Continuous wavelet transforms and Lipschitz exponents as a means for analysing seismic data

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ABSTRACT

Irregular structures and regions of abrupt change contain important information in a signal. In seismic signal analysis, the study of singularities within a trace provides a potential for extraction of critical information. The local regularity of a seismic event is determined by the wavelet transform modulus maxima and the associated Lipschitz exponent. As a means of classifying regularities of a signal and estimating the associated Lipschitz exponent, a linear and non-linear model based on the wavelet theory is reviewed and developed. For certain kind of signal events, a simple linear model can be applied in order to determine the associated Lipschitz regularity. However, in particular for band-limited signal events with some degree of smoothness a more complex non-linear model has to be applied which can be computationally expensive and difficult to analyse. Our purpose is to try to understand the key features of this more complex model of Lipschitz regularity, and develop simple and robust methods for estimation based on this understanding.

INTRODUCTION

Singularities and points of sharp variation carry critical information that are typically amongst the most important features for analysing properties of transient signals or images (Mallat and Zhong, 1992). Points of sharp variation created by shadows, occlusions, highlights are typically located at boundaries of image structures and contain different intensity profiles (Mallat and Zhong, 1992). In seismic signal analysis, regions of abrupt change classifiable as "edges", contain considerable amount of a signal's information, thus making edge detection a potentially appropriate and efficient tool for obtaining information from seismic data (Innanen, 2003). Edge detection requires analysis of local properties of corresponding edges.

Traditionally, the Fourier transform has been the main mathematical tool and technique for analysing singularities and irregular structures. However, a major drawback lies in the fact that the the Fourier transform generally provides a description of a signal's overall singularity, thus it is not well suited for finding spatial distributions and locations of singularities (Mallat and Zhong, 1992; Mallat and Hwang, 1992).

The Wavelet transform is closely related to multi-scale edge detection, characterises the local regularity of a signal by decomposing signals into fundamental building blocks localised in space and frequency. Applying advanced mathematical techniques namely continuous wavelet transform enables us to obtain the modulus maxima from seismic data and estimate the Lipschitz exponents which in turn allows us to measure the local regularity of functions and differentiate the intensity profile of different edges (Mallat and Zhong, 1992; Mallat and Hwang, 1992).

Several important physical processes can in principle affect the local regularity of a reflected event in a seismic trace: processes of absorption/wave attenuation, and reflections from targets composed of thin (sub-wavelength) layers. A robust estimation of Lipschitz exponents from seismic data, alongside prior geological information, could potentially lead to processing and inversion algorithms able to discern and characterise such targets. Algorithms of this kind would be of significant scientific and economic value. The proceeding sections provide a general review of the available literature in regards to continuous wavelet transform modulus maxima and properties of the associated Lipschitz exponent.

THEORETICAL BACKGROUND

I. Wavelet Transform

Although a powerful tool for analysing periodic functions, the Fourier transform fails to provide sufficient information in regards to the evolution of frequency content in time or local properties of the frequency content of a desired function f since it integrates $f(t)$ over all time (Daubeschies, 1992; Kaiser, 1994; Qian, 2002). One possible solution is to cut f into blocks and subsequently perform the Fourier transform on a block by block basis which will provide information in regards to the signal's frequency content or behaviour during the time frame covered by the corresponding window (Qian, 2002). This method which is referred to as the windowed Fourier transform or the short-time Fourier transform can be described by the following mathematical relation (Daubeschies, 1992),

$$(T^{win}\widetilde{f})(\omega, t) = \int_{-\infty}^{+\infty} f(s)g(s-t)e^{-i\omega s}ds, \tag{1}$$

where $f(s)$ is an arbitrary signal in the L^2 and $g(s)$ is the windowed function designed to localise signals in time.

Despite the benefits of time-frequency localisation, an underlying problem with this method relates to the flexibility or the fixed size of the window. For a small window, low frequencies are too large to be represented accurately. Additionally, one would have to use large windows for high frequencies which would result in loss of information for brief or abrupt changes in the corresponding interval (Qian, 2002).

The wavelet transform provides a solution to some of the shortcomings associated with the Fourier and the short-time Fourier transform by utilising a scalable modulated window and providing a time-scale representation of the signal by calculating the spectrum at every position and shifting the scalable window along the signal.

Mathematically, for a given function $f(t)$ the continuous wavelet transform at scale s and translation τ is given by the following relation (Qian, 2002),

$$Wf(s, \tau) = \int_{-\infty}^{+\infty} f(t)\frac{1}{\sqrt{s}}\psi^*(\frac{t-\tau}{s})dt, \tag{2}$$

where $\psi(t)$ is the *"mother wavelet"* dilated (scaled) and translated (time-shifted) satisfying

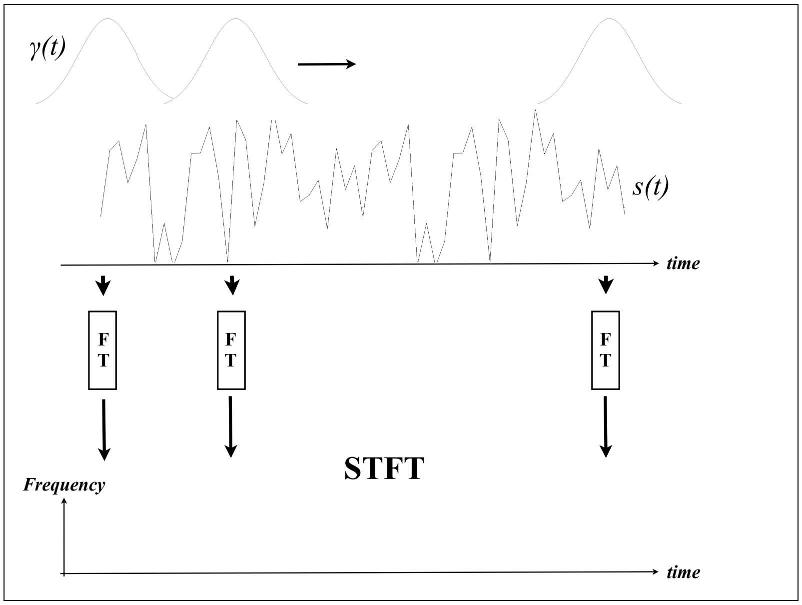

FIG. 1. Visual description of short-time Fourier transform.

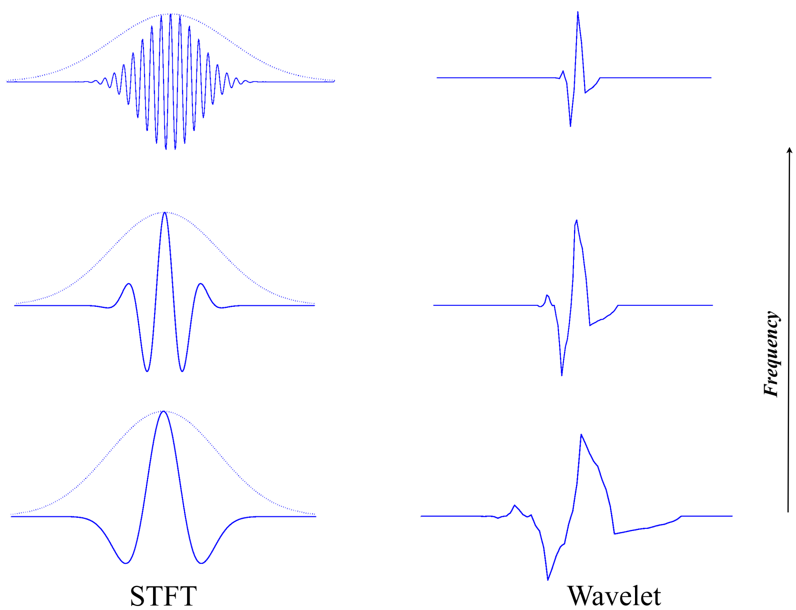

FIG. 2. The wavelet (right side) varies the width while keeping the number of oscillations constant. The short-time Fourier transform (left side) has a fixed window size independent of oscillations.

the following condition (Burden and Douglas, 2005; Qian, 2002),

$$\int_{-\infty}^{+\infty} \frac{|\tilde{\psi}(\omega)|}{|\omega|} d\omega < \infty, \tag{3}$$

where $\tilde{\psi}(\omega)$ is the Fourier transform of $\psi(t)$. Equation (3) requires that $\tilde{\psi}((0) = 0$, which implies that

$$\int_{-\infty}^{+\infty} \psi(t)dt = 0, \tag{4}$$

As a result, based on Equation (4), one could make a distinction between seemingly wavelet type functions such as scaling functions and an actual wavelet by ensuring that the function integrates to zero.

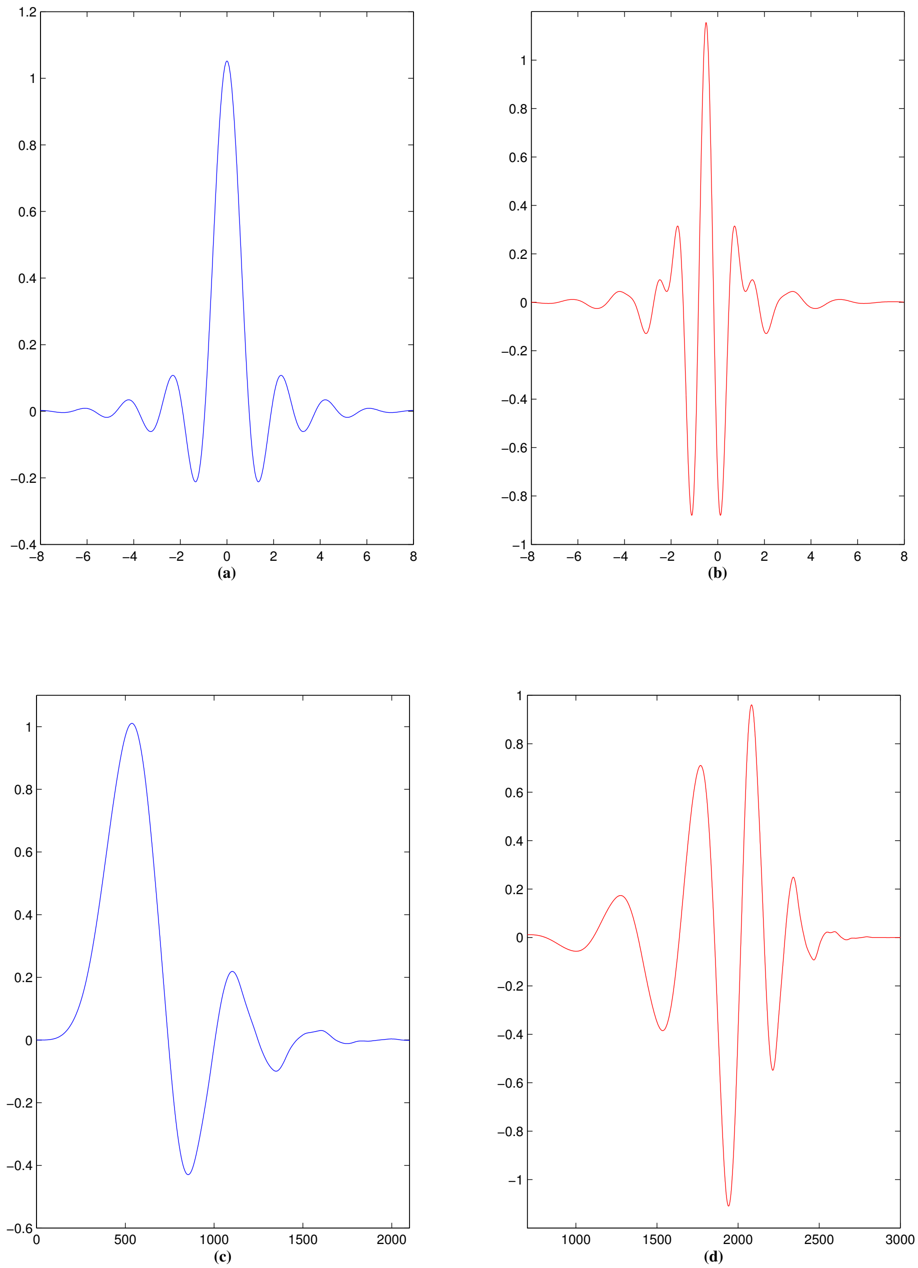

FIG. 3. (a) Meyer scaling function (b) Meyer wavelet (c) Daubeschies “db8” scaling function (d) Daubeschies “db8” wavelet.

For practical applications and in order to implement a fast numerical algorithm one would have to sample the continuous wavelet algorithm on a dyadic grid, that is $s = 2^j$, for $j \epsilon \mathbb{Z}$, where it has been proven that the wavelet transform on a dyadic grid is complete and stable (Mallat and Zhong, 1992).

II. Lipschitz Regularity

An important property associated with the continuous wavelet transform is the ability to characterise local regularities by smoothing a given signal f and detecting points of sharp variation at various scales s. The local regularity of a function f is often measured by the corresponding Lipschitz exponent (Mallat and Zhong, 1992; Hong et al., 2002). The Lipschitz exponent, a generalised measure of a function's differentiability is defined in the frequency domain by the following relation (Daubeschies, 1992),

$$\int_{-\infty}^{+\infty} |f(\omega)|(1 + |\omega|^{\alpha})d\omega < +\infty. \tag{5}$$

Based on equation (5), the Lipschitz exponent α, corresponds to the decay of the Fourier coefficients with increasing frequency and equal to the largest possible value such that the relation above holds (Innanen, 2003). However, equation (5) is not well adapted for measuring local regularity of a function at a specific point since it only provides a global regularity condition. As a result one would have to apply wavelet analysis in order to gain information in regards to local regularity of a function. Based on wavelet transform, a function $f(x)$ is said to be uniformly Lipschitz α over $[a, b]$ if and only if there exists a constant $A > 0$ such that the wavelet transform satisfies the following (Mallat and Zhong, 1992; Innanen, 2003),

$$|W_s f(x)| \leq A s^{\alpha} \tag{6}$$

where $|W_s f(x)|$ is the modulus maxima of the function $f(x)$ at various scales $s = 2^j$ for $j \epsilon \mathbb{Z}$. Equation (6) suggests that the evolution of the modulus of the wavelet coefficients across the scale depends on the local Lipschitz regularity of the desired function (Innanen, 2003). Thus, based on the following properties associated with the Lipschitz exponent, a distinction could be made between singular and differentiable function (Mallat and Zhong, 1992):

- A function $f(x)$ is singular if the associated Lipschitz exponent, α, is less than 1.
- The Lipschitz exponent α, associated with a continuously differentiable function $f(x)$ is equal to or greater than 1.
- If $f(x)$ is Lipschitz α, then its integral $g(x)$ has an associated Lipschitz exponent equal to $\alpha + 1$.
- The Lipschitz regularity of a delta function is equal to 1, since its associated modulus maxima decreases with scale.

III. Estimating the Lipschitz Exponent

In order to estimate α from the data, one could linearise equation (6) by taking the logarithms in order to obtain the following relation,

$$\log_2 |W_s f(x)| \leq \log_2 A + \alpha \log_2(s) \tag{7}$$

Finding the slope and the intercept of equation (7) yields an estimate for α and A. Although linearising equation (6) simplifies the estimation of α, nevertheless this procedure requires certain degree of caution since it involves scaling the errors associated with the numerical estimation of the modulus maxima. Additionally one could estimate α and A by forming an optimisation problem. Forming the objective function based on equation (7) would produce the following,

$$\phi(A, \alpha) = \sum_{i,j=1}^{n} (\log_2(a_i) - (\log_2 A + \alpha \log_2(s)))^2 \tag{8}$$

where $a_i = |W s_j f(x)|$ for $i, j = 1, .., n$. Minimising equation (8), provides the following system

$$\begin{pmatrix} A \\ \alpha \end{pmatrix} = \begin{pmatrix} n & \sum_{j=1}^{n} \log_2(s_j) \\ \sum_{j=1}^{n} \log_2(s_j) & \sum_{j=1}^{n} (\log_2(s_j))^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{n} \log_2(a_i) \\ \sum_{j=1}^{n} \log_2(s_j) . \sum_{i=1}^{n} \log_2(a_i) \end{pmatrix} \tag{9}$$

(For our purpose both methods have been applied in order to estimate A and α).

Furthermore, in seismic signal analysis and for our purpose, we are mainly interested in estimating Lipschitz regularities ranging from -1 to 1. Thus, our preferred wavelet may be first derivative of a Gaussian function with a single vanishing moment. A wavelet $\psi(x)$ is said to have n vanishing moments if for $k < n$ it satisfies (Mallat and Hwang, 1992),

$$\int_{-\infty}^{+\infty} x^k \psi(x) dx = 0 \tag{10}$$

Furthermore, letting $\psi(x) = \frac{d\theta(x)}{dx}$ where $\theta(x)$ is a Gaussian function and taking the continuous wavelet transform of f we obtain the following [2],

$$W_s f(x) = f * (s \frac{d\theta_s}{dx})(x) = s \frac{d}{dx} (f * \theta_s)(x). \tag{11}$$

Based on equation (11) the local extrema of $W_s f(x)$ corresponds to the inflection points of $f * \theta_s(x)$. Thus inflection points or points of sharp variation corresponding to $f * \theta_s(x)$ could be detected by estimating the local extrema of $|W_s f(x)|$ (Mallat and Zhong, 1992).

However, the use of the linear model given in (7) would be limited to single events that resemble a delta or Heaviside type function. In seismic signal analysis due to absorption and loss of energy with progression of time, a single event resembling a delta function would gradually obtain spectral characteristics of a Gaussian with increasing variance. Thus our primarily interest rests on detecting functions with a Lipschitz exponent ranging from $-1 \leq \alpha \leq 0$. Such a function could be modelled as a delta function say $h(x)$ convolved or smoothed by a Gaussian with variance σ^2 (Mallat and Zhong, 1992),

$$f(x) = h(x) * \theta_\sigma. \tag{12}$$

The continuous wavelet transform of $f(x)$ is given by,

$$W_s f(x) = s\frac{d}{dx}(h * \theta_{s_0})(x) = \frac{s}{s_0} W_{s_0} h(x) \tag{13}$$

where, $\theta_{s_0} = \theta_\sigma * \theta_s$ with $s_0 = \sqrt{s^2 + \sigma^2}$. As a result, the modulus maxima and Lipschitz regularity is given by the following relation (Mallat and Zhong, 1992; Mallat and Hwang, 1992),

$$|W_{s_0} h(x)| = \frac{s_0}{s} |W_{s_0} f(x)| \leq A s_0^\alpha \tag{14}$$

or

$$|W_{s_0} h(x)| \leq s A s_0^{\alpha - 1}. \tag{15}$$

In comparison to equation (7), the new model is non-linear and requires minimisation of the following objective function,

$$\phi(A, \alpha, \sigma) = \sum_{i,j=1}^{n} [\log_2 |a_i| - \log_2(A) - j + \frac{\alpha - 1}{2}(\log_2(\sigma^2 + 2^{2j})]^2. \tag{16}$$

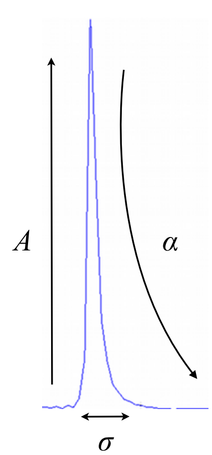

FIG. 4. Three parameters α, A and σ, measuring a pulse' decay, amplitude and width respectively.

Minimising the objective function in (16) requires a relatively expensive computational algorithm such as the conjugate gradient method or the steepest descent. The uncertainty in regards to A requires some sort of calibration in order to reduce the objective to a function of two variables (could be a time consuming procedure) or minimisation of a function of three variables.

CONCLUSION

The continuous wavelet transform and the associated Lipschitz regularity provide a potentially efficient and powerful tool for analysing singularities in a signal. For a single event, a linear model enables us to estimate the Lipschitz exponent and characterise the singularity with relative ease.

However, for practical applications, a seismic event would have to be modelled as delta function smoothed by a Gaussian, thus leading to a non-linear model. In order to estimate the Lipschitz exponent, one would have to form the objective function and minimise, using a relatively time consuming and computationally expensive method such as the steepest descent or conjugate gradient.

Moving forward, our primary concern is to analyse closely spaced events, develop a model of attenuation and dispersion in order to estimate Q values (from synthetic seismic trace and ultimately field data) based on the local regularities of the model

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