\ [(A_{36}+2A_{45})p_1p_2+(A_{35}p_1+A_{34}p_2)p_3]n_3^2\}]_e \tag{26}$$

$$= \{A_{11}n_1^2+A_{22}n_2^2+A_{33}n_3^2\}+\{E_{12}n_1^2n_2^2+E_{13}n_1^2n_3^2+E_{23}n_2^2n_3^2\}+ \\ 4\{[(A_{14}+2A_{56})n_2n_3+(A_{16}n_2+A_{15}n_3)n_1]n_1^2+ \\ [(A_{25}+2A_{46})n_1n_3+(A_{24}n_3+A_{26}n_1)n_2]n_2^2+ \\ [(A_{36}+2A_{45})n_1n_2+(A_{35}n_1+A_{34}n_2)n_3]n_3^2\}$$

where the subscript "e" denotes ellipsoidal and the constraint that the n_k have the ellipsoidal angular values has been removed. In an equivalent manner the general linearized group velocity may be written as

$$\frac{1}{V_{qP}^2(N_k)} = \frac{1}{\left[V_{qP}^2(N_k)\right]_e \{1+f_1(E_{jk},N_k,p_k)+f_2(A_{jk},N_k,p_k)\}} \tag{27}$$

Rewriting equation (27) using the approximation, $(1+a)^{-1} \approx (1-a)$ for the $\{\bullet\}$ term in the denominator, introducing the definitions of p_k presented in equation (24), and as in the phase velocity case, relaxing the ellipsoidal constraints on N_k, results in

$$\frac{1}{V_{qP}^2(N_k)} \approx \frac{\{1-f_1(E_{jk},N_k,p_k)-f_2(A_{jk},N_k,p_k)\}}{\left[V_{qP}^2(N_k)\right]_e} \tag{28}$$

The quantities $f_1(E_{jk},N_k,p_k)$ and $f_2(A_{jk},N_k,p_k)$ may be inferred from equation (26). From this it follows that

$$\frac{1}{V_{qP}^2(N_k)} \approx \left\{\frac{N_1^2}{A_{11}}+\frac{N_2^2}{A_{22}}+\frac{N_3^2}{A_{33}}\right\} - \frac{f_1(E_{jk},N_k,p_k)}{\left[V_{qP}(N_k)\right]_e} - \frac{f_2(A_{jk},N_k,p_k)}{\left[V_{qP}(N_k)\right]_e} \tag{29}$$

The substitutions $n_in_j \approx N_iN_j$ and $n_i^2 \approx N_i^2$ have been introduced, where required, into the above result. The rationale for this is that in the initial linearization process the phase vector components were used to approximate the components of the polarization vector, which in general is not aligned with either the phase or group unit vectors. At that point, the group vector components could have been used as they would serve just as well in approximating the polarization vector component. However, the use of the phase vector components was more convenient in the initial stages of the problem, being the only known quantities. Introducing this approximation into equation (29) yields the following approximation for the *qP* group velocity in a general 21 parameter medium as a function of the related ray vector angles

$$
\begin{aligned}
\frac{1}{V_{qP}^{2}\left(N_{k}\right)} \approx & \left\{\frac{N_{1}^{2}}{A_{11}}+\frac{N_{2}^{2}}{A_{22}}+\frac{N_{3}^{2}}{A_{33}}\right\}- \\
& \left\{\frac{E_{12} N_{1}^{2} N_{2}^{2}}{A_{11} A_{22}}+\frac{E_{13} N_{1}^{2} N_{3}^{2}}{A_{11} A_{33}}+\frac{E_{23} N_{2}^{2} N_{3}^{2}}{A_{22} A_{33}}\right\}- \\
& 4\left\{\left[\frac{\left(A_{14}+2 A_{56}\right) N_{2} N_{3}}{A_{22} A_{33}}+\frac{A_{16} N_{1} N_{2}}{A_{11} A_{22}}+\frac{A_{15} N_{1} N_{3}}{A_{11} A_{33}}\right] N_{1}^{2}+\right. \\
& \left[\frac{\left(A_{25}+2 A_{46}\right) N_{1} N_{3}}{A_{11} A_{33}}+\frac{A_{24} N_{2} N_{3}}{A_{22} A_{33}}+\frac{A_{26} N_{1} N_{2}}{A_{11} A_{22}}\right] N_{2}^{2}+ \\
& \left.\left[\frac{\left(A_{36}+2 A_{45}\right) N_{1} N_{2}}{A_{11} A_{22}}+\frac{A_{35} N_{1} N_{3}}{A_{11} A_{33}}+\frac{A_{34} N_{2} N_{3}}{A_{22} A_{33}}\right] N_{3}^{2}\right\} .
\end{aligned}
\tag{30}
$$

The perturbed velocity derivation above results from the fact that in ray propagation space, for some given ray, the vector beginning at the origin of the ray surface and normal to the tangent plane associated with the point at which the ray touches the ray surface is the phase velocity vector, $\mathbf{v}_{qP}\left(n_{k}\right)$. Equivalently, in slowness space, $\left(\mathbf{p}=\left[\mathbf{v}_{qP}\left(n_{k}\right)\right]^{-1}\right)$, for an arbitrary slowness vector, the vector originating at the slowness surface origin and normal to the tangent plane at the point at which the slowness vector contacts the slowness surface is the group velocity vector inverse, $\left[\mathbf{V}_{qP}\left(n_{k}\right)\right]^{-1}$. More formally, slowness space and group velocity space are dual spaces. The advantage of the expression derived above for the *qP* group velocity is that it is in terms of group (ray) angles rather than wave front normal vector components or equivalently phase velocity angles.

NUMERICAL RESULTS

The model that will be considered is the weakly anellipsoidal material where the 12 parameters beyond an orthorhombic medium are quite small. The anisotropic properties are similar in degree of anisotropy to an orthorhombic medium clay-shale associated with hydrocarbon deposits The orthorhombic bases for this model is taken from Pšenčík and Farra (2005) (their model ORTHO). The extra parameters are fairly arbitrary being scaled values from a dry sandstone model used in the paper of Pšenčík and Gajewski (1998). These additional anisotropic parameters are italicized in the figure defining the model. The model is defined by the density normalized anisotropic parameters, A_{ij}, which have the dimensions of velocity squared $\left(km/s\right)^{2}$ and given in Figure 1.

$$\begin{bmatrix} 9.80 & 3.60 & 2.25 & \mathit{0.00} & \mathit{0.00} & \mathit{0.00} \\ & 8.84 & 2.28 & \mathit{0.00} & \mathit{0.00} & \mathit{0.00} \\ & & 5.94 & \mathit{0.00} & \mathit{0.00} & \mathit{0.00} \\ & & & 2.00 & \mathit{0.00} & \mathit{0.00} \\ & & & & 1.65 & \mathit{0.00} \\ & & & & & 2.18 \end{bmatrix} \begin{bmatrix} 9.80 & 3.60 & 2.25 & \mathit{0.14} & \mathit{0.11} & \mathit{0.08} \\ & 8.84 & 2.28 & \mathit{0.02} & \mathit{-0.02} & \mathit{-0.06} \\ & & 5.94 & \mathit{0.00} & \mathit{-0.05} & \mathit{-0.10} \\ & & & 2.00 & \mathit{0.00} & \mathit{0.02} \\ & & & & 1.65 & \mathit{0.00} \\ & & & & & 2.18 \end{bmatrix}$$

$$ORTHO \qquad\qquad ORTHO \;\; (Modified)$$

FIG. 1. Density normalized anisotropic parameter specification of the modified model ORTHO. The A_{ij} have the units of $(km/s)^2$. The origin of these models is described in the text.

The group velocities for the model are computed at azimuthal angles of $\phi =$ 0, 30, 45 and 60 degrees, which do not, in general correspond to group angles of $\Phi =$ 0, 30, 45 and 60 degrees. The variation of Φ with ϕ is related to the degree of anisotropy. These angles are measured from the positive x_1 axis. The inclusion of the results, $\phi = 0$, is to provide a reference comparison from which to determine the quality of fit in the non-zero azimuthal plane examples. The group velocity approximation at $\phi = 0$ is the least affected by the inclusion of the additional 12 possible anisotropic parameters when compared to the orthorhombic problem. The approximate (V_a) and exact (V_e) group velocities are compared in Figure 2 for a polar angle range of 0 to 180 degrees for the model described above at the azimuthal phase angles previously specified. The curves in the plots on all panels are plotted such that the exact group velocity is black and the approximation is red. The group angle inputs for the approximation are obtained from those angles, computed numerically, and resulting from the phase angle input of the exact solution.

The plotting of the group velocity curves is not done in polar plots but rather in a manner that enhances the differences between exact and approximate group velocity computations. The polar angle Θ is measured from the vertical, x_3, axis. It is quite evident upon viewing the figures that for weakly anisotropic media, the match between the approximations and the exact solution is quite reasonable, which could be a subjective observation as the fit required might possibly problem specific.

The numerical measure of deviation, D_p, given in Table 1, is the a average deviation of the approximate group velocity expression (V_a) from the exact value (V_e) over a 180 degree polar angle range at N equally spaced points obtained using the formula

$$D_P = \left[\frac{1}{N} \sum_{j=1}^{N} \frac{\left| V_e - V_a \right|}{V_e} \right] \times 100\% \,. \tag{33}$$

CONCLUSIONS

A quasi-compressional *(qP)* group velocity approximation for elastic wave propagation in a general 21 parameter anisotropic medium has been presented. The solution method was facilitated by modifying the standard form of the linearized eikonal equation that is found in the literature for this medium and wave type. The eikonal equation is first put in a form such that the background slowness surface, and hence the group velocity surface, is an ellipsoid with anellipsoidal correction terms in each of the three symmetry planes, and followed by a term containing the remaining 12 anisotropic parameters. This rewriting of the eikonal equation has the effect of allowing the group velocity and slowness vector components for the degenerate (ellipsoidal) case to be determined analytically, using the method of characteristics, and as functions of group rather than phase angles. In this approximation, the exact solution for this degenerate case was then used as a trial solution to obtain the group velocity approximation for the general anisotropic case. As the approximation has an analytic solution when the anellipsoidal terms, E_{ij} , and other terms, H_k , are zero, they have been referred to as "weakly anellipsoidal" rather than "weakly anisotropic".

Comparison of the approximation, at phase azimuthal angles of 0, 30, 45 and 60 degrees, with the exact group velocity expression for a possible realistic geological model was carried out with good matches in all instances. As with any approximate method, care must be taken not to violate the original assumptions used in its development. The model used in the previous section was selected such that it lies within the set which could be designated as "weakly anellipsoidal". In geophysical applications, this assumption is infrequently contravened to a large degree in actual geological models.

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Percentage Deviation from the Exact Solution for ORTHO and ORTHO (Modified) Models

Table 1. Average percentage deviation for the model described in the text over a 180° polar angular, Θ, range, equally sampled in this angle, for azimuths of phase angles $\phi = 0$, 30, 45 and 60 degrees. Both models ORTHO and ORTHO (Modified) are considered.

	0 Degrees	**30 Degrees**	**45 Degrees**	**60 Degrees**
ORTHO	**0.2579**	**0.2344**	**0.2223**	**0.2697**
ORTHO(MOD)	**0.2806**	**0.3089**	**0.2303**	**0.2503**

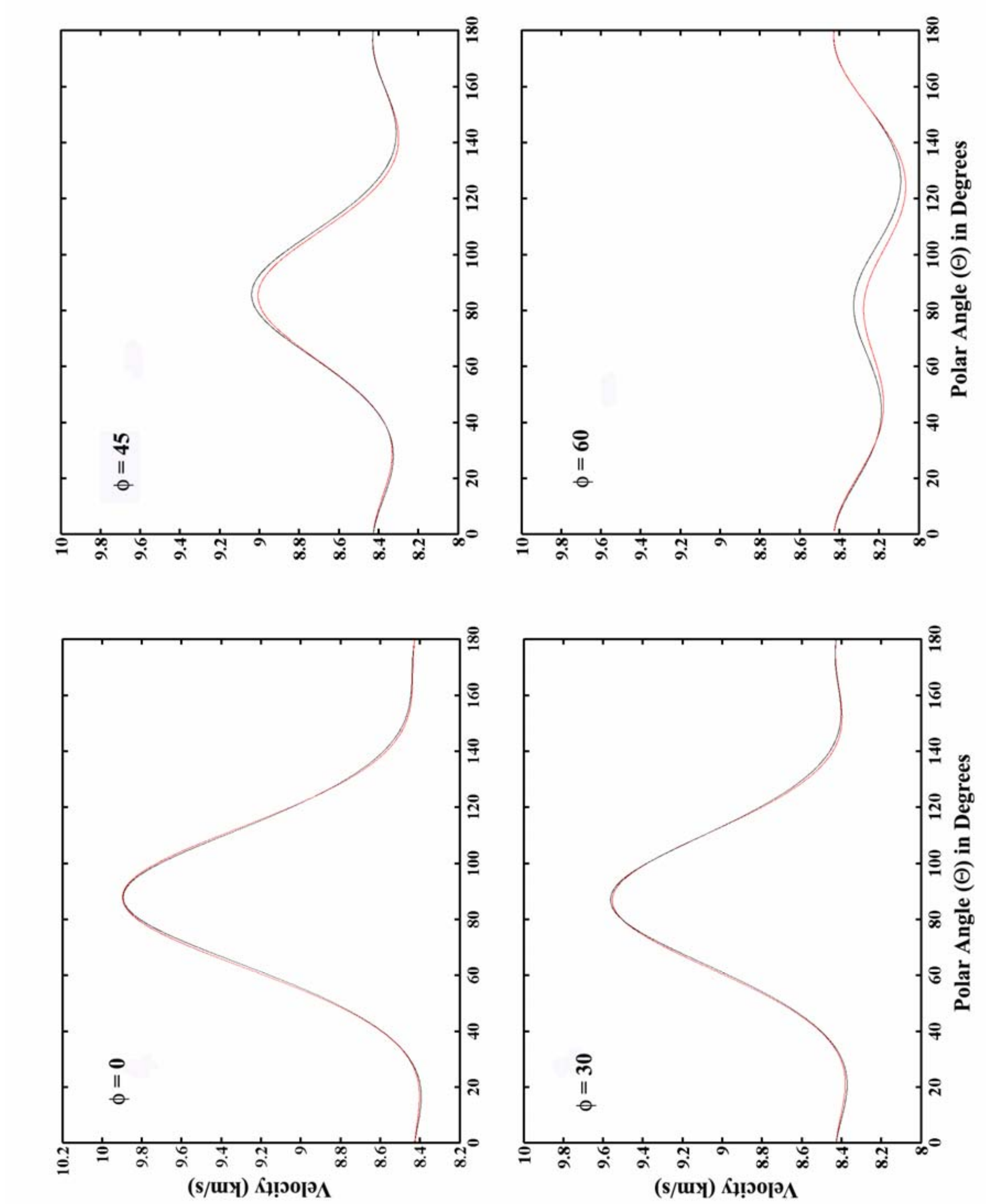

FIG. 2. ORTHO (Modified) model from Pšenčík and Farra (2006). The exact group velocity (black curve) is compared to the approximation (grey curve) for the for the polar (Θ) angle range 0 to 180 degrees and shown at four different (phase) azimuth angles. The azimuth of 0 degrees which coincides with the x_1x_3 symmetry plane The other panels are at phase azimuthal angles of 30, 45 and 60 degrees, respectively.