Porous medium – the *fast* P-wave case

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ABSTRACT

A subset of the general equations used to describe seismic wave propagation in a poroviscoelastic medium is investigated. Theoretically for an isotropic medium of this type there are four modes a propagation: a compressional $\left(P\right)$ wave, a shear $\left(S_V\right)$ wave and a shear $\left(S_H\right)$ wave as well as what in termed a *slow* compressional $\left(P_S\right)$ whose actual existence has, until fairly recently, been questioned. The scaled down version of the full poroviscoelastic equation set is one which is viscoelastic and but does not contain any shear type propagation. It is the acoustic analogue of the elastic wave equation, being a set of two coupled equations in the *fast* and *slow* compressional $\left(P\right)$ wave modes, which may be further downgraded to to a single equation in the *fast* compressional $\left(P\right)$ wave mode as the *slow* compressional $\left(P\right)$ wave mode is difficult to physically detect and as a consequence omitted, at least in this preliminary study. The relatively simple equation remaining was chosen so that a comparison with the seismic response of the acoustic wave could be done to ascertain the possible usefulness of pursuing this topic further. Apart from hydrocarbon related seismic applications, the use of this theory for near surface seismic or Ground Penetrating Radar $\left(GPR\right)$ applications to locate toxic or hazardous waste sites and possibly be of assistance in delineating the extent of seepages either from actual dumping or deteriorating containers is a possibility.

INTRODUCTION

It is difficult to discuss the problem of seismic waves propagating in a porous viscoelastic medium without simultaneously introducing concepts, and thus equations related to the interaction between the fluid and the solid matrix, that is, without speaking to at least the basic concepts of Darcy's Law. There are numerous works which concentrate on developing the theory of seismic wave propagation within the framework of what could loosely be termed either reservoir geophysics or reservoir engineering. To list some of the more prominent papers and texts that deal with this topic presents another problem, as the notations are not standardized nor are the coordinate systems, or possibly more accurately, the relative coordinate systems employed. Further, many of the theoretical developments are problem specific. As a consequence, only the basic equations will be presented here with little or no discussion of theory and referring only to the classic papers in this research area by Frankel (1944) and Biot (1956a, 1956b, 1956c, 1962a, 1962b) and leaving the investigation of matters related to Darcy's Law to the reader. For a marginally comprehensive knowledge of this area of study, topics, as mentioned in the previous sentence, must eventually be looked at in some depth.

The motivation for considering this problem is, as an example, that for a conventional hydrocarbon (oil and gas) production reservoir at some depth within the earth, fluids flow within the interstitial cavities of a matrix solid, driven conventionally, by hydrostatic pressure and is brought to the surface at specific locations (wells). To assume that the reservoir may be described by an acoustic or even elastic or anelastic medium could be considered presumptuous. Although history has shown that these are reasonable initial approximations, new technologies for monitoring the production history (time lapse seismology) and enhanced production, involving the injection of fluids, either of necessity or convenience, into the reservoir may indicate that a more relevant theory should be considered. Over the past few decades the technology associated with seismic data acquisition and primary and secondary recovery methods has increased enormously in complexity and utility. What has not kept pace, to the same extent, are fundamental seismic modeling capabilities.

In a general isotropic poroviscoelastic medium there are four modes of propagation, the conventional longitudinal *P* wave and the two shear modes S_V and S_H. As in the elastic case, the S_H propagates independently of the *P* and S_V. The *P* wave is associated with wave propagation in the matrix. In addition, there is, at least theoretically, another *P* wave, associated with the liquid, propagating in the medium with a velocity that is usually less than the shear, S_V, wave. There are a number of physical experiments where the *slow* compressional wave has been observed (for example, Coussy and Bourbie, 1984). If the medium is anisotropic, it is deemed to be anisotropic in viscoelastic parameters as well as permeability. One of the basic premises of the governing theory is that there is an interaction between a wave propagating in the matrix and in the fluid. A problem of significance is determining the viscoelastic parameters that define the medium. It is for this reason only the *fast* longitudinal case will be considered here, a modified analogue of the acoustic case in elastodynamic problems. Further, it will be assumed that that the medium is dissipative, that is, the medium is viscoelastic. As a preliminary exercise, this was thought to be a reasonable starting point.

A question posed by a number of authors (Mikhailenko, 1985) is "What media can be well described using Biot-Frankel theory?" This problem requires further investigation, both in theory and in practice. The primary problem is the necessity of determining by some means Biot's coefficients for different geological rock types, especially those in which hydrocarbon deposits are situated.

Since many reservoir environments are partially saturated with one or more fluids, incorporation of partial saturation as well as multi-fluid interaction and dissipation due to absolute movements of the fluid is necessary. Development of a more realistic physical and mathematical model is essential in understanding the propagation of seismic waves in the real environment

BIOT-FRANKEL'S EQUATIONS

The Biot – Frankel equations governing compressional (P) wave propagation may be found in Mikhailenko (1985) where

1. in a porous viscoelastic both the *fast* and *slow* *P* waves in a medium are described by

$$P\nabla^2 U + Q\nabla^2 V = \partial_t^2\left(\rho_{11}U + \rho_{12}V\right) + b\,\partial_t\left(U - V\right) + f_1\left(\mathbf{x},t\right) \tag{1}$$

$$Q\nabla^2 U + R\nabla^2 V = \partial_t^2\left(\rho_{12}U + \rho_{22}V\right) - b\,\partial_t\left(U - V\right) + f_2\left(\mathbf{x},t\right) \tag{2}$$

2. or in a single (*fast*) P type porous viscoacoustic medium by

$$P\nabla^2 U = \rho\,\partial_t^2 U - b\,\partial_t U + f\left(\mathbf{x},t\right). \tag{3}$$

The damping coefficient b in equations (1-2) is related to Darcy's coefficient of permeability k and porosity ϕ by $b = \eta\phi^2/k$ with η being the fluid viscosity. The quantity $U\left(\mathbf{x},t\right)$ is the dilation ($U\left(\mathbf{x},t\right) = \nabla\cdot\mathbf{u}\left(\mathbf{x},t\right)$) in the solid and $V\left(\mathbf{x},t\right)$ is the dilation ($V\left(\mathbf{x},t\right) = \nabla\cdot\mathbf{v}\left(\mathbf{x},t\right)$) in the liquid and it has been assumed that the source is located in some region or inclusion of convenience of the medium. Biot's coefficients, P, Q and R $\left(P = \lambda + 2\mu\right)$, which may be functions of position, must satisfy the inequality $PR - Q^2 > 0$. In the limit, $R, Q, \rho_{12}, \rho_{22} \to 0$, a scalar compressional wave equation (3) is all that remains.

To not include the "*slow*" $P-$wave in the derivation has been judged by a number of authors to produce a questionable synthetic seismic response. This topic is summarized quite well by Hassanzadeh. (1991) "Synthetic seismograms computed using this method indicate that in the presence of heterogeneities the Biot mechanism (namely, the macroscopic differential fluid-solid movement, controlled primarily by hydraulic permeability) contributes considerably to the dispersion and dissipation of compressional waves. The dispersion and dissipation of the *fast* wave is due to the conversion of a small amount of its energy into the *slow* wave each time it crosses an interface between two dissimilar materials. In cases involving several layers and multiple reflections, as in the crosswell geometry, the cumulative effect of this kind of conversion is significant. The amount of energy lost as a result of this kind of conversion is influenced by the permeability of the medium. No equivalent single-phase model can adequately describe these effects on seismic waves propagating through fluid filled porous media."

In the "viscous" case, $b = \eta\phi^2/k$ where $\eta-\text{viscosity}$, $\phi-\text{porosity}$ and $k-\text{permeability}$. Equations (1), (2) and (3) for b are valid for the low-frequency range where the flow in the porous medium is of the Poiseuille type[1]. (The upper limit of frequency for this flow type is called Biot's frequency, defined as $2\pi f_B = \eta\phi^2/k\rho_f$) Most studies have indicated that normal seismic frequencies lie within this range and provide a reasonably good approximation (Dutta and Ode, 1979, 1983 and Bourbie et al., 1987).

The coefficients, ρ_{11}, ρ_{12} and ρ_{22}, which have the dimensions of density and may also be spatially dependent, satisfy the inequalities

[1] Non – turbulent flow.

$$\rho_{11} > 0, \quad \rho_{12} > 0, \quad \rho_{22} > 0 \tag{4}$$

$$\rho_{11}\rho_{22} - \rho_{12}^2 > 0 \tag{5}$$

In the limit of the non-porous case, $\rho_{11} \to \rho$, where ρ is the density of an elastic medium.

In an infinite medium, apart from source excitation, the problem may be fully specified by the initial conditions

$$U\big|_{t=0} = \partial_t U\big|_{t=0} = V\big|_{t=0} = \partial_t V\big|_{t=0} = 0 \tag{6}$$

and assuming that the solution is such that radiation conditions are physically realizable.

The source, which will be chosen as a point source of *P* waves, is located within the medium (now taken to be a vertically inhomogeneous half space) or at the interface, $z = 0$. In the latter case, the free surface boundary (stress continuity) conditions are of the form

$$\partial_z U\big|_{z=0} = \delta(x)\gamma_S f(t) \tag{7}$$

$$\partial_z V\big|_{z=0} = \delta(x)\gamma_f f(t) \tag{8}$$

where γ_S and γ_f are constants are specified in terms of the porosity (ϕ) and effective density of the matrix solid, together with factors inherent to the fluid; $\gamma_S = (1-\phi)$ and $\gamma_f = \phi$ with $\gamma_S + \gamma_f = 1$. The above equations would be applicable to a layer at the earth's surface that is fully or partially fluid saturated. The reason for continuing to refer to this last type of problem is that it is familiar, and as the equations for wave propagation in this medium type are the same as for a hydrocarbon reservoir at depth, may serve as a point of reference.

The wave propagation will initially be assumed to be confined to a spatial plane defined as $(0 \le x \le a; 0 \le z \le b\ )$ and all of these four boundaries are initially assumed to be perfectly reflecting. These conditions require that some measures such as absorbing boundaries (Clayton and Enquist, 1977 and Reynolds, 1978) or attenuating boundaries (for example, Cerjan et al., 1985) be incorporated in the solution method so that spurious reflections from them will not contaminate the wavefield propagating within the spatial plane.

At the surface, $z = 0$, the stress free boundary conditions are

$$\partial_z U_{i,0}^m = \partial_z V_{i,0}^m = 0 \tag{9}$$

$$\partial_z U\big|_{z=0} = \partial_z V\big|_{z=0} = 0\ . \tag{10}$$

At the free surface, equation (3) is taken to be satisfied by a perfectly reflecting boundary condition.

PLANE WAVE SOLUTION

Before investigating equation (3) in a finite difference context, it may be useful to introduce a plane wave solution and see what evolves. Rewriting this equation

$$P\nabla^2 U = \rho\,\partial_t^2 U - b\,\partial_t U + f\left(\mathbf{x},t\right). \tag{11}$$

and assuming a plane wave solution of the form

$$U\left(x,t\right) = A\exp\left[i\omega t - i\omega px\right]. \tag{12}$$

results in (neglecting the source term)

$$P\left(1/V\right)^2 = \rho + i\,b/\omega = 0 \qquad \text{where} \quad V_0^2 = P/\rho\,. \tag{13}$$

Rewriting (11) has

$$\frac{1}{V^2} = \frac{\rho}{P} + \frac{ib\rho}{\omega\rho P} = \frac{1}{V_0^2} + \frac{ib}{\omega\rho V_0^2} = \frac{1}{V_0^2}\left[1 + \frac{i}{\omega\rho/b}\right]. \tag{14}$$

Comparing (14) with the formula for complex velocity in an anelastic medium (Aki and Richards, 1980) which is given by

$$\frac{1}{v^2} = \frac{1}{V^2}\left(1 + \frac{i}{Q}\right) \;\leftarrow\; \textit{Anelastic case}\,. \tag{15}$$

Here V is some real phase velocity and v is the complex anelastic velocity characterizing the medium, results in the value of Q_P for the scalar poroviscoelastic *fast* $P-$wave case being given by

$$Q_P^{(f)} = \omega\rho/b\,. \tag{16}$$

if the approximate single $P-$wave acoustic case being considered here is assumed a valid initial strategy. The expression obtained for Q_P above is closely related to Biot's frequency defined earlier, with the exception that ρ_f is replaced by $\rho = \phi\,\rho_S + \left(1-\phi\right)\rho_f$ in equation (16) – the weighted average of the fluid and solid or the bulk density.

VISCOACOUTIC MEDIUM

Two different $2.5D$ finite integral transform – finite difference solution methods for the poroacoustic wave equation will be presented here. The first involves assuming radial symmetry; that is, there are no lateral inhomogeneities and any quantities defining may vary only in the depth $\left(z\right)$ coordinate. The result is a finite difference problem in depth and time. Apply a finite Hankel transform w.r.t. the radial coordinate of $U\left(r,z,t\right)$ in equation (3) with a source term given by $f\left(r,z,t\right) = \delta\left(r\right)\delta\left(z-z_S\right)f\left(t\right)/2\pi$. To obtain a solution, it is most often assumed that the time dependence of the source wavelet, $f\left(t\right)$ is band limited. This type of source minimizes the number of term needed to approximate the infinite series summation.The finite Hankel transform is defined by

$$S\left(k_i,z,t\right)=\int_0^a U\left(r,z,t\right)J_0\left(k_i r\right)rdr \tag{17}$$

where the k_i are the solutions of the transcendental equation $J_0\left(k_j a\right)=0$. At the pseudo – boundary, $r=a$, the boundary condition is that of total reflection. The inverse of (17) is given by

$$U\left(r,z,t\right)=\sum_{i=1}^{\infty}\frac{S\left(r,k_i,t\right)J_0\left(k_i r\right)}{\left[J_1\left(k_i a\right)\right]^2} \tag{18}$$

(See Appendices A and B.) The transformed equation (3) has the modified form

$$\bar{P}\frac{\partial^2 S}{\partial z^2}-\bar{P}k_j^2 S-b\partial_t S-\rho\partial_{tt}S=\frac{\delta\left(z-z_S\right)f\left(t\right)}{2\pi} \tag{19}$$

where an over scored quantity indicates the harmonic average. Introducing second order accurate finite difference analogues in depth and time yield the following finite difference problem

$$\begin{aligned}S_m^{\ell+1}=&\left\{\left[S_{m+1}^{\ell}+S_{m-1}^{\ell}\right]-\left[\bar{P}k_j^2dt^2+2\bar{P}\Delta^2-2\rho_{11}\right]S_m^{\ell}+\right.\\&\left.\left[bdt/2-\rho_{11}\right]S_m^{\ell-1}-\delta\left(z-z_S\right)dt^2 f\left(t\right)/2\pi\right\}\Big/\left[\rho_{11}+b\,dt/2\right]\end{aligned} \tag{19}$$

which is to be solved subject to zero initial conditions

$$U\left(r,z,t\right)\Big|_{t=0}=\left.\frac{\partial U\left(r,z,t\right)}{\partial t}\right|_{t=0}=0\quad\rightarrow\quad S\left(k_j,z,t\right)\Big|_{t=0}=\left.\frac{\partial S\left(k_j,z,t\right)}{\partial t}\right|_{t=0}=0 \tag{20}$$

The time step is denoted as dt and the quantity Δ is defined as dt/h where h is the sptial step in the vertical $\left(z\right)$ direction.

Starting again with equation (3) and apply a finite Fourier transform w.r.t. the coordinate y to $U\left(x,y,z,t\right)$ in the equation

$$P\nabla^2 U=\rho\,\partial_t^2 U-b\,\partial_t U+f\left(\mathbf{x},t\right). \tag{21}$$

where the source term is given by $f\left(x,y,z,t\right)=\delta\left(x-x_s\right)\delta\left(y-y_s\right)\delta\left(z-z_S\right)f\left(t\right)$. In this instance it is assumed that the medium does not vary spatially in the y direction. The forward Fourier cosine transform is defined by

$$S\left(x,p,z,t\right)=\int_0^c U\left(x,y,z,t\right)\,\cos\left(\pi py/c\right)dy \tag{22}$$

And the inverse transform is given by the infinite series

$$U(x,y,z,t)=\frac{S(x,0,z,t)}{c}+\frac{2}{c}\sum_{n=1}^{\infty}S(x,p,z,t)\cos\left(\frac{p\pi y}{c}\right) \tag{23}$$

The assumption of a band limited source wavelet is again made so that a finite number of terms to adequately approximate the infinite series summation may be obtained.

$$\begin{aligned} P\frac{\partial^2 S}{\partial x^2}+P\frac{\partial^2 S}{\partial z^2}-P(2\pi p/c)^2 S-b\partial_t S-\rho\,\partial_{tt}S= \\ \delta(x-x_S)\cos(\pi p y_s/c)\,\delta(z-z_S)\,f(t)=F \end{aligned} \tag{24}$$

Second order accurate finite difference analogues in depth (z), the horizontal coordinate (x) and time result in the following finite difference analogue:

$$\begin{aligned} S_{m,n}^{\ell+1}=\Big\{\bar{P}\Delta^2\left[S_{m,n+1}^{\ell}+S_{m+1,n}^{\ell}+S_{m-1,n}^{\ell}+S_{m,n-1}^{\ell}\right]- \\ \left[\bar{P}(\pi p\,dt/c)^2+4\bar{P}\Delta^2-2\rho\right]S_{m,n}^{\ell}+\left[(b\,dt/2)-\rho\right]S_{m,n}^{\ell-1}-F\Big\}\Big/\left[\rho+(b\,dt/2)\right] \end{aligned} \tag{25}$$

where as before an over scored quantity indicates the harmonic average. The terms b and ρ should be expressed as arithmetic averages. The spatial steps in the (x) and (z) dimensions are chosen to be equal and equal to h, The time step is dt with $\Delta=dt/h$. The free surface is chosen to be perfectly reflecting so that the finite difference analogue here is of a very simple form

DISCUSION AND CONLUSIONS

A plane wave analysis of the poroviscoacoustic wave equation in a fluid filled porous medium was briefly considered here. Although use of this equation, rather than the coupled poroviscoelastic equations where both the slow and fast $P-$waves are inherent, has been questioned by some authors. The acoustic type damped wave, which describes only the propagation of the fast $P-$wave and does not consider any interaction that the two $P-$waves may introduce, does present some insight into the attenuation mechanism. When compared to an anelastic acoustic style of equation, the quantities involved, which defined the quality factor $Q_P^{(f)}$ may be determined. In addition, the relationship between the quantities that define $Q_P^{(f)}$ and *Biot's frequency* become evident. Also presented are two hybrid finite integral transform – finite difference for the computation of synthetic traces in a radially symmetric medium with depth dependent model parameters and in a $2.5D$ medium where the parameters may be spatially dependent in the (x,z) plane, but constant in the $y-$direction. The use of these relatively simple approaches for the determination of the seismic response in a poroviscoacoustic medium may be considered as tutorial steps in the problem of treating the the coupled slow/fast $P-$waves which more adequately describes $P-$wave propagation in a more correct poroviscoelastic formulation.

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APPENDIX A: FINITE HANKEL TRANSFORM

Although the two following finite Hankel transform methods may be found in the literature (Sneddon, 1972, for example), it was felt that for completeness they should be included here, at least in an abbreviated theorem formulation. The finite Hankel transform of the first kind is a direct application of the following theorem.

Theorem I: If $f\left(x\right)$ satisfies Dirichlet's conditions in the interval $\left(0,a\right)$ and if its Hankel transform in that range is defined to be

$$H_{\mu}^{(1)}\left[f(x)\right] \equiv f_{\mathrm{J}}\left(\xi_j\right) = \int_0^a x\, f(x) J_{\mu}\left(\xi_j x\right) dx \tag{A.1}$$

where ξ_j is a root of the transcendental equation

$$J_{\mu}\left(\xi_j a\right) = 0 \tag{A.2}$$

then, at any point in the interval $(0, a)$ at which the function $f(x)$ is continuous ,

$$f(x) = \frac{2}{a^2} \sum_{j=1}^{\infty} f_{\mathrm{J}}\left(\xi_j\right) \frac{J_{\mu}\left(\xi_j x\right)}{\left[J_{\mu+1}\left(\xi_j x\right)\right]^2} \tag{A.3}$$

where the sum is taken over all the positive roots of equation (A.2).

The finite Hankel transform and inverse of the second kind used in the text are given as follows:

Theorem II: If $f(x)$ satisfies Dirichlet's conditions in the interval $(0, a)$ and if its Hankel transform in that range is defined to be

$$H_{\mu}^{(1)}\left[f(x)\right] \equiv f_{\mathrm{J}}\left(\xi_j\right) = \int_0^a x\, f(x) J_{\mu}\left(\xi_j x\right) dx \tag{A.4}$$

in which ξ_j is a root of the transcendental equation

$$\xi_j J'_{\mu}\left(\xi_j a\right) + h J_{\mu}\left(\xi_j a\right) = 0 \tag{A.5}$$

then, at each point in the interval $(0, a)$ at which the function $f(x)$ is continuous ,

$$f(x) = \frac{2}{a^2} \sum_{j=1}^{\infty} \frac{\xi_j^2 f_{\mathrm{J}}\left(\xi_j\right)}{h^2 + \left(\xi_j^2 - \mu^2/a^2\right)} \frac{J_{\mu}\left(\xi_j x\right)}{\left[J_{\mu}\left(\xi_j x\right)\right]^2} \tag{A.6}$$

where the sum is taken over all the positive roots of (A.5) and h is determined from a boundary operator **N** at $x = a$ defined as

$$\mathbf{N}[f] = \frac{df(a)}{dx} + h\, f(a) = 0\,. \tag{A.7}$$

APPENDIX B: TERMS IN INVERSE SUMMATION SERIES

The analytic Fourier transform of the Gabor wavelet

$$f(t) = \cos(\omega_0 t) \exp\left[-\left(\frac{\omega_0 t}{\gamma} \right)^2 \right] \tag{B.1}$$

is, apart from some constant multiplicative terms,

$$F(\omega) = \frac{\pi^{1/2}}{\omega_0} \exp\left[-\frac{\gamma^2}{4} (1 + \omega/\omega_0)^2 \right] \cosh\left(\frac{\omega \gamma^2}{\omega_0} \right). \tag{B.2}$$

The horizontal wave number, k, in a coordinate system with cylindrical symmetry is related to the angular frequency as

$$k = \frac{\omega}{v} \tag{B.3}$$

where v is velocity and ω is the circular frequency. It will be assumed that some upper bound, ω_u, on the band limited spectrum of the source wavelet has been determined, often through numerical integration of the spectrum and then reintegration to the value ω_u up to which about 99.99% of the initial integration. Once ω_u has been determined, the value of k_u may be obtained as

$$k_u = \frac{\omega_u}{v_{\min}} \tag{B.4}$$

with $v_{\min}$ being the minimum velocity P or S_V encountered on the spatial grid which is one dimensional. It is known from numerical experiments that a good approximation for the duration of the Gabor wavelet in the time domain is γ/f_0. For some arbitrary k_i in the inverse series,

$$k_i = \frac{\zeta_i}{a} \tag{B.5}$$

where the values of ζ_i are the roots of the transcendental equation

$$J_1(\zeta_i) = 0 \tag{B.6}$$

so that

$$k_u = \frac{\omega_u}{v_{\min}} = \frac{\zeta_u}{a} \tag{B.7}$$

or equivalently

$$\zeta_u = \frac{a\,\omega_u}{v_{\min}} \tag{B.8}$$

indicating that the number of terms which must be considered to adequately approximate the infinite series summation increases linearly with *a*. It may be seen upon examination of equation (C.2) that the spectral width of the Gabor wavelet decreases with increasing values of γ. With the value of $\gamma = 4$ used here, $\omega_u \approx 2\omega_0$, so that with the predominant wavelength defined in terms of the predominant circular frequency and the minimum velocity encountered, $\lambda_0 = v_{\min}/f_0$ equation (B.7) becomes

$$\zeta_u = 4\pi\alpha \tag{B.9}$$

In the above equation, $\alpha = a/\lambda_0$, a dimensionless quantity relating the predominant wavelength with the pseudo – boundary introduced at $r = a$. For large values of *i*, the relation approximate relation $\zeta_i \approx \pi i$ holds (Abramowitz and Stegun, 1980). Thus the number of terms *N*, required to approximate the infinite series is, with $\zeta_u \approx \pi N$, given as

$$N \approx 4\alpha\ . \tag{B.10}$$

For comparison purposes, going through the derivation with $\gamma = 5$ results in the value of *N* being given as

$$N \approx 8\alpha/5 \tag{B.11}$$

which is less than that estimated for $\gamma = 4$, as would be expected.