

Thomas Bayes versus the wedge model: An example inference using a geostatistical *prior* function

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ABSTRACT

The Bayesian inference is used to estimate model parameters in a synthetic example. The model is a thin bed that follows a wedge shaped trend. However rather than a uniform trend, the model's thickness at each lateral position across the wedge is randomized. Additionally, the velocity at each model location is normally random with a specified mean and variance. Synthetic seismic data are generated with a zero-offset convolutional model and are then used to invert for bed-thickness. The Bayesian *likelihood* function is defined using the known relationship between bed-thickness, wedge velocity, wavelet tuning frequency, and seismic amplitude. Population of the amplitude/bed-thickness joint probabilistic density *likelihood* function is achieved using Monte Carlo methods, and the Bayesian *prior* is weighted by the known geostatistical trend of the wedge. The inversion result is a probabilistic density function describing the probabilities of wedge thicknesses. Results show that the *maximum a posteriori* parameter estimate is more accurate than the estimate found with the raw statistical trend or with a deterministic method of data inversion. Additionally, the *posterior* probabilistic density function can be used to perform statistical analysis, opening a pathway for quantified analysis of parameter uncertainty.

INTRODUCTION

This paper explores the use of the Bayesian inference to solve for model parameters in a simple example problem. Bayes's theorem is stated in equation (1) where $P(A)$ is the probabilistic density function (PDF) that describes the event A . $P(B)$ is similarly defined. $P(A|B)$ is the probabilistic density function that describes event A given the occurrence of event B . $P(B|A)$ is similarly defined.

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}. \quad (1)$$

The Bayesian inference can be used to solve for the probabilistic distribution of a parameter estimate. The inverse problem is formalized as:

$$P(\mathbf{m}|\mathbf{d}) = \frac{P(\mathbf{d}|\mathbf{m})P(\mathbf{m})}{P(\mathbf{d})}. \quad (2)$$

where \mathbf{m} is the model parameter(s) to be solved for and \mathbf{d} is the data.

Each of the PDF's of Bayes's theorem has a name. The PDF $P(\mathbf{m}|\mathbf{d})$ of Bayes's theorem is called the '*posterior*' distribution. $P(\mathbf{d}|\mathbf{m})$ is called the '*likelihood*' function and $P(\mathbf{m})$ is called the '*prior*' probability distribution. The denominator in equation (2) is a constant that can be calculated as:

$$P(\mathbf{d}) = \int_{\text{range of models}} P(\mathbf{d} | \mathbf{m})P(\mathbf{m})d\mathbf{m} = c . \quad (3)$$

where c normalizes the right hand side of (2) so that the integral of the PDF $\int_{-\infty}^{\infty} P(\mathbf{m})d\mathbf{m}$ over all models is unity.

A PDF of the possible models \mathbf{m} can be found if the *likelihood* function can be established and data has been “measured”. This paper explores a method of populating the *likelihood* function as a joint PDF (JPDF) between the model space and the data space.

To evaluate the *likelihood* $P(\mathbf{d}|\mathbf{m})$, the functional relationship between \mathbf{d} and \mathbf{m} must be known. In the case that the inverse problem to be solved is an Earth parameter and the data is a seismic attribute, the *likelihood* function encapsulates the physics relating the Earth property to the seismic response. For example, a rock physics model might relate porosity to the seismic amplitude at the top of a sandstone reservoir. However, since the amplitude will depend not only on the porosity, but also on fluid content, shale content, fluid pressure, etc., there is no unique mapping from porosity to amplitude. Any given porosity value might give a range of amplitude values. This covariance between seismic attribute and Earth parameter is captured by the *likelihood* JPDF $P(\mathbf{d}|\mathbf{m})$.

In addition to the *likelihood* function, the Bayesian inference requires a prior PDF for the model. The *prior* can be treated as “*uninformative*”, where all model parameters are equally likely. However, if a reliable prior estimate of the model PDF can be established, the resulting *posterior* can be designed to lend favor to results that are more probable given the prior knowledge (Scales and Tenorio, 2001). In the case of subsurface properties, such *a priori* information might come from a geological model or geostatistics.

THE EXAMPLE PROBLEM

The example problem solved in this paper is an estimate of the thickness of a thin bed. The data are synthetic zero-offset seismograms synthesized from a velocity model that has a wedge shape. However, the thickness of the wedge, as well as the P-wave velocity of the wedge material is modelled with some random variation. At each lateral position across the wedge, the bed-thickness and velocity are assigned a mean value and a Gaussian variance. From these statistical parameters, the wedge thickness and velocity are randomly assigned.

The model, which is 100m across, is illustrated in FIG. 1. 50 m of homogeneous overburden with a constant density and velocity of 2400 kg m⁻³ and 3300 m s⁻¹, respectively. The wedge has an average trend of increasing in mean thickness from 8.0 m thick on the left side to 18.0 m thick on the right side. The lateral distance x runs from $x=1\text{m}$ to $x=100\text{m}$. However, the thickness is allowed to vary normally about the mean trend with a standard deviation of 2 m. The velocity is also set to vary normally with a mean of 2500 m s⁻¹ and standard deviation of 100 m s⁻¹. The density in the wedge is uniformly 1700 kg m⁻³. The layer under the wedge has the same velocity and density as

the overburden layer and extends to 120 m of depth. Finally, a synthetic density and sonic log at each lateral position is created.

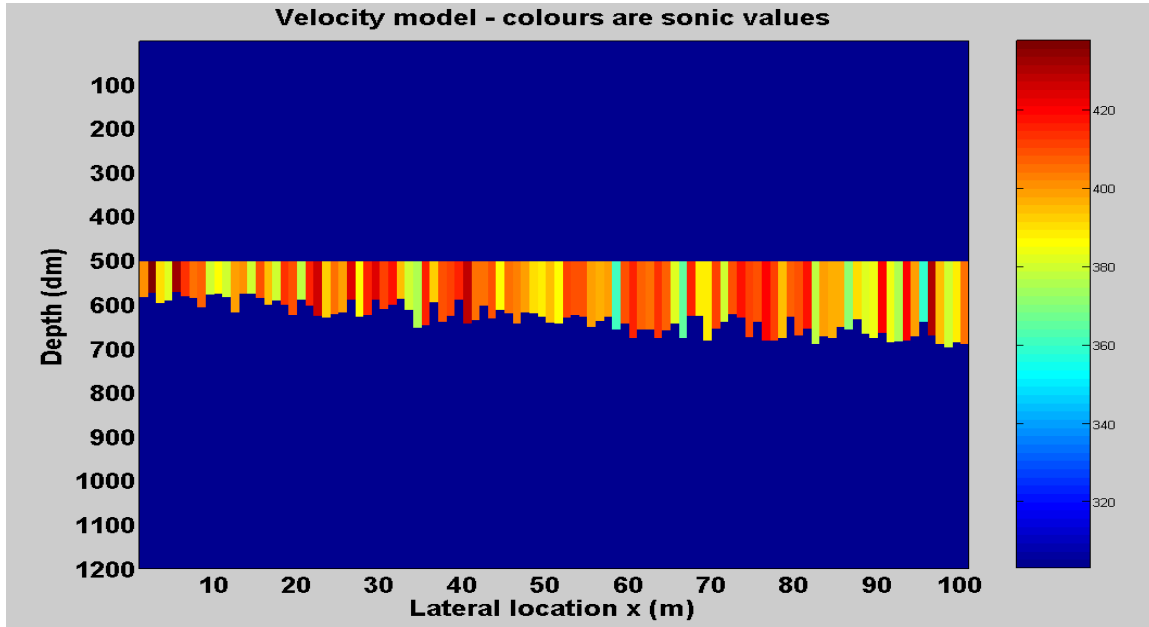


FIG. 1. The velocity model of the wedge. The color bar indicates the sonic values (microseconds).

The data fabricated from the model, is a set of zero-offset synthetic seismograms. Convolution of each reflectivity log with a 40 Hz Ricker wavelet generates the data illustrated in FIG. 2. The tuning thickness for a Ricker wavelet is (Kallweit and Wood, 1982):

$$time_tuning_thickness = \frac{2}{2.6f_d} \quad (4)$$

where f_d is the dominant frequency. For a 40 Hz Ricker wavelet the tuning thickness in time is 19 ms. Considering that two standard deviations from the mean thickness at the wedge end (18m) is 24 m and two standard deviations below the mean velocity is 2300 m/s, a reasonable estimation of greatest time thickness of the wedge is 10.4 ms. Therefore the model wedge is both below the tuning thickness and below the Widess (1973) criteria for resolution of T/2 where T is the tuning thickness of the wavelet in time.

When the thickness of a bed is less than the resolution limit, the seismic reflection amplitude (A) of the event at the bed top and bed bottom is given by Widess (1973):

$$A = 2A_o \sin \left[\frac{2\pi(t)f_t}{v} \right] \quad (5)$$

where A_o is the amplitude of the reflection if there was a single reflection (i.e. a two layer model), t is the bed-thickness, f_t is the tuning frequency of the wavelet, and v is the velocity of the bed. For the 40 Hz Ricker wavelet, the tuning thickness can be found with

equation (4) and is 52 Hz. The data is therefore a single amplitude at each lateral position along the model.

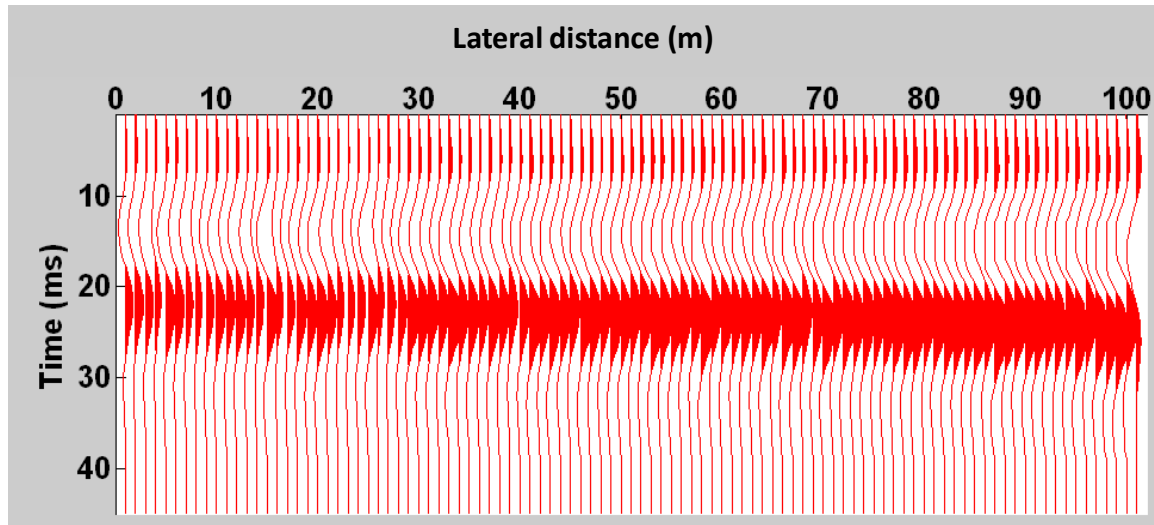


FIG. 2. The zero-offset synthetic seismogram data set.

To execute the inversion, several assumptions are made:

- The dominant frequency of the wavelet is known.
- The overall trend in the wedge thickness is known.
- The variance (standard deviation) of the wedge thickness is known.
- The mean velocity in the wedge layer is known.
- The variance (standard deviation) of the wedge velocity is known.

The trend in the thickness of the wedge and the variance are representative of data that might come from geostatistical analysis. Likewise, the mean and variance in the velocity of the model could also be representative of statistical information derived from wells in a field data set.

THE PRIOR

The *prior*, $P(\mathbf{m})$, can be calculated because the overall trend in the wedge thickness is known. The variability is normally distributed, with a mean or expected value that is given by the lateral location along the model length. Thus the *prior* is evaluated individually at each lateral model location. For example, FIG. 3 illustrates the *prior* at the lateral location of $x = 5$ m.

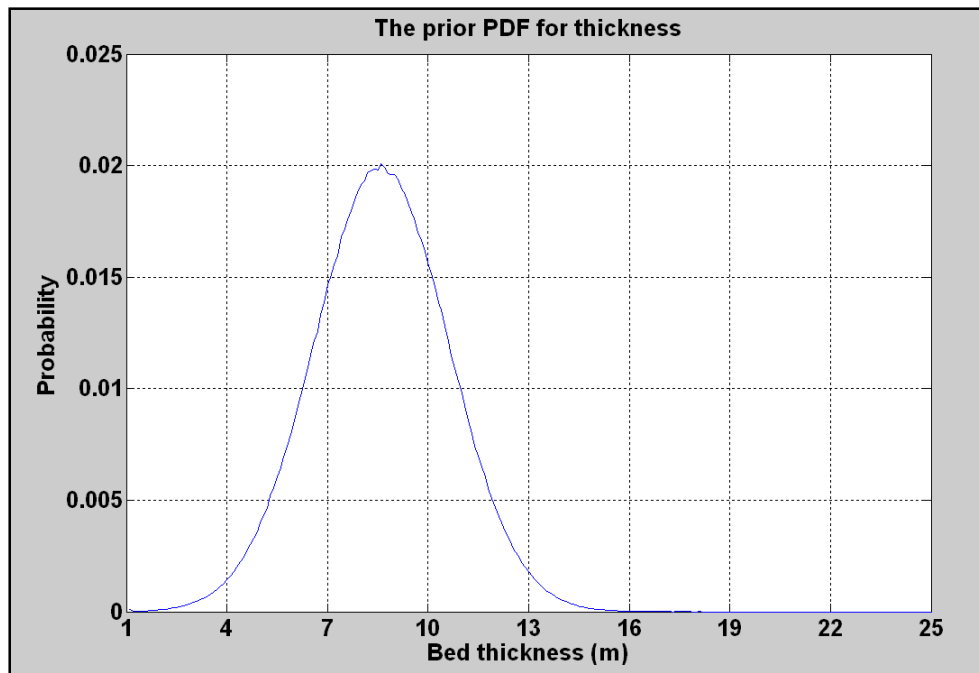


FIG. 3. An example *prior* PDF of bed-thickness (at lateral location $x = 5$ m). The expected mean thickness at this location is 8.5 m.

THE LIKELIHOOD

The *likelihood*, $P(\mathbf{d}|\mathbf{m})$, is the JPDF of the bed-thickness and the corresponding reflection amplitude. A Monte Carlo sample of the bed-thickness and velocity is drawn from the PDF of each. The PDF for each is illustrated in FIG. 4. Then, using equation (5), the associated amplitude is calculated for each random draw. With repeated sampling, the JPDF between bed-thickness and amplitude is built as illustrated in FIG. 5. The JPDF has been slightly smoothed using a Gaussian filter.

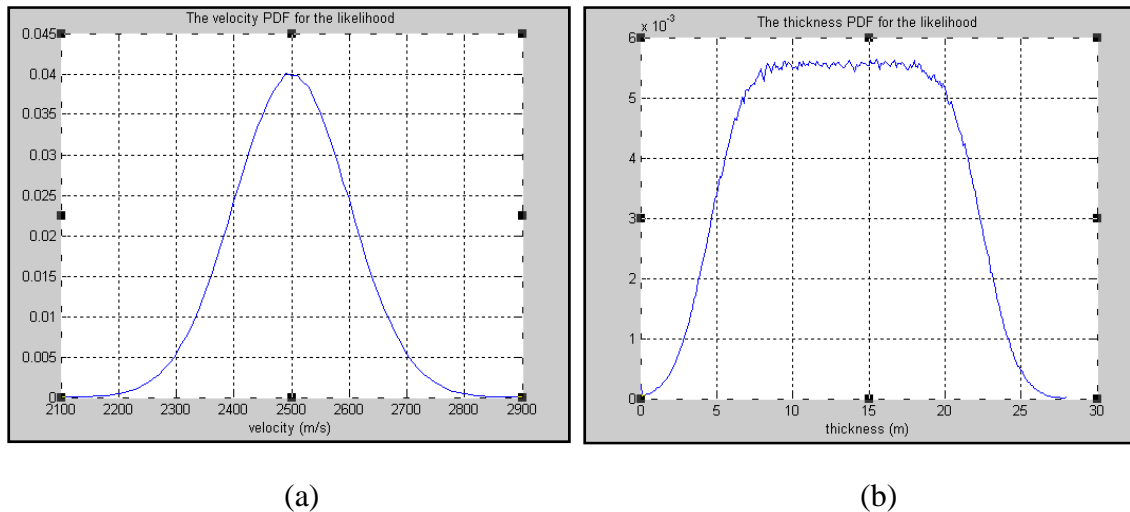


FIG. 4. The PDF of (a) the bed velocity and (b) the bed-thickness used to generate the simulations.

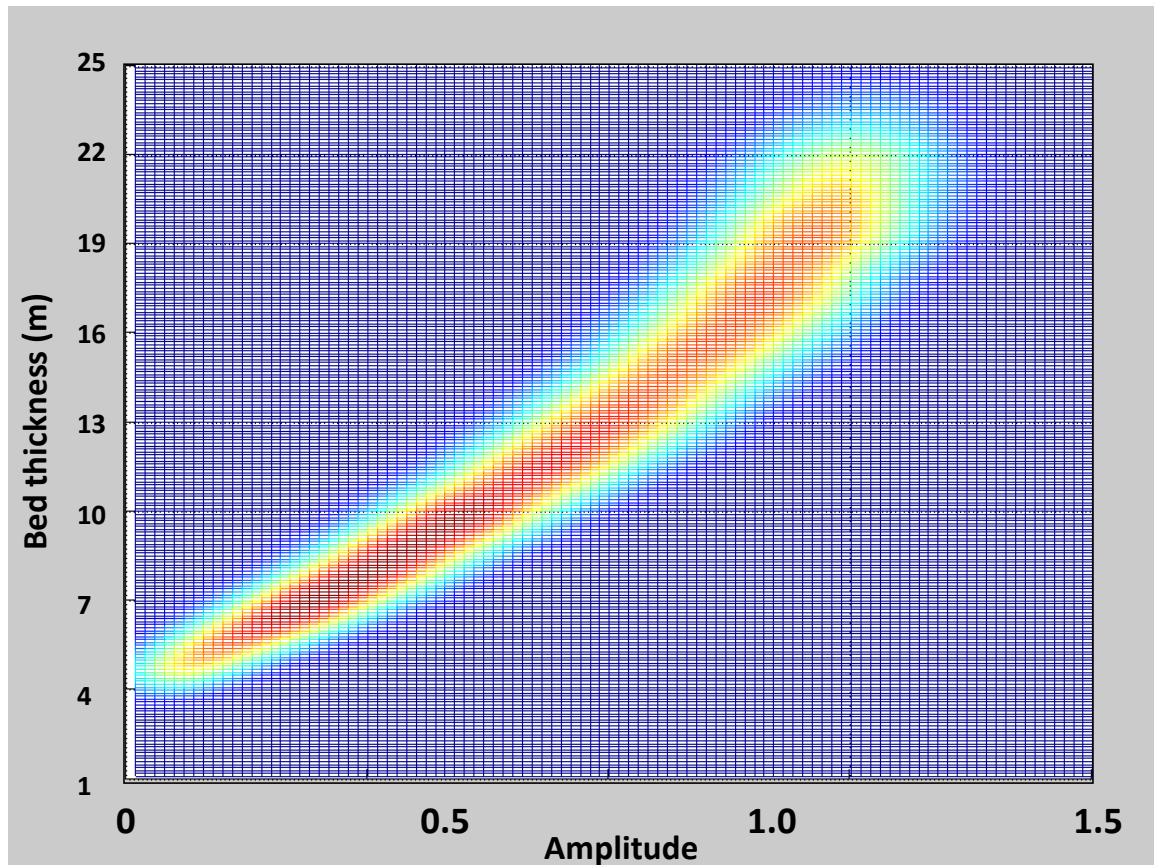


FIG. 5. The *likelihood* JPDF of bed-thickness and seismic amplitude.

Although it is easiest to randomly select from a Gaussian function, the PDF for the thickness and velocities need not be normal. Also, in the case of real field data, the underlying PDF of the data might not be known and may not be Gaussian. Field data might be distributed in a skewed or multimodal distribution and are very likely to be sparsely sampled. However, Silverman (1986) shows that a general PDF can be generated from a sparse, discrete data set by taking the summation of a set of normal distributions, each with a mean centered on the individual data sample values. The method is illustrated in FIG. 4 where the general PDF (turquoise curve) is built by summing the Gaussian curves associated with three discrete data points. A random draw could be sampled from the general PDF by randomly sampling from each of the discrete Gaussians and taking the average of the samples.

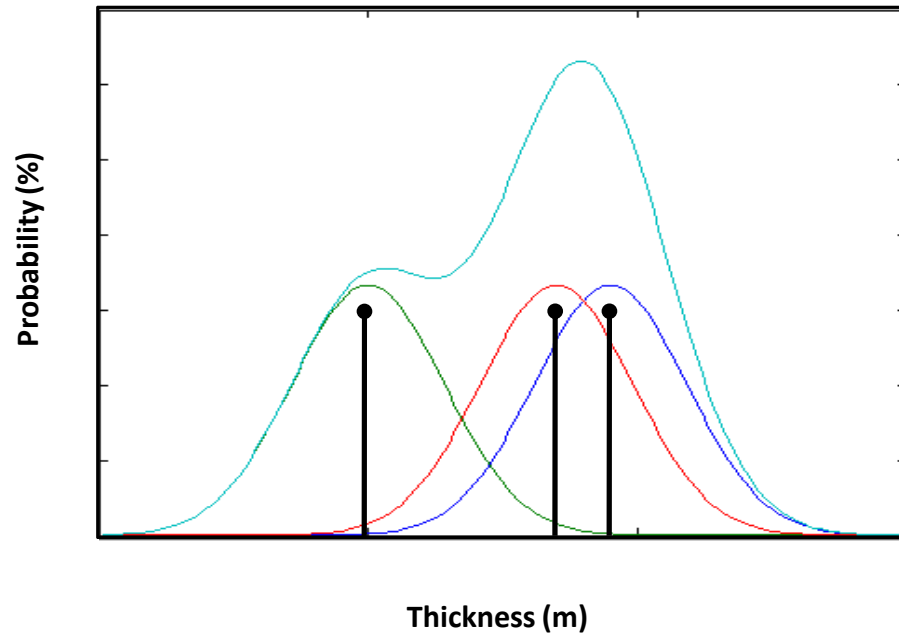


FIG. 6. An example of building a general PDF from three discrete data samples. The turquoise curve is the summation of the discrete normal curves centered on each discrete data point.

As illustrated in FIG. 4b, the PDF of bed-thickness is treated as a variable that has equal probability of occurring between 8 and 18 m, and with a drop off in probability beyond these bounds. Monte Carlo sampling is drawn from a set of underlying, equally spaced bed-thickness Gaussian distributions.

THE POSTERIOR

In Bayes' theory, the *posterior* is proportional to the product of the *likelihood* and the *prior* as in equation (2). An example of the *posterior* at one lateral location along the model is illustrated in FIG. 7.

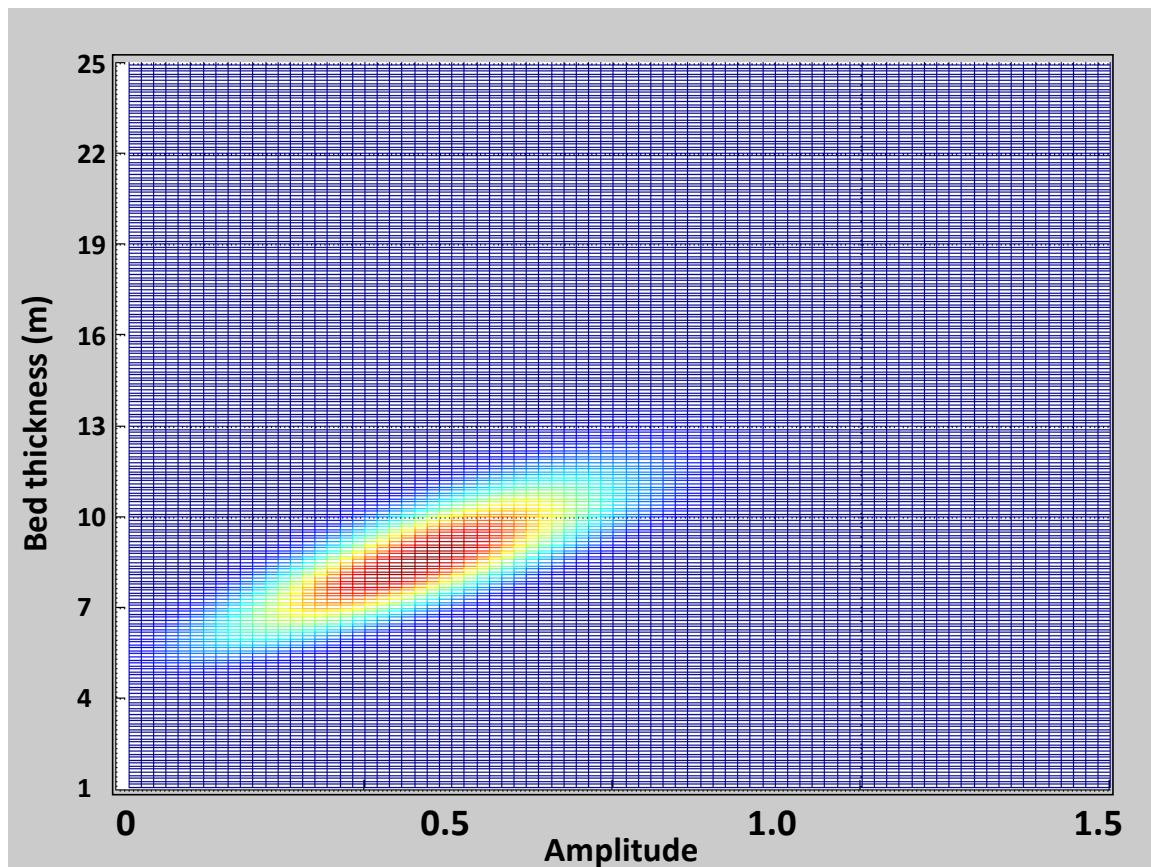


FIG. 7. An example of the product of the *prior* and the *likelihood* at a lateral location along the model ($x=5m$).

THE INFERENCE

At each location x , the amplitude from the data set picks out the appropriate distribution for the possible bed-thicknesses. The resulting PDF is normalized to account for constant c of equation (3). FIG. 8 illustrates the inverted PDF of bed-thickness at each lateral location across the survey.

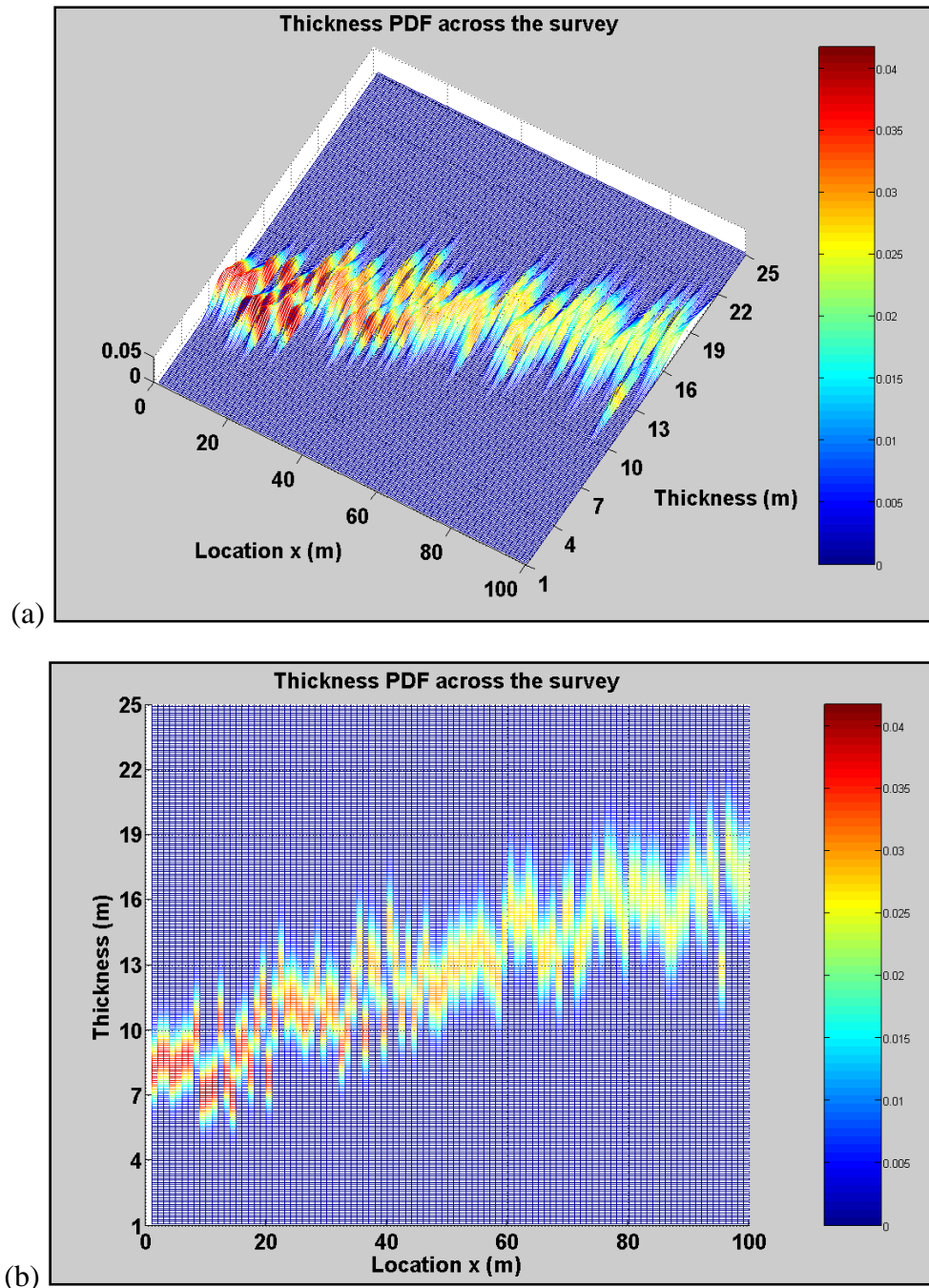


FIG. 8. Inversion PDF of bed-thickness at each lateral location. (a) and (b) are different views of the same data.

For comparison, an inversion was generated with a uninformative *prior*; where all thicknesses are equally probable. The result is shown in FIG. 9. Especially at the thick end of the model, the *posterior* PDF is much broader than in the inversion conducted with the informative *prior* (compare with FIG. 8).

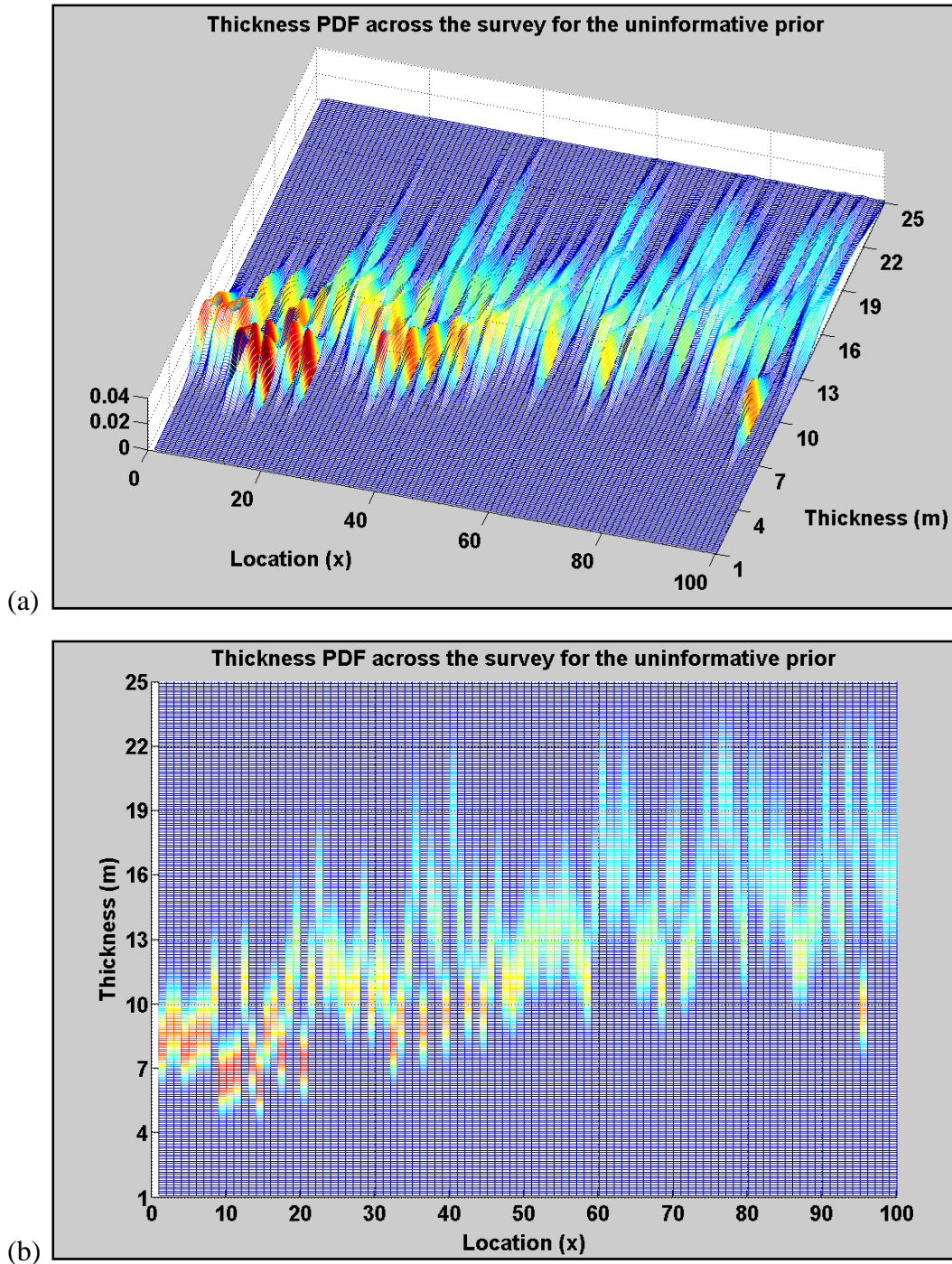


FIG. 9. The inversion PDF calculated using a non-informative *prior*. (a) and (b) are different views of the same data.

DISCUSSION

The result of the inversion is the *posterior* PDF which requires further analysis to understand completely. For example, FIG. 10 shows the *prior* PDF and the *posterior* at location $x=95\text{m}$ where the velocity and thickness of the wedge are both very anomalous relative to the expected velocity and thickness. The actual velocity is 2788 m s^{-1} whereas the expected velocity is 2500 m s^{-1} ; and the actual thickness is 13.7 m whereas the expected thickness based on the *prior* is 17.5 m . Therefore, the *prior* over predicts the bed-thickness. Also, the seismic data at this location would imply a very thin bed-thickness. Using equation (5) and assuming the wedge has the average velocity, the predicted bed-thickness would be 9.8 m which is a large underestimation of the true bed-thickness. Here Bayes triumphs. The thickness of maximum probability from the *posterior* PDF using the weighted *prior* is 12.7 m . Although this is not the exact solution, it is certainly closer than the estimate made by the *prior* or the deterministic estimate. Additionally, while the Bayesian inference result does show a significant probability of the true thickness.

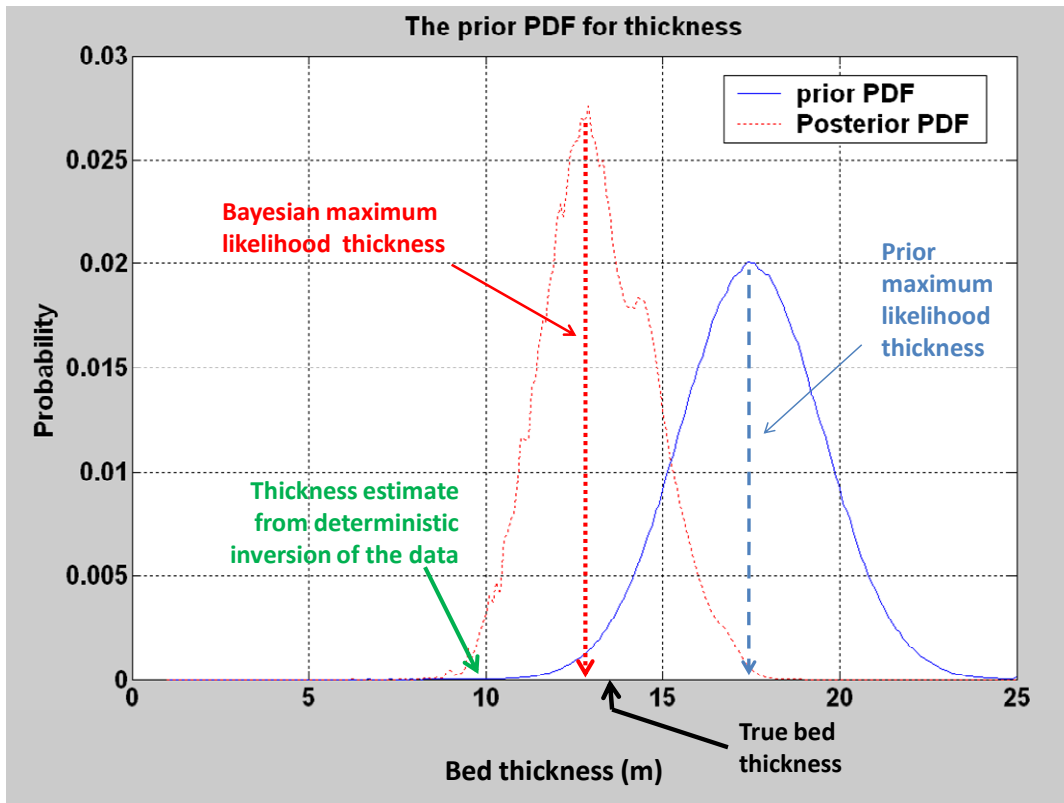


FIG. 10. The *prior* PDF and the inversion PDF for bed-thickness at $x=95$. The actual model thickness at this location is 13.3 m with an anomalously high velocity of 2815 m/s .

While the largest value of $P(\mathbf{m} | \mathbf{d})$, called the *maximum a posteriori* (MAP), can be used as the most probable single value estimate of \mathbf{m} (FIG. 11), additional insight can be found in the statistics of the model PDF or the model cumulative probability function (FIG. 12). For example, the percentile distribution of possible thickness values can be found. Also, the PDF can be used to establish the probability that the bed-thickness is at least as thick as a given value.

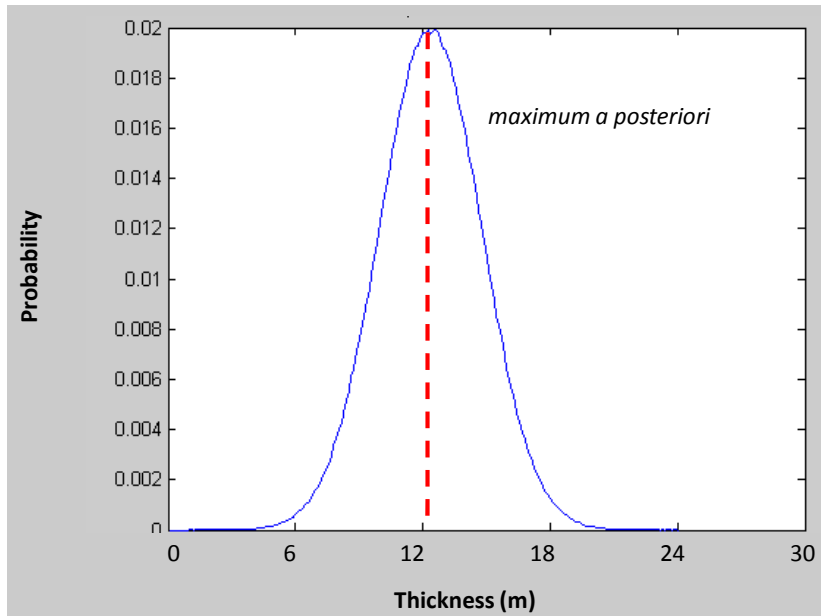


FIG. 11. An example of the *maximum a posteriori*.

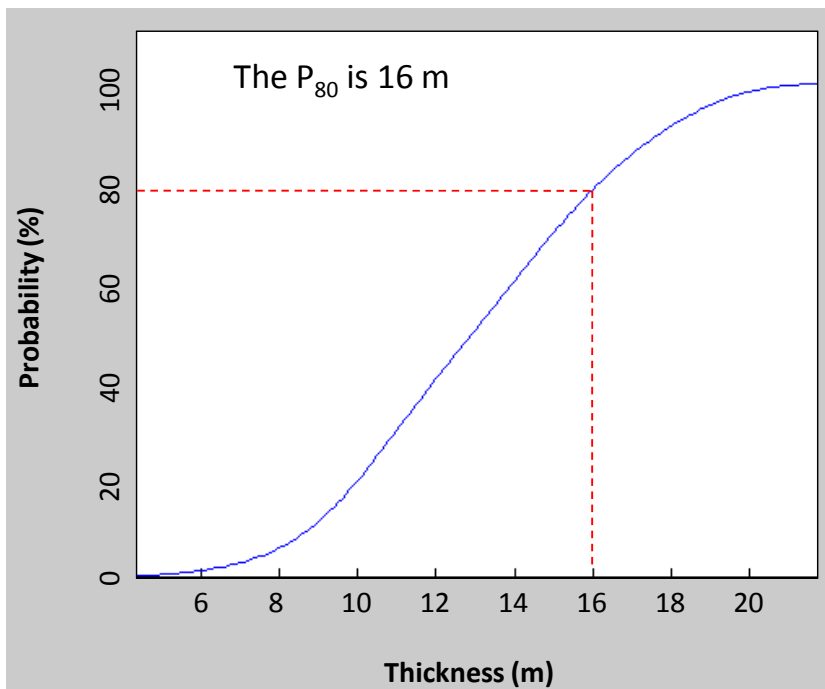


FIG. 12. An example of the cumulative probability function.

Illustrating these forms of analysis using the results from the example problem, FIG. 13 shows the estimated probability that the bed-thickness is greater than a 15 m at each location across the model and FIG. 14 shows the 60th percentile thickness estimate across the wedge.

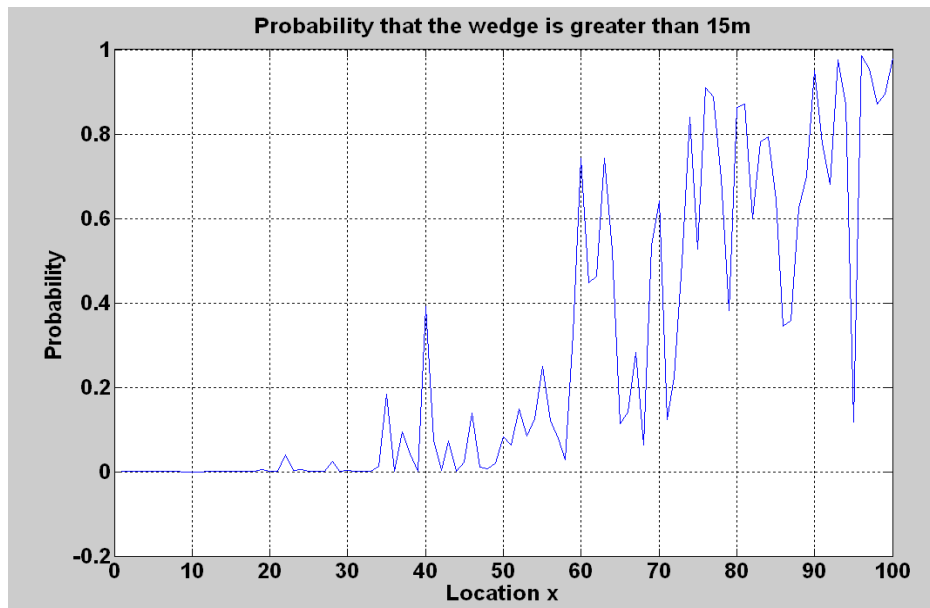


FIG. 13. The probability that the wedge is greater than 15m thick at each survey location.

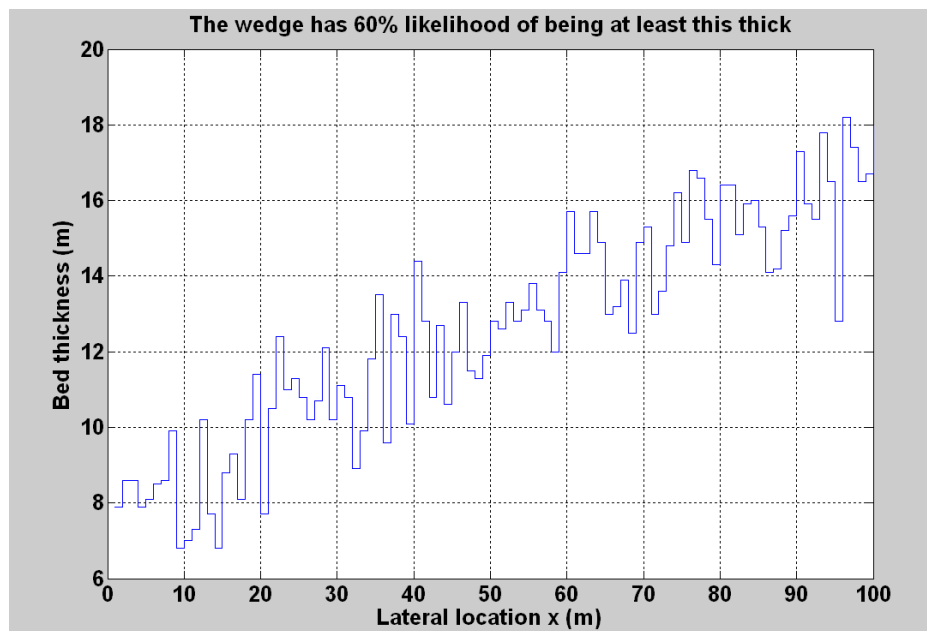


FIG. 14. The model has a 60% chance of being at least this thick at each model location.

There are four methods to estimate bed-thickness in this example:

1. Using the *prior* statistical estimate alone; where the mean is the best estimate.
2. A deterministic estimate using the seismic data amplitudes and equation (5) with the assumption of the average velocity.
3. Using the Bayesian inference with the uninformative *prior*.

4. Using the Bayesian inference with the informative *prior*.

Table 1 contains an example comparison of the various method of estimating the bed-thickness. It shows the number of times that the estimated bed-thickness was greater than 15 m, when the bed-thickness actually was greater than 15 m (a true positive). The table also counts the number of false positives; when the estimated bed-thickness was greater than 15 m but the bed was not actually that thick. Likewise, the table shows the count of true negatives and false negatives. In addition, the table shows the results from the *posterior* PDF for probability thresholds of 70%, 60%, and 30% probability (that the wedge is at least 15m thick).

The informative *prior* Bayesian inversion finds more true positives and fewer false positives than the estimate using the *prior* alone, using the deterministic inversion or using the Bayesian inference performed with the uninformative *prior*. Bayes wins again. Also, although the probabilistic estimates of “when the bed-thickness is greater than 15m with 60% or 70% probability” identify fewer true positives, they also identify fewer false negatives. This last result could be significant if the purpose of the model inversion is to quantify and reduce risk associated with false positives.

Table 1. Comparison of the various methods to estimate the number of times that the wedge exceeds 15m in thickness.

	Times the estimate was less than 15m when the wedge actually was less than 15m	Times the estimate was less than 15m when the wedge was actually greater than 15m	Times the estimate was greater than 15m when the wedge was actually less than 15m	Times the estimate was greater than 15m when the wedge was actually greater than 15m
<i>Informative prior</i>				
<i>Prior</i> estimate only	60	9	8	23
Linear inversion	58	6	10	26
MAP	65	4	3	28
PDFprobability>0.7	68	11	0	21
PDFprobability>0.6	67	8	1	24
PDFprobability>0.3	60	3	8	29
<i>Uninformative prior</i>				
MAP	66	9	2	23
PDFprobability>0.7	66	12	2	20
PDFprobability>0.6	66	9	2	23
PDFprobability>0.3	53	4	15	28

OTHER EXAMPLES FROM LITERATURE

Sava and Mavko (2007) derived a JPDF of fracture density and seismic data signature using a Monte Carlo simulation of the rock physics parameters. The JPDF related crack density to seismic AVAZ (amplitude variation with azimuth). The PDFs of the rock properties such as crack density, crack aspect ratio, fluid properties in the cracks were known and the JPDF of P-wave speed, S-wave speed, and density are derived from log data. Monte Carlo sampling was used to populate the JPDF between the crack density and AVAZ seismic response. The *prior* was established with the assumption that the probability of encountering fractured rock would decrease with increasing distance from a known fault. Multiplying the crack density/AVAZ JPDF with the *prior* that was established based on geological trends generated the *posterior* JPDF.

When inverting for porosity, shale content, and water saturation in a reservoir, Spikes et al. (2007) constructed the *likelihood* PDF by comparing the seismic traces to an exhaustive set of synthetic traces generated from a complete but discretized set of rock property values. The subset of synthetic traces that had a sufficiently high cross correlation with the real traces were included in the PDF calculation. The result was a PDF for the possible geological models that could have yielded the measured trace. Mukerji et al. (2001) used a Monte Carlo simulation to generate a PDF relationship between seismic facies and offset dependant amplitudes. The inversion result was a PDF of facies models. Further discussion of these techniques can be found in Aster et al. (2005).

CONCLUSIONS

The Bayesian inference provides a powerful technique to integrate geostatistical information into a parameter estimate based on geophysical data. Geostatistics can be used to weight the *prior* in the inference and thus move the *posterior* PDF toward the *a priori* more likely solutions. This yields an improved maximum probability parameter estimate and the *posterior* PDF can be used to perform probabilistic analysis of model scenarios and possibilities.

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