A multiple model and Padé approximation

Dali Zhang, Michael P. Lamoureux, and Gary F. Margrave

ABSTRACT

The paper considers rational Padé approximation of the Z-transform function of a time-dependent minimum phase signal. We present a derivation of reflection and transmission coefficients in layered media which are related to infinite impulse response (IIR) filters as a rational function of a special form. The rational ($[p, q]$-Padé) approximation of the Z-transform function is formulated as a constrained least squares minimization problem with regularization constraints provided by the minimum phase signal. Numerical simulations for reconstruction of the Z-transform function and its use as an inverse filter for a multiple model demonstrate the effectiveness of the presented approach.

INTRODUCTION

The numerical experiments described in this paper were conceived as a test of how the technique of least squares Padé approximation could be used in a typical seismic data processing application. The Padé approximation is a general purpose technique for approximating an analytic function in the complex plane using a numerically derived rational function – the quotient of two polynomials in complex variable z. It has been used extensively in numerical modelling and analysis of composite materials (Milton, 2002; Zhang and Cherkaev, 2008, 2009) and is widely applicable to many data processing tasks (Nita, 2008) because of the general form of functions that may be approximated. As such, it seems to be a promising technique for use in seismic data processing.

The Padé method features a numerical approach that works directly on the data – there is no need to transform to the Fourier (or other) domain. Also, the approximation problem is transformed to a linear least squares solution by lifting the denominator of a rational function into the data terms. And finally, the method offers direct control of the coefficients in the rational approximation, allowing for constraints to be placed on certain terms as motivated by the physics of the model being studied. On the other hand, rational function can have poles, which may lead to instabilities in the representing Padé function. Data has noise, so solutions must be regularized in order to reduce other instabilities. Finally, a reasonable Padé approximation in certain physical models may have very high order, requiring some care in achieving a feasible numerical technique.

For these reasons, it is useful to attempt using this Padé method on a seismic data processing task. For this work, we select the problem of modelling (and removing) reflection multiples in a 1D seismic reflection and/or transmission data survey.

The paper is organized as follows. We review the derivation of reflection multiples for a double interface, and observe the multiple signal can be modelled by an infinite impulse response (IIR) filter of a simple form, with coefficients determined by the reflection and transmission parameters. We set up the Padé approximation problem using the seismic data directly, with some choice on the rational function form to reduce the dimension of the solution space. We then report on the results of some numerical experiments in building the Padé approximating filter, and its use as an inverse filter to remove the multiples.

MULTIPLE REFLECTIONS AND TRANSMISSIONS

The aim of this section is to derive multiple reflection and transmission coefficients for one dimensional seismic plane waves propagating with interfaces.

Single interface

We derive reflection and transmission coefficients from the 1D wave equation for a single interface (see Figure 1). With an incoming wave $e^{i(\omega t+k_1 x)}$ on the right of the interface at $x = 0$, a transmitted and reflected wave is generated as

$$Te^{i(\omega t+k_2 x)}, \quad Re^{i(\omega t-k_1 x)} \tag{1}$$

on the left with the wave number k_1 and right side of the interface with the wave number k_2 , respectively. The continuity of the total waveforms at the point $x = 0$ gives the equation

$$Te^{i\omega t} = e^{i\omega t} + Re^{i\omega t} \tag{2}$$

which reduces to

$$T - R = 1. \tag{3}$$

The continuity of the normal derivatives also gives

$$ik_2 Te^{i\omega t} = ik_1 e^{i\omega t} - ik_1 Re^{i\omega t} \tag{4}$$

which reduces to

$$rT + R = 1, \tag{5}$$

where $r = k_2/k_1$ is the relative index of refraction.

Solving the two equations (3) and (5) for two unknowns R and T in terms of r, the reflection and transmission coefficients are given by

$$R = \frac{1-r}{1+r}, \quad T = \frac{2}{1+r}. \tag{6}$$

It is important to note that these coefficients R and T depend on which way the wave is travelling. Referring to Figure 1, a wave traveling from right to left will generate reflection and transmission coefficients

$$R_{\leftarrow} = \frac{1-r}{1+r}, \quad T_{\leftarrow} = \frac{2}{1+r}, \tag{7}$$

where $r = c_{right}/c_{left}$ is the relative index of refraction as the wave travels from right to left. If we reverse directions, the relative index of refraction is flipped, and thus we have

$$R_{\rightarrow} = -R_{\leftarrow}, \quad T_{\rightarrow} = rT_{\leftarrow}. \tag{8}$$

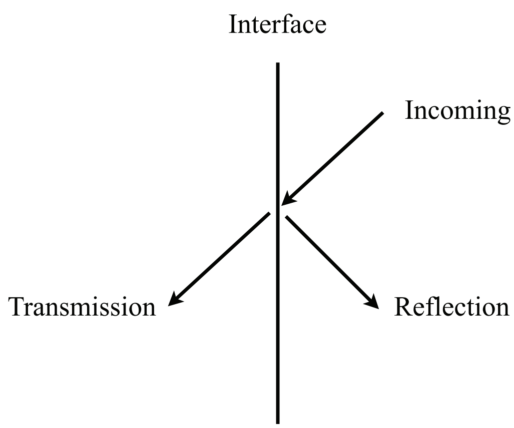

FIG. 1. A reflection and transmission event across a single interface

Two interfaces

In the case of two interfaces, there will be multiple internal reflections. Referring to Figure 2, we see that the effective reflectivity will come from a infinite sum of reflected components. Similarly, the effective transmission will also come from an infinite sum. Start by computing the effective reflectivity. The first bounce gives a reflection $R_1 = R^1_{\leftarrow}$, the reflection off the first interface, for a wave initially travelling from right to left. The second reflection comes from a transmission through the first interface, a delay as it travels through the gap between interfaces, a bounce off the second interface, delay again as it travels through the gap, then a transmission through the first interface. So we have

$$R_2 = T^1_{\rightarrow} D R^2_{\leftarrow} D T^1_{\leftarrow}. \tag{9}$$

Similarly, we can find the third reflection component as

$$R_3 = T^1_{\rightarrow} D R^2_{\leftarrow} D [R^1_{\rightarrow} D R^2_{\leftarrow} D] T^1_{\leftarrow}, \tag{10}$$

and the fourth component as

$$R_4 = T^1_{\rightarrow} D R^2_{\leftarrow} D [R^1_{\rightarrow} D R^2_{\leftarrow} D]^2 T^1_{\leftarrow}. \tag{11}$$

Continuing this way gives an infinite series with total reflectivity

$$\begin{aligned}
R_{total} &= R^1_{\leftarrow} + \sum_{n=0}^{\infty} T^1_{\rightarrow} D R^2_{\leftarrow} D [R^1_{\rightarrow} D R^2_{\leftarrow} D]^n T^1_{\leftarrow} \\
&= R^1_{\leftarrow} + \frac{T^1_{\rightarrow}}{R^1_{\rightarrow}} \sum_{n=1}^{\infty} [R^1_{\rightarrow} D R^2_{\leftarrow} D]^n T^1_{\leftarrow} \\
&= R^1_{\leftarrow} - \frac{T^1_{\rightarrow}}{R^1_{\rightarrow}} T^1_{\leftarrow} + \frac{T^1_{\rightarrow}}{R^1_{\rightarrow}} \sum_{n=0}^{\infty} [R^1_{\rightarrow} D R^2_{\leftarrow} D]^n T^1_{\leftarrow} \\
&= R^1_{\leftarrow} - \frac{T^1_{\rightarrow}}{R^1_{\rightarrow}} T^1_{\leftarrow} + \frac{T^1_{\rightarrow}}{R^1_{\rightarrow}} \left(1 - R^1_{\rightarrow} D R^2_{\leftarrow} D\right)^{-1} T^1_{\leftarrow}.
\end{aligned}$$

Noting that the coefficients T's and R's are just constants, the reflectivity R_{total} can be written as in the form

$$R_{total} = a + b\left(1 - cD^2\right)^{-1}, \tag{12}$$

for some constants a, b, c.

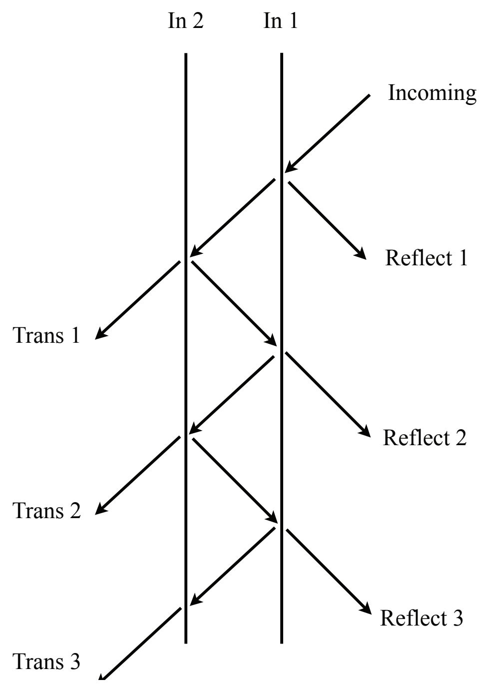

FIG. 2. Reflection and transmission events across two interfaces

A similar analysis shows that the transmission terms can be derived as

$$\begin{aligned} T_1 &= T^2_{\leftarrow} D T^1_{\leftarrow} \\ T_2 &= T^2_{\leftarrow} D [R^1_{\rightarrow} D R^2_{\leftarrow} D] T^1_{\leftarrow} \\ T_3 &= T^2_{\leftarrow} D [R^1_{\rightarrow} D R^2_{\leftarrow} D]^2 T^1_{\leftarrow} \end{aligned}$$

and so on, so the effective transmission T_{total} is given as

$$\begin{aligned} T_{total} &= \sum_{n=1}^{\infty} T^2_{\leftarrow} D [R^1_{\rightarrow} D R^2_{\leftarrow} D]^n T^1_{\leftarrow} \\ &= T^2_{\leftarrow} D \left(1 - R^1_{\rightarrow} D R^2_{\leftarrow} D\right)^{-1} T^1_{\leftarrow}. \end{aligned} \tag{13}$$

Again noting the constants in (13), we see that the transmission term T_{total} can also be written as in the form of

$$T_{total} = b' D \left(1 - cD^2\right)^{-1} \tag{14}$$

for some constants b', c.

MODELLING WITH IIR FILTERS

The two responses R_{total} and T_{total} in the previous section can be modelled with Infinite Impulse Response (IIR) filters (Karl, 1989) of very similar form. For the reflectivity R_{total} given in (12), the related IIR filter is given by the rational function

$$G(z) = a + \frac{b}{1 - cz^{2d}} = \frac{\alpha + \beta z^{2d}}{1 + \eta z^{2d}} \tag{15}$$

where $\alpha = a + b$, $\beta = -ac$, and $\eta = -c$ are related to the multiple reflectivities, and d is an integer that models the delay of the signal through the gap. For the transmission T_{total} in (14) the related IIR filter is given by the rational function

$$H(z) = \frac{b' z^d}{1 - cz^{2d}} \tag{16}$$

where d is an integer that models the delay of the signal through the gap. Note that $c = R^1_{\rightarrow} R^2_{\leftarrow} < 1$, so the filters $G(z)$ and $H(z)$ are stable. We also note that the integer d can be computed from the width of the gap, the velocity of sound in the gap, and the sample rate of the sampled signal. It is given as

$$d = \frac{\text{length}}{\text{velocity}} * \text{ sample rate.} \tag{17}$$

PADÉ APPROXIMATION

In this section, we present a new numerical inversion method for constructing Padé approximation of the Z-transform function $G(z)$. The approach is based on the rational approximation of the relaxation spectrum for viscoelastic media introduced in (Zhang et al., 2010). The rational ($[p, q]$-Padé) approximation of $G(z)$ (Baker Jr. and Graves-Morris, 1996) is

$$G(z) \simeq G_{[p,q]}(z) = \frac{a(z)}{b(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_p z^p}{b_0 + b_1 z + b_2 z^2 + \cdots + b_q z^q} \qquad (p \leq q) \tag{18}$$

where a_l $(l = 0, 1, \ldots, p)$ and b_j $(j = 0, 1, \ldots q)$ are real coefficients of two polynomials $a(z)$ and $b(z)$ of orders p and q, respectively. We see that the function $G(z)$ in (15) has at least one pole and all poles are nonzero, and use a standard normalization of the polynomial coefficient $b_0 = 1$ in the denominator $b(z)$.

We note that the approximation $G_{[p,q]}(z)$ of the IIR filter $G(z)$ implies that the input wavelet $\{w_l\}_{l=0}^{\infty}$ and the output signal $\{s_k\}_{k=0}^{\infty}$ must satisfy the recursion formula

$$s_k = \sum_{l=0}^{p} a_l w_{k-l} - \sum_{j=1}^{q} b_j s_{k-j}, \quad k = 0, 1, 2, ... \tag{19}$$

From a computational point of view, we need to choose a finite number $N \gg p + q + 1$ in order to reconstruct the full Padé coefficients a_l's and b_j's given the values of the input wavelet $\{w_l\}_{l=0}^{\infty}$ and the output signal $\{s_k\}_{k=0}^{\infty}$. Therefore, the system (19) for the unknown coefficients a_l's $(l = 0, 1, ..., p)$ and b_j's $(j = 1, 2, ..., q)$ becomes

$$a_0 w_k + a_1 w_{k-1} + ... + a_p w_{k-p} - b_1 s_{k-1} - b_2 s_{k-2} - ... - b_q s_{k-q} = s_k \\ p \le q < N, \quad k = 0, 1, 2, ..., N. \tag{20}$$

It should also be noted that both numerator and denominator of the rational IIR filter function $G(z)$ in (15) contain only two terms, i.e., a constant term, and a term with the highest order of $2d$ degree. The coefficients of the terms $z^1, z^2, ..., z^{(2d-1)}$ in the numerator and denominator of $G(z)$ are zero. Thus, we can reduce the number of the unknown $[p, q]$-Padé coefficients a_l's and b_j's in (18) for numerical computation. Some coefficients a_l's and b_j's in the middle terms of the polynomials $a(z)$ and $b(z)$ of $G_{[p,q]}(z)$ can be assumed to be zero. More precisely, for a fixed integer number m (e.g., $m = 5$ or $m = 6$), we suppose that some polynomial coefficients of $a(z)$ and $b(z)$ are zero, i.e.,

$$\begin{aligned} a_m = a_{m+1} = ... = a_{p-(m-2)} = a_{p-(m-1)} = 0, \\ b_{m+1} = b_{m+2} = ... = b_{q-(m-2)} = b_{q-(m-1)} = 0. \end{aligned} \tag{21}$$

Then the unknown coefficients a_l's and b_j's in the system (20) become a_0, a_1, ..., $a_{(m-1)}$, a_{p-m}, ..., a_p, and b_1, b_2, ..., b_m, b_{q-m}, ..., b_q. The linear system of equations (20) under the assumption (21) can be rewritten as follows

$$\mathbf{Ac} := [\mathbf{A_1} \quad \mathbf{A_2}] \begin{bmatrix} \mathbf{c_1} \\ \mathbf{c_2} \end{bmatrix} := \mathbf{s} \tag{22}$$

where

$$\mathbf{A_1} = \begin{pmatrix} w_0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ w_1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & w_0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & w_1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & w_0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & w_1 & w_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & w_0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ w_N & \cdots & w_{N-(m-1)} & w_{N-p+m} & w_{N-p+(m-1)} & \cdots & w_{N-p} \end{pmatrix} \tag{23}$$

$$
\mathbf{A_2} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
-s_0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
-s_1 & -s_0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & -s_0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & -s_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & -s_0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & -s_1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -s_0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
-s_{N-1} & -s_{N-2} & \cdots & -s_{N-m} & -s_{N-q+m} & \cdots & -s_{N-q}
\end{pmatrix} \tag{24}
$$

$$
\mathbf{c} = [\mathbf{c_1} \quad \mathbf{c_2}]^\top, \quad \mathbf{s} = (s_0, s_1, s_2, ..., s_N)^\top
$$

$$
\mathbf{c_1} = (a_0, a_1, ..., a_{(m-1)}, a_{p-m}, ..., a_p)^\top, \mathbf{c_2} = (b_1, b_2, ..., b_m, b_{q-m}, ..., b_q)^\top, \tag{25}
$$

The symbol $[\cdot]^\top$ indicates a transposed matrix. It is clear that in order for the reduced Padé coefficients a_l's $(l = 0, 1, ..., m-1, p-m, ..., p)$ and b_k's $(k = 1, 2, ..., m, q-m, ..., q)$ to be uniquely determined, the total number of the impulse-response signal $\{s_k\}_{k=0}^{N}$ is required to be greater to the number of coefficients, i.e., $N \gg p+q+1 > 4m+2$. The reconstruction problem of determining the column real coefficient vector $\mathbf{c} = [\mathbf{c_1} \quad \mathbf{c_2}]^\top$ in (22)-(25) is an inverse problem. It is ill-posed and requires regularization to develop a stable numerical algorithm.

To construct a real solution vector $\mathbf{c}$ of the reduced Padé coefficients for the inverse problem (22), we introduce a penalization term in the Tikhonov regularization functional $\mathcal{T}^\lambda(\mathbf{c}, \mathbf{s})$ (Tikhonov and Arsenin, 1977), so that the problem (22) can be formulated as the following constrained least squares minimization problem with the regularization parameter $\lambda > 0$ chosen properly (Tikhonov and Arsenin, 1977):

$$
\min_{\mathbf{c}} \mathcal{T}^\lambda(\mathbf{c}, \mathbf{s}) = \min_{\mathbf{c}} \{ \, || \, \mathbf{Ac} - \mathbf{s} \, ||^2 + \lambda^2 ||\mathbf{c}||^2 \}
$$

$$
\text{subject to } \; |u_k - r_0| < \delta_0, \; |v_j - r_1| < \delta_1, \; k = 1, 2, \ldots, p, \; j = 1, 2, \ldots, q. \tag{26}
$$

Here $|| \cdot ||$ is the usual Euclidean norm, parameters u_k and v_j in the constraints (26) are zeros and poles of the reconstructed $[p, q]$-Padé approximation $G_{[p,q]}(z)$ of $G(z)$, $r_0 = (-\alpha/\beta)^{1/2d}$, and $r_1 = (-1/\eta)^{1/2d}$. To find the minimizer of the problem (26), we solve its Euler equation; the solution is given by

$$
\mathbf{c} = \{\mathbf{A}^\top \mathbf{A} + \lambda \mathbf{I}_{4m+2}\}^{-1} \{\mathbf{A}^\top \mathbf{s}\} \tag{27}
$$

where $\mathbf{I}_{4m+2}$ denotes the $(4m+2) \times (4m+2)$ identity matrix.

After reconstruction of the real coefficient vector $\mathbf{c}$, we can extend it to a full coefficients of the rational function approximation $G_{[p,q]}(z)$ by inserting zero coefficients, this gives $[p, q]$-Padé approximation of $G(z)$. The reconstructed function $G_{[p,q]}(z)$ can be used to estimate the reflectivity parameters and to identify the impulse wavelet using inverse filtering.

NUMERICAL SIMULATIONS

We show preliminary results of numerical reconstruction of Z-transform function $G(z)$ given in (15). To simulate the synthetic data - impulse response signal $\{s_j\}_{j=0}^{\infty}$ using formula (19), a Ricker wavelet with the dominant frequency 25Hz was generated using the CREWES Matlab function [wavelet,tw]=ricker(dt,fdom,tlength) for the input signal, and the parameters a, b and c in (15) were chosen as

$$a = \frac{4}{45}, \quad b = \frac{11}{18}, \quad c = \frac{9}{10}, \quad d = 50 \tag{28}$$

leading to the polynomial coefficients a_0, a_{100}, b_{100} represented by

$$\alpha = 0.7, \quad \beta = -0.08, \quad \eta = -0.9, \tag{29}$$

respectively. The desired temporal sample rate of the Ricker wavelet is dt =0.003 seconds, the wavelet length is tlength=3.0 seconds, so that the total number of data is $N = 1001$.

In Figure 3 we show the results of reconstruction of the reduced Padé coefficients compared with the true polynomial coefficients of G(z) for the order of $p = q = 104$ chosen in the inversion algorithm. Here $m = 6$, so there were $4m + 2 = 26$ Padé coefficients to be determined in the vector **c**. The values of a_0, a_{100}, b_{100} are reconstructed very accurately when there is no noise in the data. The valid recovered poles of the approximation $G_{[p,q]}(z)$ of $G(z)$ using the constraints in (26) are illustrated in Figure 4. In this numerical example, we chose $\delta_1 = \delta_2 = 0.06$ in the constraints of (26), all poles and zeros of the function $G(z)$ lie on the unite circle with radius $r_1 = 1.0011$ and $r_2 = 1.022$ in the complex z-plane, respectively. The true and computed output signals $\{s_j\}_{j=0}^{\infty}$ (the IIR filtered Ricker wavelet) using the recovered $[p, q]$-Padé coefficients fit fairly well for data with 8% noise presented in Figure 5. It is seen from Figure 5 that the IIR filtered Ricker wavelet exhibits multiple reflections in the 1D synthetic seismogram. We also used the recovered function $G_{[p,q]}(z)$ as an inverse filter to reconstruct the original input Ricker wavelet illustrated in Figure 6. The reconstruction of the input Ricker wavelet is almost identical, with no difference between the theoretical and reconstructed functions seen in the figure when there is no noise in the data. Even for input data with adding 8% noise, the original input Ricker wavelet in the time interval $[0, 1.5]$ seconds was recovered very well using the reconstructed function $G_{[p,q]}(z)$ as an inverse filter. However, there are some oscillation events that occurred with amplitudes changing rapidly after $t = 0.15$ seconds, which need to be further studied.

CONCLUSIONS

We developed a new numerical inversion method for reconstruction of the Z-transform function of a time-dependent minimum phase signal using Padé approximation. The approach is based on rational ($[p, q]$-Padé) approximation of the Z-transform function in the complex plane. The problem is formulated as a constrained least squares minimization problem with regularization constraints provided by the minimum phase signal (all poles and zeros of Z-transform function lie outside the unit circle). The method was tested using a Ricker wavelet to generate a minimum phase signal (IIR filtered wavelet) as synthetic input data. The performed numerical experiments for reconstruction of the Z-transform function and its use as an inverse filter show the effectiveness of the presented approach.

The Padé approximation technique may provide a method to pull out the effective filter that produces the multiple reflections in seismic data processing.

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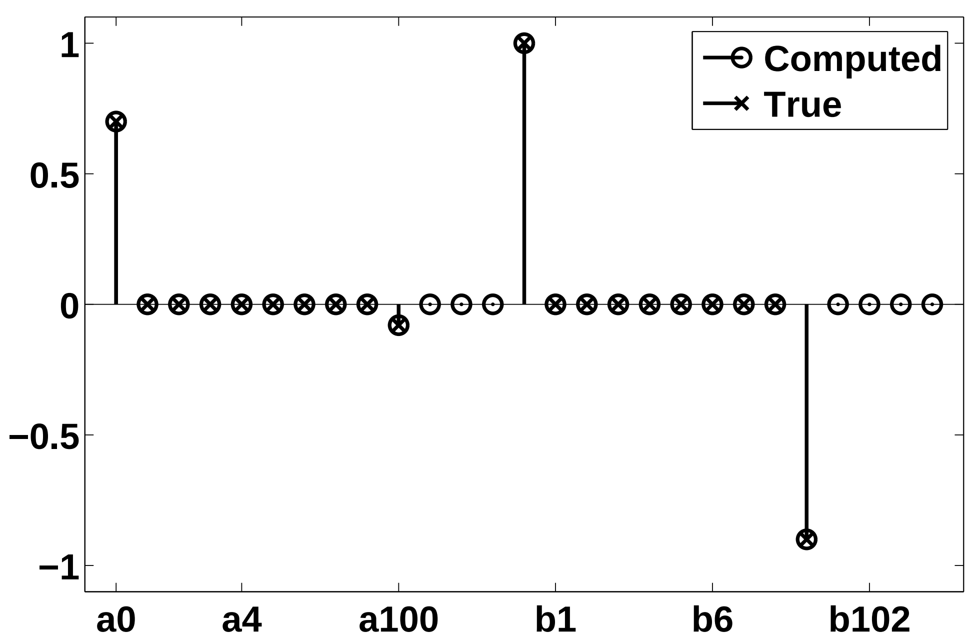

FIG. 3. Reconstruction of Padé coefficients for data with no noise.

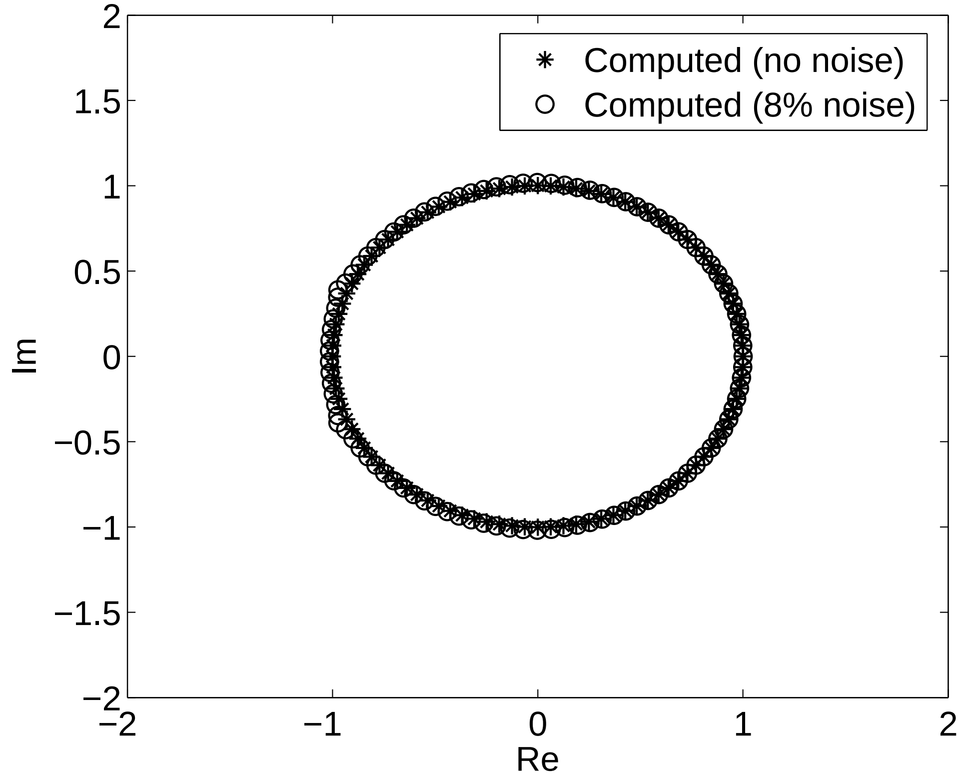

FIG. 4. Calculation of poles for function G(z).

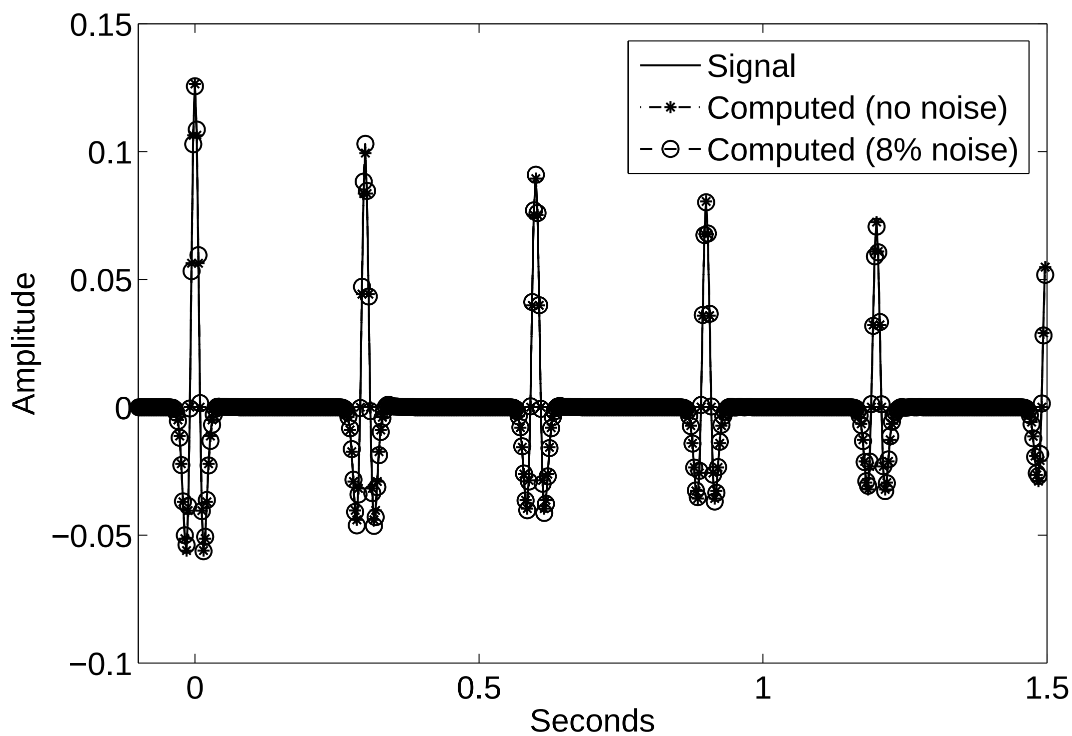

FIG. 5. Reconstruction of the IIR filtered Ricker wavelet (25 Hz dominant frequency).

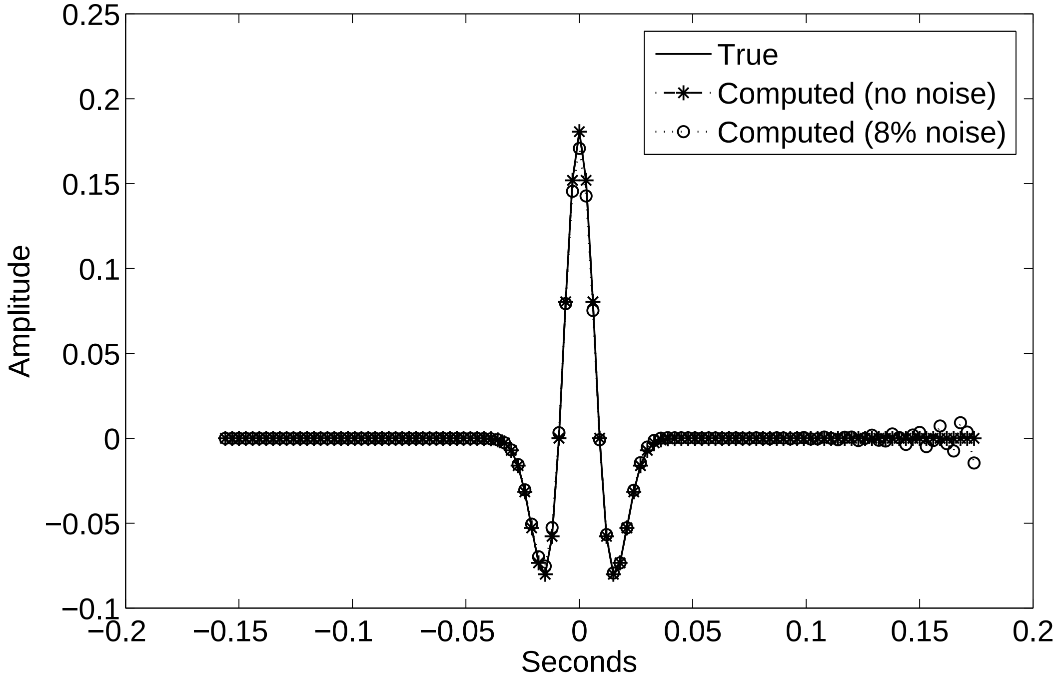

FIG. 6. Reconstruction of the Ricker wavelet with a 25 Hz dominant frequency.