Unphysical negative values of the anelastic SH plane wave energy-based transmission coefficient

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ABSTRACT

Computing reflection and transmission coefficients in viscoelastic media is complicated since proper signs have to be chosen for the vertical slowness of the scattered waves. Research has shown that the commonly used methods for deciding the sign of the vertical slowness in the elastic media do not produce satisfactory results in viscoelastic case. New methods have been suggested by researchers to solve the problem, but none of them are quite perfect since some undesirable results still exist in these methods. ERC or the Extended Radiation Condition suggested by Krebes and Daley (2007) is one of the methods that give reasonable results. It has been found that when ERC is used, in some cases, the energy-based transmission coefficient becomes negative in a specific range of supercritical angles. In this work the conditions under which T becomes negative is determined and examined with numerical examples. It will be shown that only for the case in which $V_2 > V_1$ and $Q_2 > Q_1$, under specific conditions T is negative for specific range of supercritical incidence angles.

INTRODUCTION

The theory of seismic wave propagation has been developed based on the wave propagation in elastic medium. A medium is said to be elastic if it poses a natural state to which it will revert when applied forces are removed (Aki and Richards, 1980). In such an idealized medium the stresses and strains occurring within a propagating wave only cause reversible changes in the medium; hence wave motion will continue indefinitely once it has been initiated by some specific source. In contrast to such idealized behavior as a wave propagates through a real material, its amplitude attenuates as a result of a “internal friction” (Aki and Richards, 1980). Such media are said to be anelastic as the configuration of material particles is dependent on the history of applied stress. In anelastic media the wave motion is spatially attenuated and the wave energy is absorbed as the wave spreads away from a source. Thus the waves experience frequency-dependent attenuation and dispersion. The anelasticity of a material is described by the attenuation loss factor $1/_Q$, where Q is the quality factor. The attenuation loss factor for a perfect elastic media is zero that means $Q = \infty$ (Krebes, 2009).

The anelasticity of the earth is modeled by the theory of viscoelasticity. In this theory the earth behavior is modeled as a combination of an elastic solid and a viscous fluid (Carcione, 1988).The elastic solid is modeled by a spring element, and the viscous fluid is modeled by a dashpot. Although the properties of the wave propagation in the linear viscoelastic media are completely different from those in the elastic media, the wave propagation formulas for the perfectly elastic medium can be used for linear viscoelastic media, except that the velocities and angles should be replaced by the complex ones. This is referred to as “*the elastic-viscoelastic correspondence principle*” (Buchen, 1971; Borcherdt, 1973; Aki and Richards, 1980; Hearn and Krebes, 1990). Similarly the plane wave reflection and transmission (R/T) coefficients for linear viscoelastic media are calculated using the same formulas as for the perfectly elastic medium, except that the velocities and angles should be replaced by the complex ones. The complex quantities used in a linear viscoelastic medium adds to the complexity of the calculation of (R/T) coefficients and makes them different from the elastic ones. In the determination of (R/T) coefficients with the continuity criterion (assumption of continuously variation of (R/T) coefficients by offset), Hearn and Krebes (1990) showed that the phase curve of the viscoelastic (R/T) coefficients does not tend to the elastic one when the elasticity is removed. Also the determination of the proper sign for the vertical slowness in a viscoelastic medium is ambiguous. Ruud (2006) considered an energy flux based criterion and also the steepest descent path for determining the proper sign for vertical slowness; Ruud (2006) obtained desirable results in which the phase curve agreed quite well with that of the elastic case. Krebes and Daley (2007), hereafter referred to as $K\&D$, suggested some new approaches to the problem of determining the proper sign for vertical slowness. One of these approaches is an extension of the radiation condition referred to as "ERC". The radiation condition describes the amplitude decay while the wave is travelling away from an interface. It is found that the radiation condition is not always applicable in the viscoelastic case. $K\&D$ showed by numerical examples that the energy based SH-wave transmission coefficient becomes negative for a specific range of supercritical incidence angles when ERC is used. In this report we investigate the conditions under which these coefficients become negative and numerical examples will be presented. A more complete presentation of the results of this report can be found in Krebes and Moradi (2011).

THEORY

According to the elastic-viscoelastic correspondence principle a plane wave propagating in a linearly viscoelastic medium is defined as:

$$u = U\ e^{\ i(\mathbf{k}.\mathbf{x}-\omega t)} \tag{1}$$

where **k** is the complex wavenumber

$$\mathbf{k} = \mathbf{P} + \mathrm{i}\mathbf{A} \tag{2}$$

where **A** is the attenuation vector that is perpendicular to the planes of constant amplitude, $\mathbf{A}.\mathbf{x}$ = *constant* and, **P** is the propagation vector, which is perpendicular to the planes of constant phase, $\mathbf{P}.\mathbf{x}$ =*constant*

Consider a plane SH wave striking a plane boundary separating two viscoelastic media. The displacement reflection and transmission coefficients (Aki and Richards, 1980) are given by

$$C_R = \frac{M_1\eta_1 - M_2\eta_2}{M_1\eta_1 + M_2\eta_2} \text{ and } \qquad C_T = \frac{2M_1\eta_1}{M_1\eta_1 + M_2\eta_2} \tag{3}$$

where $M = \rho\beta^2$ is the complex shear modulus, with β and ρ being the complex *S*-wave velocity and density respectively and

$$\frac{1}{\beta^2} = \frac{1}{V^2}\left(\frac{Q}{q}\right)\left[1 + \frac{i}{Q}\right] \tag{4}$$

where V is the shear wave speed and

$$q = (Q/2)\left[1 + \sqrt{1 + Q^{-2}}\right] \tag{5}$$

η_1 and η_2 are the vertical slownesses of the reflected and transmitted waves respectively. In fact if we decompose the wavenumber into horizontal and vertical components we would have

$$\mathbf{k} = \omega(p\hat{\boldsymbol{x}} + \eta\hat{\boldsymbol{z}}) \tag{6}$$

where p and η are the horizontal and vertical components of the slowness vectors respectively

$$\eta_n = \frac{\cos j_n}{\beta_n} = \pm\sqrt{\frac{1}{\beta_n^2} - p^2} \quad , \; p = \frac{\sin j_n}{\beta_n} \quad n = 1,2 \tag{7}$$

where j_n is the angle that the slowness vector of the wave makes with the z-axis in medium *n*.

Note that the displacement reflection and transmission coefficients in equation (3) have the same form as the elastic ones except that the velocities and the angles are replaced by the complex ones.

One of the difficulties in the viscoelastic case is to determine whether the wave is downgoing or upgoing. The sign of the vertical slowness in equation (7) corresponds to upward or downward propagation. In the elastic case this is straightforward since the vertical slowness η is real, so the positive sign could be chosen for the downgoing wave considering the radiation condition (which states the amplitude decreases away from the interface). Also for supercritical incidence angles where η becomes imaginary, the positive square root can be chosen again for the downgoing wave. In the viscoelastic case the choice of the sign of the square root is complicated since the vertical slowness is complex and it is not clear that which sign could be referred to as the positive or negative square root. The convention which is usually used is that the solution with the positive real part is referred to as the positive square root and the one with the negative real part is referred to as the negative square root.

It is found that the radiation condition does not work satisfactorily in the viscoelastic case for choosing the proper sign of the vertical slowness since the amplitude of the wave might grow with distance from the boundary *(Ruud, 2006);* because when $Im(\eta^2) < 0$, the imaginary and the real parts of the vertical slowness have the opposite signs which means a downgoing wave attenuates upward. In general, for a downgoing wave, the term *"physical"* is used when the positive sign of the square root ($Im(\eta) > 0$) is chosen and the *"non-physical"* term is used when the negative one is chosen ($Im(\eta) < 0$) *(Ruud, 2006).* However we cannot say that always the downward attenuation should be chosen for a downgoing wave whenever $Im(\eta^2) < 0$, in fact the problem is more complicated than that.

Borcherdt (1977) expressed the continuity of the normal energy flux across the boundary for an SH plane wave as follows:

$$R + T + I = 1, \tag{8}$$

where

$$R = |C_R|^2 \ , \qquad T = |C_T|^2 \left[\frac{Re(M_2\eta_2)}{Re(M_1\eta_1)}\right], \tag{9}$$

and

$$I = -2Im(C_R)\left[\frac{Im(M_1\eta_1)}{Re(M_1\eta_1)}\right] \tag{10}$$

R, T and *I* are energy based reflection, transmission and *interaction* coefficients. *I* is a normal energy flux caused by interaction between stress and velocity fields of the incident and reflected waves which disappears in perfectly elastic media. The reflection transmission coefficients, for the P and SV waves could be found in literature (Aki and Richards, 1980). The same formula as for the elastic case can be used, except that the velocity and angles have to be replaced by the complex ones (Hearn & Krebes, 1990; Aki and Richards, 1980).

As discussed above, determining the sign of the vertical slowness is ambiguous in viscoelastic media. Different approaches to the problem have been recently presented by researchers. Krebes and Daley (2007) discussed the sign of η_2^2 and its relation with the medium parameters in further details which is helpful in finding a proper sign for the vertical slowness. From equation (7) we have, for a homogeneous incident wave (**P** parallel to **A**):

$$\eta_2^2 = \frac{1}{\beta_2^2} - p^2 = \frac{1}{\beta_2^2} - \frac{\tilde{r}^2}{\beta_1^2} \tag{11}$$

where

$$\tilde{r} = \sin j_1 \tag{12}$$

Separating real and imaginary parts of η_2^2 gives

$$Re(\eta_2^2) = \frac{Q_2}{V_2^2 q_2} - \frac{\tilde{r}^2 Q_1}{V_1^2 q_1} \tag{13}$$

$$Im(\eta_2^2) = \frac{1}{V_2^2 q_2} - \frac{\tilde{r}^2}{V_1^2 q_1} \tag{14}$$

From equations (13) and (14) we have:

$$Re(\eta_2^2) \geq 0 \quad \text{for} \quad \tilde{r}^2 \leq \frac{q_1 Q_2}{q_2 Q_1}\,\tilde{r}_c^2 \tag{15}$$

$$Im(\eta_2^2) \geq 0 \quad \text{for} \quad \tilde{r}^2 \leq \frac{q_1}{q_2}\,\tilde{r}_c^2 \tag{16}$$

where $\tilde{r}_c = \frac{V_1}{V_2} = \sin j_{1c}$, and j_{1c} is the elastic critical angle of incidence .We can assume $Q \gg 1$ which is typical in the earth's subsurface; then equations (15) and (16) become

$$Re(\eta_2^2) \geq 0 \quad \text{for} \quad \tilde{r}^2 \leq \tilde{r}_c^2 \qquad Q \gg 1 \tag{17}$$

$$Im(\eta_2^2) \geq 0 \quad \text{for} \quad \tilde{r}^2 \leq \frac{Q_1}{Q_2}\,\tilde{r}_c^2 \qquad Q \gg 1 \tag{18}$$

These equations state that both $Im(\eta_2^2)$ and $Re(\eta_2^2)$ decrease from an initially positive value to a negative value as the incidence angle increases. If $Q_2 > Q_1$, $Im(\eta_2^2)$ becomes negative before $Re(\eta_2^2)$ does because it decreases faster than $Re(\eta_2^2)$.

For $Q \gg 1$ this could be summarized as follows

$$\text{for} \qquad \tilde{r} \leq \sqrt{\frac{Q_1}{Q_2}}\;\tilde{r}_c \qquad Im(\eta_2^2) \geq 0,\; Re(\eta_2^2) \geq 0 \tag{19}$$

$$\text{for} \qquad \sqrt{\frac{Q_1}{Q_2}}\;\tilde{r}_c < \tilde{r} < \tilde{r}_c \qquad Im(\eta_2^2) < 0\,,\; Re(\eta_2^2) \geq 0 \tag{20}$$

$$\text{and for} \qquad \tilde{r}_c \leq \tilde{r} \qquad Im(\eta_2^2) < 0\,,\; Re(\eta_2^2) < 0 \tag{21}$$

Krebes and Daley (2007*)* also suggested an approach to the problem which is an extension of the elastic radiation condition. In this approach the positive root square is chosen up to the point $\tilde{r} = \tilde{r}_c$ and the negative one for $\tilde{r} > \tilde{r}_c$ (the one in which $Im(\eta_2^2) > 0$). This criterion states that if both $Im(\eta_2^2)$ and $Re(\eta_2^2)$ are negative, the square root in which $Im(\eta_2) > 0$ has to be chosen, otherwise the positive square root should be chosen. This approach is referred to as the "*Extended Radiation Condition*" or "*ERC*". However it is found via numerical experimentation that in some conditions for a specific range of supercritical angles, $R + I > 1$ and $T < 0$, which is unphysical since the energy-based coefficients could not be negative.

DISCUSSION

Here we investigate that under which conditions and for what medium parameters T becomes negative when ERC is used. It can be inferred from equation (9) that T is negative when $Re(\mu_2\eta_2) < 0$, because the other terms in T, i.e. $Re(M_1\eta_1)$ and $|C_T|^2$ are always positive. We also know that

$$Re(M_2\eta_2) = Re(M_2)Re(\eta_2) - Im(M_2)Im(\eta_2) \tag{22}$$

where

$$M_2 = M_2 v_2^2 = M_2 V_2^2\left[1 - \frac{i}{Q_2}\right],\;\; Q \gg 1 \tag{23}$$

and

$$\eta_2 = \left(\frac{1}{v^2} - p^2\right)^{\frac{1}{2}} = \left(\frac{1}{v^2} - \frac{\tilde{r}^2}{v_1^2}\right)^{\frac{1}{2}} \tag{24}$$

$$= \left(\frac{1}{V^2}\left[1+\frac{i}{Q_2}\right]-\frac{\tilde{r}^2}{V_1^2}\left[1+\frac{i}{Q_1}\right]\right)^{\frac{1}{2}} \tag{25}$$

$$= \left(\left[\frac{1}{V^2}-\frac{\tilde{r}^2}{V_1^2}\right]+i\left[\frac{1}{V_2^2 Q_2}-\frac{\tilde{r}^2}{V_1^2 Q_1}\right]\right)^{\frac{1}{2}} \tag{26}$$

From complex variables theory we know that if Z is a complex number where

$$Z = a + ib \tag{27}$$

then the root square of Z would be

$$Z^{\frac{1}{2}} = \pm\left\{\frac{1}{2}\left[a+(a^2+b^2)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \pm i\left\{\frac{1}{2}\left[-a+(a^2+b^2)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \tag{28}$$

If we suppose

$$a = Re(\eta_2^2) = \left[\frac{1}{V_2^2}-\frac{\tilde{r}^2}{V_1^2}\right] \tag{29}$$

and

$$b = Im(\eta_2^2) = \left[\frac{1}{V_2^2 Q_2}-\frac{\tilde{r}^2}{V_1^2 Q_1}\right] \tag{30}$$

then

$$Re(\eta_2) = \pm\left\{\frac{1}{2}\left[a+(a^2+b^2)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \tag{31}$$

$$Im(\eta_2) = \pm\left\{\frac{1}{2}\left[-a+(a^2+b^2)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \tag{32}$$

To investigate the conditions under which T is negative, ERC should be used for choosing the proper sign of the vertical slowness. It is obvious that

$$Im(\eta_2^2) = 2Re(\eta_2)Im(\eta_2) \tag{33}$$

So the real and imaginary parts of η_2 would have the same signs when $Im(\eta_2^2) > 0$, and the opposite signs when $Im\left(\eta_2^2\right) < 0$. Consequently, the signs for real and imaginary parts of η_2 could be as follows

for $b > 0$ and all a: $[Re(\eta_2), Im(\eta_2)]$ is $[+,+]$or$[-,-]$ (34)

for $b < 0$ and all a: $[Re(\eta_2), Im(\eta_2)]$ is $[+,-] or [-,+]$ (35)

for $b = 0$ and $a \geq 0$: $\eta_2 = \pm a^{\frac{1}{2}}$ (η_2 is pos. or neg. real or zero) (36)

for $b = 0$ and $a < 0$: $\eta_2 = \pm i(-a)^{\frac{1}{2}}$ (η_2 is pos. or neg. imaginary) (37)

As defined before, the positive square root is the one for which $Re(\eta_2) > 0$. If $Re(\eta_2) = 0$, then the positive root square is the root for which $Im(\eta_2) > 0$.

The ERC criterion (Krebes and Daley, 2007) could be summarized as follows:

If $Im(\eta_2^2) < 0$ and $Re\left(\eta_2^2\right) < 0$, choose the negative square root (the one which for $Im(\eta_2) > 0$); otherwise choose the positive square root.

Therefore, according to the ERC the choices for the root square sign are either the negative sign in equation (35) or the positive sign in equations (34) - (37).

Now we examine the conditions under which $Re(M_2\eta_2) < 0$ and whether they are consistent with the ERC or not. From equations (22), (23) we get

$$Re(M_2\eta_2) = \rho_2 V_2^2[Re(\eta_2) + Q_2^{-1}Im(\eta_2)] \tag{38}$$

The cases for which $Re(\mu_2\eta_2)$ could possibly be negative are:

(a) $Re(\eta_2) < 0,\ Im(\eta_2) < 0$

(b) $Re(\eta_2) = 0,\ Im(\eta_2) < 0$

(c) $Re(\eta_2) < 0,\ Im(\eta_2) = 0$

(d) $Re(\eta_2) > 0,\ Im(\eta_2) < 0$

(e) $Re(\eta_2) < 0,\ Im(\eta_2) > 0$

In case (a), $Re(\eta_2) < 0$ and $Im(\eta_2) < 0$, which means the negative square root has to be chosen in equation (34) .This could never happen in ERC. Also in cases (b) and (c), the negative square root should be chosen in equations (37) and (36) respectively, which can never happen if ERC is used. Therefore cases (a), (b) and (c), could be simply eliminated. In case (d), $Re(\eta_2) > 0$ and $Im(\eta_2) < 0$ which means the positive square root is chosen in equation (35) which is consistent with ERC. To examine whether T is negative or not, we suppose $Re(\eta_2 M_2) < 0$, then we have

$$Re(\eta_2) < -Q_2^{-1}Im(\eta_2) \tag{39}$$

According to ERC in this case, $(\eta_2) > 0$ and $Im(\eta_2) < 0$, therefore

$$|Re(\eta_2)| < Q_2^{-1}|Im(\eta_2)| \ \Rightarrow$$

$$\left\{\frac{1}{2}\left[a + (a^2+b^2)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} < Q_2^{-1}\left\{\frac{1}{2}\left[-a + (a^2+b^2)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \tag{40}$$

Since $Re(\eta_2^2) < 0$ $(a \geq 0)$ and $Q_2^{-1} \ll 1$, the right hand side is always less than the left hand side, hence, the assumption that $Re(\eta_2\mu_2) < 0$ is not true; consequently $T \geq 0$ and this case could be eliminated as well.

In case (e), $Re(\eta_2) < 0$ and $Im(\eta_2) > 0$ which means the negative square root is chosen in equation (35). In this case if we suppose that $T < 0$ then

$$Re(\eta_2) < -Q_2^{-1} Im(\eta_2) \quad \Rightarrow$$
$$-|Re(\eta_2)| < -Q_2^{-1}|Im(\eta_2)| \tag{41}$$

Since $Re(\eta_2^2) < 0$ and $Im(\eta_2^2) < 0$ ($a < 0, b < 0$) according to ERC we would have

$$-\left\{\frac{1}{2}\left[-|a| + (a^2+b^2)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} < -Q_2^{-1}\left\{\frac{1}{2}\left[|a| + (a^2+b^2)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \Rightarrow$$
$$(1 - Q_2^{-2})^2(a^2+b^2) > (1 + Q_2^{-2})a^2 \quad \Rightarrow$$
$$1 + \frac{b^2}{a^2} > \left(\frac{1-Q_2^{-2}}{1+Q_2^{-2}}\right)^2 \tag{42}$$

Supposing $\alpha \equiv Q_2^{-2}$, we get

$$\left(\frac{1+\alpha}{1-\alpha}\right)^2 = (1-\alpha)^2\left(\frac{1}{1-\alpha}\right)^2 \tag{43}$$

Since $\alpha \ll 1$ then

$$= (1+\alpha)^2(1+\alpha+\alpha^2+\alpha^3+\cdots)^2 \approx (1+\alpha)^2(1+\alpha)^2 = (1+\alpha)^4$$
$$= 1 + 4\alpha + 6\alpha^2 + 4\alpha^3 + \alpha^4 \approx 1 + 4\alpha \;\Rightarrow$$
$$\left(\frac{1+\alpha}{1-\alpha}\right)^2 \approx 1 + 4\alpha \tag{44}$$

Then equation (42) becomes

$$1 + \frac{b^2}{a^2} > 1 + 4\mathrm{Q}_2^{-2} \quad \Rightarrow \quad b^2 > 4Q_2^{-2}a^2 \tag{45}$$

As mentioned before, b and a are both negative, so

$$b < 2Q_2^{-1}a$$

Substituting values for *a* and *b* from equations (29) and (30) gives

$$\left[\frac{1}{V_2^2 Q_2} - \frac{\tilde{r}^2}{V_1^2 Q_1}\right] < 2Q_2^{-1}\left[\frac{1}{V_2^2} - \frac{\tilde{r}^2}{V_1^2}\right] \;\Rightarrow$$
$$\left[2 - \frac{Q_2}{Q_1}\right]\tilde{r}^2 < \tilde{r}_c^2, \tag{46}$$

To determine when this inequality is satisfied, we examine it for four different possible cases:

Case1: $V_2 > V_1, Q_2 > Q_1$

Case 2: $V_2 > V_1, Q_2 \le Q_1$

Case 3: $V_2 \le V_1, Q_2 > Q_1$

Case 4: $\mathrm{V}_2 \le \mathrm{V}_1, \mathrm{Q}_2 \le \mathrm{Q}_1$

Case 1: This is the most common case in the Earth's subsurface. There are three possibilities in this case:

(a) $\left(2-\frac{Q_2}{Q_1}\right)>0$

(b) $\left(2-\frac{Q_2}{Q_1}\right)<0$

(c) $\left(2-\frac{Q_2}{Q_1}\right)=0$

Here we investigate these conditions:

(a) Since $\frac{Q_1}{Q_2}<1\ and\ \tilde{r}_c<1$ we have

$$\sqrt{\frac{Q_1}{Q_2}}\tilde{r}_c<\tilde{r}_c<\tilde{r}\leq 1, \quad \mathrm{a}<0, b<0, \tag{47}$$

also $\left(2-\frac{Q_2}{Q_1}\right)>0$, therefore

$$1<\frac{Q_2}{Q_1}<2 \tag{48}$$

From equations (46) and (47) we get

$$\tilde{r}_c<\tilde{r}<\frac{\tilde{r}_c}{\sqrt{2-\frac{Q_2}{Q_1}}}\equiv\tilde{r}'\equiv\sin j_1{}' \tag{49}$$

Accordingly, if T is negative, then equations (48) and (49) hold. This analysis could be done backward too (because $\alpha\ll 1$ -- see Krebes and Moradi, 2011); therefore, we can say that if equations (48) and (49) hold, then $T<0$ for case 1.

Note that the maximum value that j_1 could get is 90°, meaning that $\tilde{r}\leq 1$. Equation (49) shows that if $\tilde{r}'\leq 1$ then for case 1 we have

$$\frac{\tilde{r}_c}{\sqrt{2-\frac{Q_2}{Q_1}}}\leq 1 \quad\Rightarrow\quad \frac{Q_2}{Q_1}\leq 2-\tilde{r}_c^2 \quad\Rightarrow$$

$$1<\frac{Q_2}{Q_1}\leq 2-\tilde{r}_c^2 \tag{50}$$

meaning that if $\tilde{r}'\leq 1$, then T is negative for $\tilde{r}_c<\tilde{r}<\tilde{r}'$, and positive for $\tilde{r}'<\tilde{r}<1$. Hence, T becomes negative only for a specific range of supercritical incidence angles; but if $\tilde{r}'>1$ then

$$2-\tilde{r}_c^2<\frac{Q_2}{Q_1}<2 \tag{51}$$

Therefore, T will be negative for the entire supercritical zone.

After all, for case 1 we can say that if equation (50) holds, $T<0$ for $\tilde{r}_c<\tilde{r}<\tilde{r}'$ and if equation (51) holds, $T<0$ for entire supercritical zone.

These equations were examined numerically for the medium parameters in table 1 for SH wave. For the values in this table equation (48) becomes $1 < 1.33 < 1.75$ meaning that $T < 0$ for a specific range of supercritical incidence angles $30° < j_1 < 38°$ which is consistent with the results obtained by K&D. Figure 1 shows the corresponding energy flux based coefficients R, T and I, where T is negative for $30° < j_1 < 38°$.

(b) If $\left(2 - \frac{Q_2}{Q_1}\right) > 0$ then $\frac{Q_2}{Q_1} > 2$, therefore equation (46) states that a negative value is less than a positive value which is true. Also as discussed in case1 (a), since $\tilde{r}' > 1$, it can be inferred that for all supercritical angles $T < 0$.

(c) If $\left(2 - \frac{Q_2}{Q_1}\right) = 0$ then $\frac{Q_2}{Q_1} = 2$, consequently from equation (46) we get $\tilde{r}_c^2 > 0$ which is true and $\tilde{r}' > 1$; so $T < 0$ for all supercritical angles.

Case 2: As $Q_2 \leq Q_1$ then $0 < \frac{Q_2}{Q_1} < 1$ meaning that $\left(2 - \frac{Q_2}{Q_1}\right)$ is always positive and this is the only possibility in this case. For $Q_2 < Q_1$, $\tilde{r}_c < \sqrt{\frac{Q_1}{Q_2}}\tilde{r}_c < \tilde{r} \leq 1$, and for $Q_2 = Q_1$, $\tilde{r}_c < \tilde{r} \leq 1$. Then equation (46) becomes

$$\frac{\tilde{r}^2}{\tilde{r}_c^2} < \frac{1}{2 - \frac{Q_2}{Q_1}} \tag{52}$$

The right hand side is between 1/2 and 1 which means $\frac{\tilde{r}}{\tilde{r}_c} < 1$ that is not true because $\frac{\tilde{r}}{\tilde{r}_c}$ is always greater than 1; consequently T is always positive in this case. Figure 2 shows the calculated *R, T* and *I* coefficients using parameters in table 1, with the Q values switched i.e., for case 2. This figure confirms that T is positive for all supercritical incidence angles.

Cases 3 and **4:** in these cases $V_2 \leq V_1$, consequently $a \geq 0$, that could never happen if ERC is used; therefore, T is positive for all incidence angles and these cases could be eliminated as well. Figures 3 and 4 are examples for cases 3 and 4 respectively. In figure 3 the parameters in table 1 are used, but with the V values switched. In figure 4 the same table is used but with both the V values and the Q values switched. Both figures confirm that T has positive value for all supercritical angles.

RESULTS

After all, the conditions under which the SH viscoelastic energy-based transmission coefficient becomes negative when ERC is used, were obtained and could be summarized as follows:

Assuming $V_2 > V_1$ and $Q_2 > Q_1$ for an incident plane SH wave, if ERC is used, then

$$\text{If} \qquad 1 < \frac{Q_2}{Q_1} \le 2 - \frac{V_1^2}{V_2^2} \tag{53}$$

$$\text{then } T < 0 \text{ for } j_{c1} < j_1 < j_1{}' \tag{54}$$

$$\text{where } \sin j_{c1} = \frac{V_1}{V_2} \text{ and } \sin j_1{}' = \left(\frac{V_1}{V_2}\right) \Big/ \sqrt{2 - \frac{Q_2}{Q_1}} \tag{55}$$

$$\text{and if} \qquad \frac{Q_2}{Q_1} > 2 - \left(\frac{V_1}{V_2}\right)^2 \tag{56}$$

$$\text{then } T < 0 \text{ for } j_{c1} < j_1 < 90° \tag{57}$$

CONCLUSION

We have investigated the conditions under which the SH viscoelastic energy-based transmission coefficient becomes negative when ERC is used for choosing the sign of the vertical slowness. It is found that only for the case in which $V_2 > V_1$ and $Q_2 > Q_1$, if $\frac{Q_2}{Q_1} > 2 - \left(\frac{V_1}{V_2}\right)^2$, the SH-wave energy-based transmission coefficient becomes negative for all supercritical incidence angles; and if $1 < \frac{Q_2}{Q_1} \le 2 - \frac{V_1^2}{V_2^2}$, it becomes negative for a part of the supercritical incidence angles. Using these conditions we are able to predict negative *T* values, but the interpretation of these unphysical values remains uncertain.

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Table 1. Medium parameters used in the examples. ρ is the density, and V is phase velocity (Krebes and Daley, 2007)

Medium	$\boldsymbol{\rho}$ $\left[\frac{g}{cm^3}\right]$	$\boldsymbol{V_P}$ $\left[\frac{km}{s}\right]$	$\boldsymbol{V_S}$ $\left[\frac{km}{s}\right]$	$\boldsymbol{Q_P}$	$\boldsymbol{Q_S}$
1	2.1	2.5	1.0	25	15
2	2.2	5.0	2.0	40	20

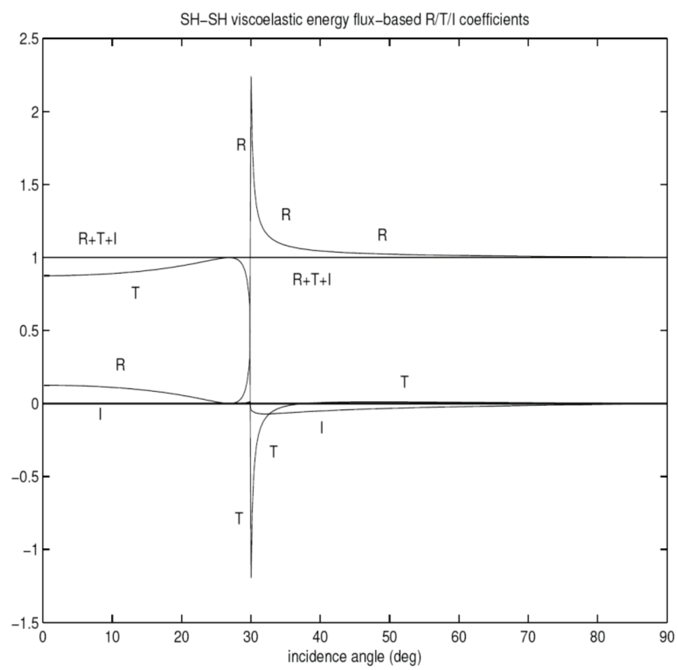

FIG. 1. SH viscoelastic energy based coefficients R, T and I for the model in table 1. Near the critical angle T takes on some unphysical negative values (from Krebes and Moradi, 2011)

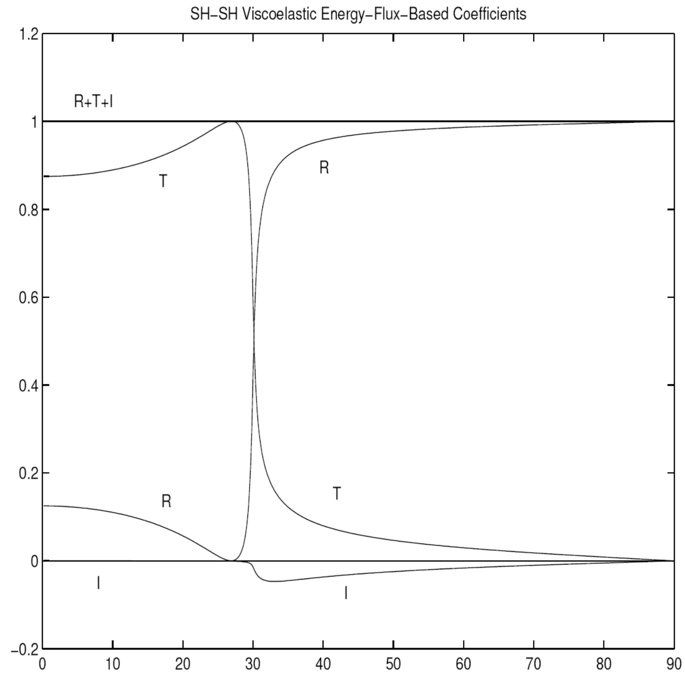

FIG. 2. The SH energy based coefficients R, T and I for the model in table 1, but with the Q-values switched. ERC is used to determine the sign of the vertical slowness component (from Krebes and Moradi, 2011).

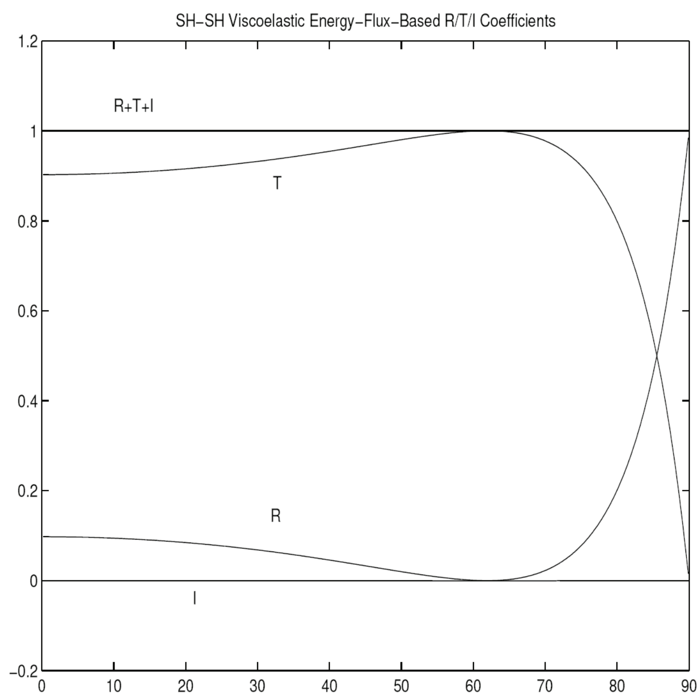

FIG. 3. The SH energy based coefficients R, T and I for the model in table 1, but with the V-values switched. ERC is used to determine the sign of the vertical slowness component (from Krebes and Moradi, 2011)

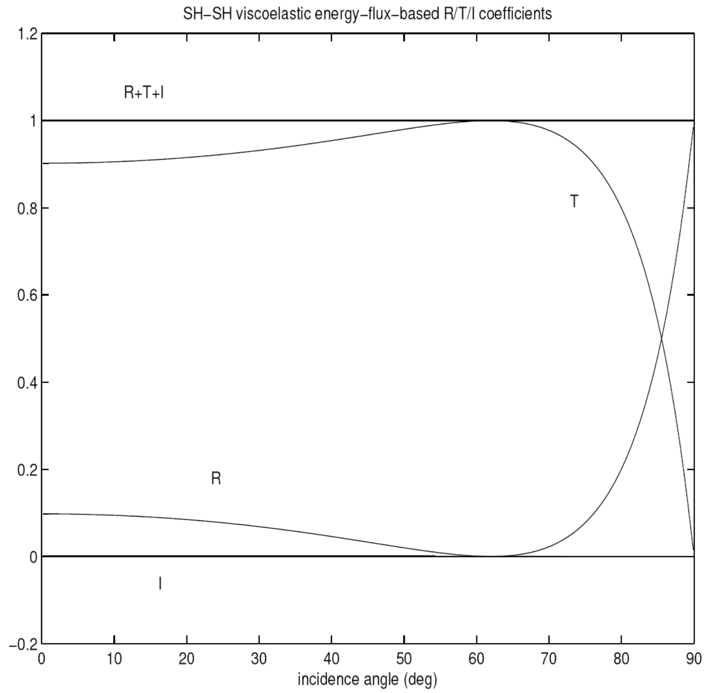

FIG. 4. The SH energy based coefficients R, T and I for the model in table 1, but with both the Q-values and V-values switched. ERC is used to determine the sign of the vertical slowness component (from Krebes and Moradi, 2011).