Reflection coefficients through a linear velocity ramp, in 1D

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ABSTRACT

A seismic wave propagating from one region of constant velocity to another, through a smooth transition zone, will differentially reflect or transmit across the zone, depending on the relative sizes of the transition zone and the wavelength of the propagating wave. This work presents an exact analytic solution for the case of a linear ramp velocity in the transition zone, and demonstrates that for long wavelengths, the ramp looks essentially like a jump discontinuity in the medium, with the corresponding reflection and transmission coefficients. For short wavelengths, the ramp provides essentially 100% transmission and no reflection. Energy conservation is verified for all wavelengths.

A careful consideration is given to the two cases of varying the velocity parameter, one via variations in the density of the propagation medium, the other in varying the modulus of elasticity. The results are different, in particular there is a sign difference in the reflection coefficient, and a large amplitude difference in the transmission coefficient.

We also present the numerical result for the transmission and reflection of a delta spike through the velocity ramp, and observe the reflection is a broadened "boxcar" response, while the transmission results in a spike.

INTRODUCTION

For testing numerical procedures in mathematical modeling, it is useful to create some analytic solutions that are exact answers for specific physical models, which can be used as an direct comparison with the numerical results obtained by approximate procedures. Of course, in seismic imaging, we are intensely interested in the numerical solution of the wave equation representing the propagation of seismic energy through complex geological structures. Exact analytic solutions are hard to come by in such situations, so simpler models will have to suffice.

Exact solutions to the one dimension wave equation are often possible; these can be used to construct more general solutions in higher dimensions, but also have direct utility of their own. In this short paper, we construct an exact solution for a wave propagating through a linearly varying velocity field, and use it to answer specific physical questions.

Three questions are resolved. First of all, the reflection and transmission coefficients of a monochromatic wave impinging on the interface should depend on frequency: we compute an exact answer. We verify that in the limiting case of large wavelengths, the ramp looks effectively like a jump discontinuity, with the corresponding reflection and transmission coefficients for this discontinuous case. For very short wavelengths, we verify that there is effectively 100% transmission of energy, and zero reflection.

Second, we examine what is the effect of varying the density parameter in the wave equation, versus varying the modulus of elasticity. The solutions are different, and can be seen in the computation of the reflection and transmission coefficients. In both cases, a conservation of energy law is observed across the interface.

Finally, we show the result of propagating a pulse through the velocity ramp. Curiously, the transmitted signal is still a pulse, while the reflected signal broadens into a boxcar.

VELOCITY MODEL AND LOCAL SOLUTIONS

The one dimensional "elastic" wave equation for a displacement field $u(x,t)$, of a disturbance travelling along a weighted string under tension, with density ρ and modulus of elasticity (bulk modulus) K is given in the standard form

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right). \tag{1}$$

We use this form as the 1D model, since this is the model directly applicable to elastic waves in three dimensions. For clarity, though, we note that the acoustic wave equation appears in a similar form,

$$\frac{1}{K(x)}\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{1}{\rho(x)}\frac{\partial u}{\partial x}\right). \tag{2}$$

where here $u(x,t)$ represents acoustic pressure, ρ is density of the medium, and $K(x)$ comes from the equation of state. In the case of constant coefficients, these two equations are equivalent.

It is important to note that in the elastic equation (1), it is the modulus of elasticity that appears inside the derivative, while in the acoustic equation (2), it is the density that appears inside the derivative. For non-constant ρ, K, the position of these coefficients within the derivative has important physical consequences – in particular on the sign of reflection coefficients. However, throughout this article, we will only work with equation (1). The ratio $K/\rho = c^2$ is identified as the velocity of propagation, squared.

We model a situation where a wave propagates from a region of constant velocity to another constant region, through a smooth transition zone. For simplicity, we select a linear velocity ramp for which exact analytic solutions can be obtained. As we are interested only in general behaviour, we renormalize to convenient units and select a continuous velocity field, constant $c = 1$ on the left ($x < 1$), constant $c = 2$ on the right ($x > 2$), with a linear ramp $c(x) = x$ in between, as shown in Figure 1.

Setting density $\rho(x) = c^{-2}(x)$ and constant modulus $K = 1$, we obtain a first case of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2}, \tag{3}$$

solved via separation of variables, with $u(x,t) = X(x)T(t)$ to obtain basic oscillatory solutions

$$\begin{aligned} u(x,t) &= e^{i\omega(\pm x - t)} && x < 1, && (4)\\ &= x^{1/2\pm\sqrt{1/4-\omega^2}}e^{-i\omega t} && 1 < x < 2, && (5)\\ &= e^{i\omega(\pm x/2 - t)} && x > 2. && (6) \end{aligned}$$

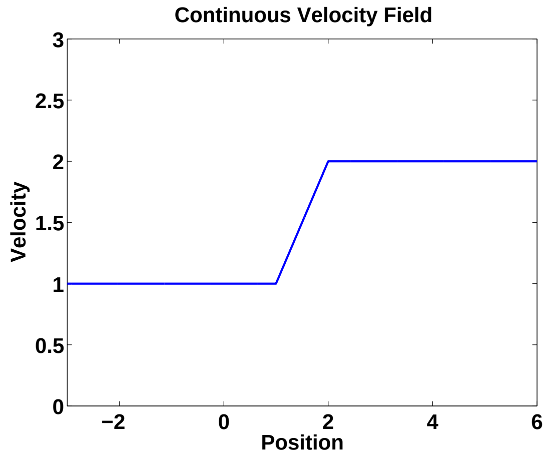

FIG. 1. Velocity field $c = 1$ to $c = 2$, with a linear ramp in-between.

It is convenient to express the middle solution in the form

$$u(x, t) = x^n e^{-i\omega t}, \tag{7}$$

for exponents $n = 1/2 \pm \sqrt{1/4 - \omega^2}$. We call this the varying density case. Figure 2 shows a typical wave in the transition zone. Notice that from left to right, the spatial wavelength is increasing, as is the amplitude.

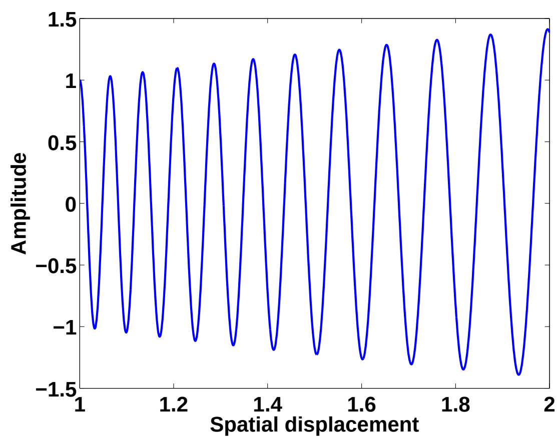

FIG. 2. Waveform in the transition zone, varying density case.

A second solvable case of the wave equation is obtained by taking density $\rho = 1$ constant, and varying modulus in the form $K(x) = c^2(x)$, yielding the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right) \tag{8}$$

which has solutions of a similar form,

$$\begin{aligned} u(x,t) &= e^{i\omega(\pm x - t)} && x < 1, && (9)\\ &= x^n e^{-i\omega t} && 1 < x < 2, && (10)\\ &= e^{i\omega(\pm x/2 - t)} && x > 2. && (11) \end{aligned}$$

with the change that here, the exponents are $n = -1/2 \pm \sqrt{1/4 - \omega^2}$. We call this the varying modulus case. Figure 3 shows a typical wave in the transition zone. Notice that from left to right, the spatial wavelength is again increasing, but the amplitude is now decreasing, in contrast to the varying density case.

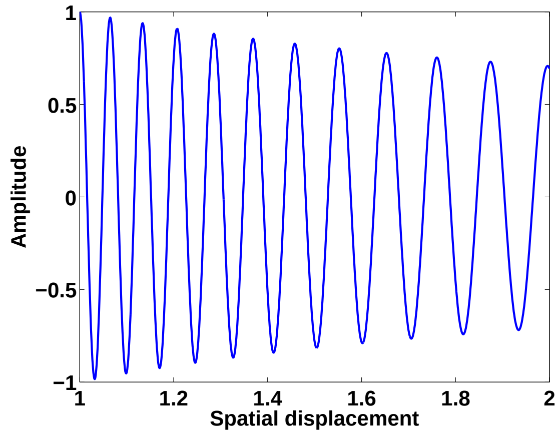

FIG. 3. Waveform in the transition zone, varying modulus case.

REFLECTION AND TRANSMISSION COEFFICIENT - VELOCITY JUMP

As an illustrative example, we consider the case of a reflection and transmission through a simple velocity jump, of the form

$$c(x) = 1 \qquad x < 0, \tag{12}$$
$$= 2 \qquad x > 0. \tag{13}$$

Solutions are of the form

$$u(x,t) = e^{i\omega(\pm x - t)} \qquad x < 0, \tag{14}$$
$$= e^{i\omega(\pm x/2 - t)} \qquad x > 0. \tag{15}$$

To compute reflection and transmission coefficients, set an incoming wave on the left of the form $e^{i\omega(x-t)}$, and hypothesize a reflected wave of the form $Re^{i\omega(-x-t)}$ and transmitted wave on the right of the form $Te^{i\omega(x/2-t)}$.

The initial wave and the reflected wave on the left sum together to give

$$u_{left} = e^{i\omega(x-t)} + Re^{i\omega(-x-t)} \tag{16}$$

The transmitted wave stands alone, giving

$$u_{right} = Te^{i\omega(x/2-t)}. \tag{17}$$

To obtain the coefficients R, T, we impose continuity on displacement u across the interface $x = 0$,

$$u_{left} = u_{right} \text{ at } x = 0 \tag{18}$$

and continuity of force

$$K_{left}(\partial_x u)_{left} = K_{right}(\partial_x u)_{right} \text{ at } x = 0, \tag{19}$$

yielding

$$\begin{aligned} 1 + R &= T & (20) \\ K_{left}(i\omega - i\omega R) &= \frac{K_{right}}{2} i\omega T. & (21) \end{aligned}$$

There are two cases to consider. First, when the modulus K is constant, and $\omega \neq 0$, the system reduces to

$$\begin{aligned} 1 + R &= T & (22) \\ 1 - R &= \frac{1}{2}T, & (23) \end{aligned}$$

which has solution $R = 1/3, T = 4/3$.

In the second case, with varying modulus ($K_{left} = 1, K_{right} = 4$), the system reduces to

$$\begin{aligned} 1 + R &= T & (24) \\ 1 - R &= 2T, & (25) \end{aligned}$$

which has solution $R = -1/3, T = 2/3$. The negative reflection coefficient indicates that there is a flip in polarity upon reflection.

Observe that in both cases, we have a conservation of energy constraint, namely

$$R^2 + \frac{K_{right}}{c_{right}} T^2 = 1. \tag{26}$$

REFLECTION AND TRANSMISSION COEFFICIENTS - VELOCITY RAMP, VARYING DENSITY

To compute reflection and transmission coefficients across a velocity ramp, set an incoming wave on the left of the form $e^{i\omega(x-t)}$, and hypothesize a reflected wave of the form $Re^{i\omega(-x-t)}$ and transmitted wave on the right of the form $Te^{i\omega(x/2-t)}$. In the transition region, set the wave to a linear combination of the solutions $x^{1/2\pm\sqrt{1/4-\omega^2}} e^{-i\omega t}$. This gives three regional solutions,

$$\begin{aligned} u_{left} &= e^{i\omega(x-t)} + Re^{i\omega(-x-t)}, & (27) \\ u_{trans} &= Ax^{n1}e^{-i\omega t} + Bx^{n2}e^{-i\omega t}, & (28) \\ u_{right} &= Te^{i\omega(x/2-t)}, & (29) \end{aligned}$$

with $n1 = 1/2 + \sqrt{1/4 - \omega^2}, n2 = 1/2 - \sqrt{1/4 - \omega^2}$. This corresponds to the varying density case identified in Section 2.

To find the four coefficients R, T, A, B, set four continuity conditions

$$\begin{aligned} u_{left} &= u_{trans} \text{ at } x = 1, & (30) \\ u_{trans} &= u_{right} \text{ at } x = 2, & (31) \\ K_{left}\left(\partial_x u_{left}\right) &= K_{trans}\left(\partial_x u_{trans}\right) \text{ at } x = 1, & (32) \\ K_{trans}\left(\partial_x u_{trans}\right) &= K_{right}\left(\partial_x u_{right}\right) \text{ at } x = 2. & (33) \end{aligned}$$

This yields the four equations

$$\begin{aligned} e^{i\omega} + Re^{-i\omega} &= A + B, & (34) \\ A2^{n1} + B2^{n2} &= Te^{i\omega}, & (35) \\ i\omega e^{i\omega} - i\omega Re^{-i\omega} &= An1 + Bn2, & (36) \\ n1A2^{n1-1} + n2B2^{n2-1} &= T(i\omega/2)e^{i\omega}. & (37) \end{aligned}$$

In matrix form, these equations are

$$\begin{bmatrix} 1 & 1 & -e^{-i\omega} & 0 \\ 2^{n1} & 2^{n2} & 0 & -e^{i\omega} \\ n1 & n2 & i\omega e^{-i\omega} & 0 \\ n1 \cdot 2^{n1-1} & n2 \cdot 2^{n2-1} & 0 & -0.5i\omega e^{i\omega} \end{bmatrix} \begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} e^{i\omega} \\ 0 \\ i\omega e^{i\omega} \\ 0 \end{bmatrix}. \quad (38)$$

Invert the matrix to obtain the solution, and note that the coefficients A, B, R, T depend on frequency ω:

$$\begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \mathbf{M}^{-1} \begin{bmatrix} e^{i\omega} \\ 0 \\ i\omega e^{i\omega} \\ 0 \end{bmatrix}. \quad (39)$$

Figure 4 shows a plot of the reflection and transmission coefficients, as well as a check on conservation of energy, verifying that

$$|R(\omega)|^2 + \frac{1}{2}|T(\omega)|^2 = 1. \quad (40)$$

Observe that for frequency $\omega \approx 0$, we obtain

$$|R(\omega)| \approx 1/3, \qquad |T(\omega)| \approx 4/3, \quad (41)$$

which agrees with the solution for a simple jump, as discussed in the last section. Physically, this says for long wavelength inputs, the ramped velocity change looks a lot like a simple jump.

We also observe that for frequency ω large, we obtain

$$|R(\omega)| \approx 0, \qquad \frac{1}{2}|T(\omega)| \approx 1, \quad (42)$$

which says that there is very little energy in the reflected signal, and lots of energy in the transmitted signal. That is to say, short wavelength signals pass through the ramp with very little change in energy, although the amplitude changes by a factor of $\sqrt{2}$.

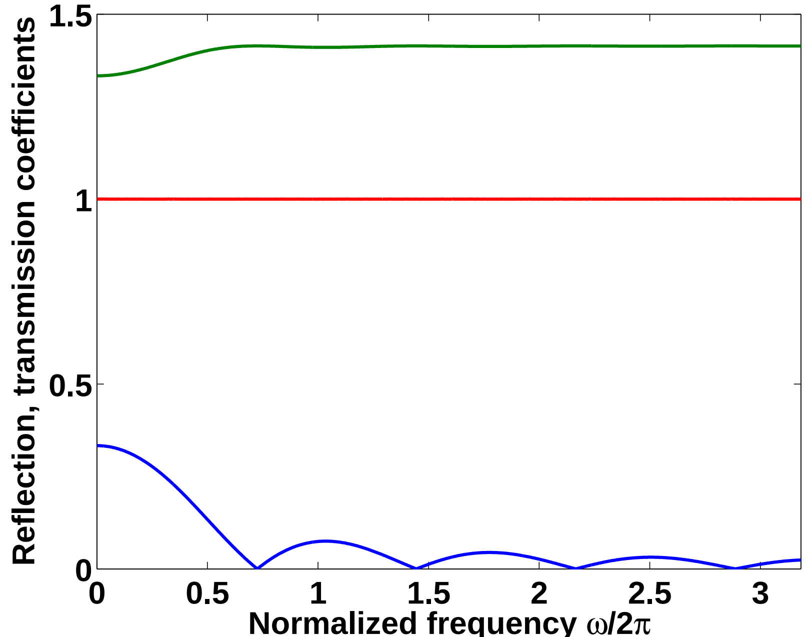

FIG. 4. Frequency dependent coefficients: reflection (lower), transmission (upper), and energy check (middle).

It is also interesting to note that for certain wavelengths, there is no reflection at all – and thus full transmission. This is a resonance phenomena due to the relative lengths of the waves, and the transition zone.

REFLECTION AND TRANSMISSION COEFFICIENTS - VELOCITY RAMP, VARYING MODULUS

We can solve for reflection and transmission coefficients for the second variant of the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right), \tag{43}$$

using the same system of equations, except change coefficients $n1, n2$ to

$$n1 = -1/2 + \sqrt{1/4 - \omega^2}, \qquad n1 = -1/2 - \sqrt{1/4 - \omega^2}. \tag{44}$$

Notice that the modulus of elasticity $K(x) = c^2(x)$ is continuous across all interfaces, so it can be eliminated from the system of equations 30–33.

Figure 5 shows a plot of the reflection and transmission coefficients, as well as a check on the conservation of energy, with

$$|R(\omega)|^2 + 2|T(\omega)|^2 = 1, \tag{45}$$

where the coefficient $2 = K/c$ comes from the right hand side of the wave field.

We observe the reflection coefficients look very similar to the previous case. The transmission coefficient differs by a factor of two.

In particular, for ω small, we obtain a value of $T(\omega) \approx 2/3$, which agrees with the jump case (Section titled "Velocity Jump") using a varying bulk modulus. For large ω, there is almost no reflected energy, and the transmitted signal carries almost all the energy, while its amplitude is reduced by a factor of $1/\sqrt{2}$.

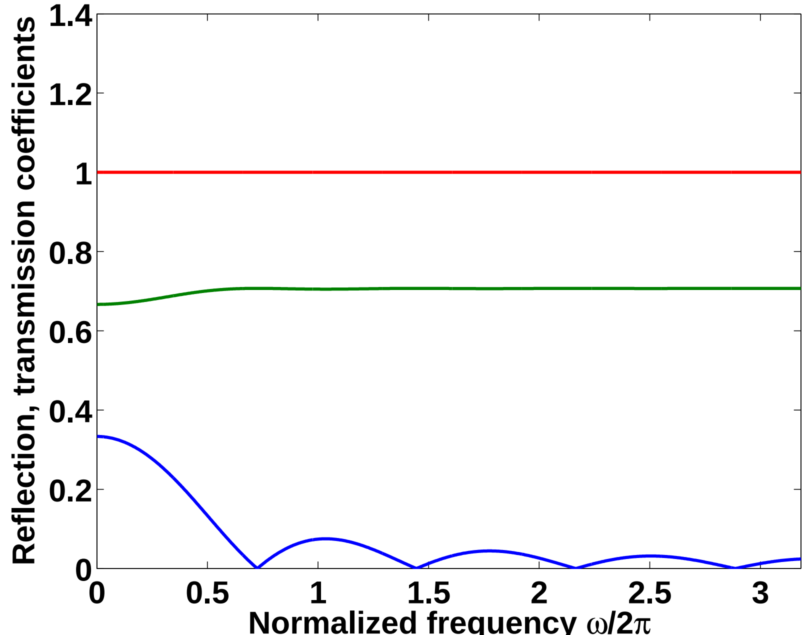

FIG. 5. Frequency dependent coefficients: reflection (lower), transmission (middle), and energy check (top).

COMPARISON OF REFLECTION COEFFICIENTS

The two versions of the wave equation seem to give similar reflection coefficients, at least when the absolute values are displayed (Figures 4 and 5). One can plot the difference of absolute values, and see they are identical up to machine error, as shown in Figure 6

This suggests the two results might be closely related, so some careful computations are in order. The next section verifies that the results are just the negative of each other.

ANALYTIC SOLUTION TO REFLECTION COEFFICIENT

Using Cramer's rule, and a little help from Mathematica™, one can obtain an exact solution for the reflection coefficient

$$R(\omega) = \frac{e^{2i\omega}(2^{n1} - 2^{n2})(n1 + n2)}{2^{n2}(2i\omega + 2\sqrt{1/4 - \omega^2}) + 2^{n1}(-2i\omega + 2\sqrt{1/4 - \omega^2})}. \tag{46}$$

In the varying density case, we have $n1 + n2 = 1$, and setting $a = \sqrt{1/4 - \omega^2}$, we obtain

$$R(\omega) = \frac{e^{2i\omega}(2^a - 2^{-a})}{2^{-a}(2i\omega + 2a) + 2^a(-2i\omega + 2a)}. \tag{47}$$

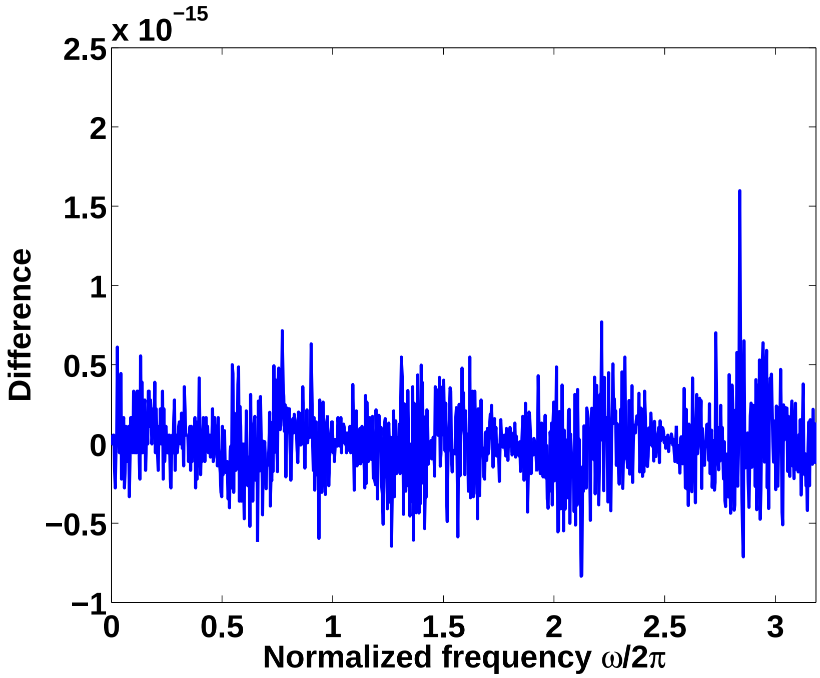

FIG. 6. Difference between the two reflection coefficients.

Note that $R(0) = 1/3$, so there is a positive reflection in this varying density case, at low frequencies.

In the varying modulus case, we have $n1 + n2 = -1$, and we get exactly the negative of this previous solution,

$$R(\omega) = -\frac{e^{2i\omega}(2^a - 2^{-a})}{2^{-a}(2i\omega + 2a) + 2^a(-2i\omega + 2a)}. \tag{48}$$

Here, we find $R(0) = -1/3$, so there is a negative reflection in this varying modulus case, at low frequencies.

A similar procedure will give the analytic solution for the transmission coefficients.

PHASE COEFFICIENTS, REAL VALUES

Keep in mind we have to be careful *where* we evaluate the phase, to give a useful answer. With a bit of experimentation, one can find that the phase alternates between plus and minus one, if we normalize into the form

$$R'(\omega) = R(\omega)e^{-2.695i\omega}. \tag{49}$$

That is, this adjusted function $R'(\omega)$ is real-valued, which is what one would expect for a physical reflection coefficient. The physical significance of this numerically obtained number -2.695 is probably related to the fact that it is very close to

$$2 + \log(2) = 2.6931 \tag{50}$$

which is very suggestive. Figure 7 demonstrates that this phase correction causes the sign of the normalized phase to flip from $+1$ to -1 exactly at the zero crossings of $R(\omega)$.

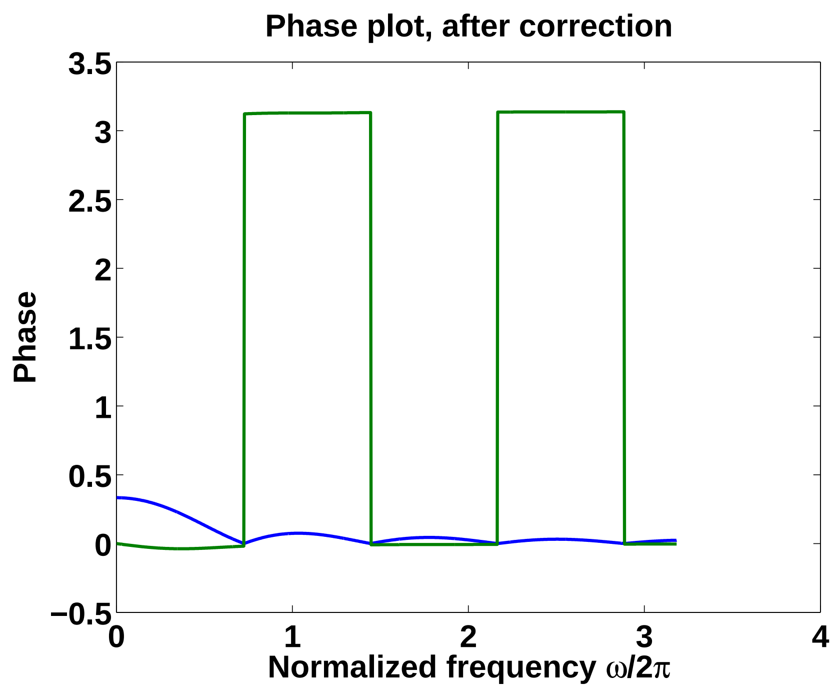

FIG. 7. Phase angle of the reflection coefficient (on top of abs coeff).

VARYING DENSITY AND BULK MODULUS

One can carry out the same analysis where both density and modulus are varying, in the form

$$\rho(x) = x^{\alpha-2} \tag{51}$$
$$K(x) = x^{\alpha} \tag{52}$$

for some fixed α, with x in the interval $[1, 2]$, and extend by continuity to constants on the rest of the real line. Solutions in the transition region are of the form

$$u(x, t) = x^n e^{-i\omega t}, \tag{53}$$

where $n = (1-\alpha)/2 \pm \sqrt{(1-\alpha)^2/4 - \omega^2}$. We skip the details – the same matrix inversion is required.

The case $\alpha = 1$ would be particularly interesting, for in this situation a monochromatic waveform in the transition zone would maintain a constant amplitude, in contrast to the increase (Figure 2) or decrease (Figure 3) in amplitude shown earlier. This interesting case is not discussed further here.

TRANSMISSION, REFLECTION OF A DELTA SPIKE

It is interesting to ask how an arbitrary waveform impinging on the velocity ramp is transformed into two resulting waveforms, the reflected waveform and the transmitted waveform. In principle, this can be computed by convolving the initial waveform with the filter response of the ramp, for both reflection and transmission.

The filter response is simply the result of initiating a delta spike on the left of the velocity ramp, and allowing it to travel into the ramp, creating a reflected and a transmitted waveform.

Conveniently, we have already computed the filter response for the ramp in the frequency domain, as Equations 46 and 47 represent analytic solution for reflection. Taking the inverse Fourier transform will give the filter response in space.

However, computing the exact inverse Fourier transform by hand is beyond what we are able to do in this short paper. As an alternative, we can simply sample in the frequency domain, then numerically compute the inverse Fast Fourier Transform. Figure 8 shows the result, which is the impulse response of the ramp in reflecting a delta spike. Observe that exactly one reflection is obtained, but it is much wider than a delta spike. Perhaps there is a physical explanation for this broadening of the waveform.

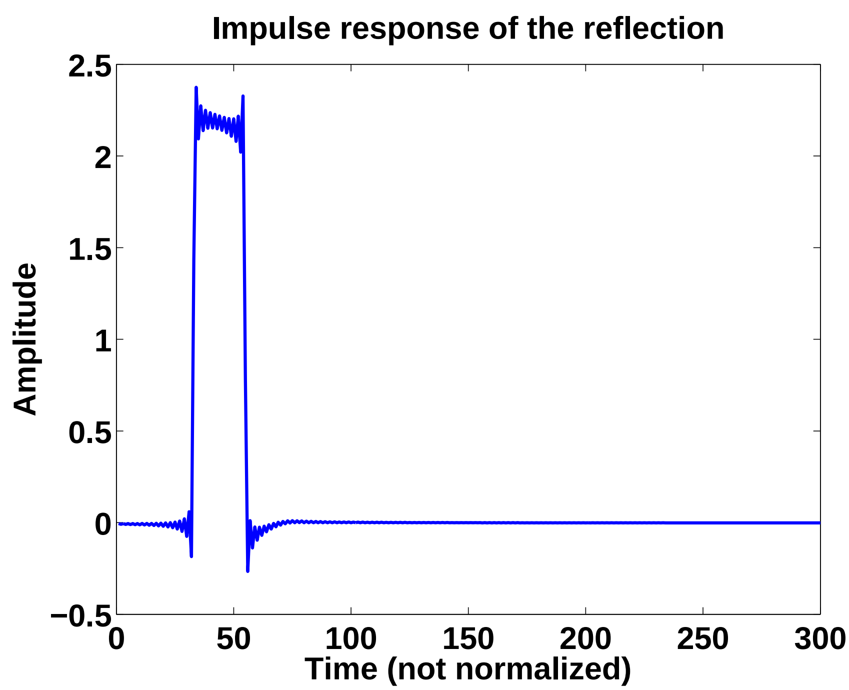

FIG. 8. Impulse response of a delta spike, reflectivity.

The same computation can be performed on the transmission coefficients, resulting in the transmitted waveform shown in Figure 9. Observe there is a much narrower spike in this transmitted case (same units in the two figures).

FUTURE WORK

It would be useful to set up physically relevant models, with realistic physical velocity, density, and modulus of elasticity, rather than the "dimension-free" units here. Some MATLAB™ code that allows for a selection of parameters would be a useful tool to investigate the frequency response in various physical settings. The mathematics would remain the same; however we could make useful observations on the size of a transition zone that would be observable for typical seismic frequencies in use.

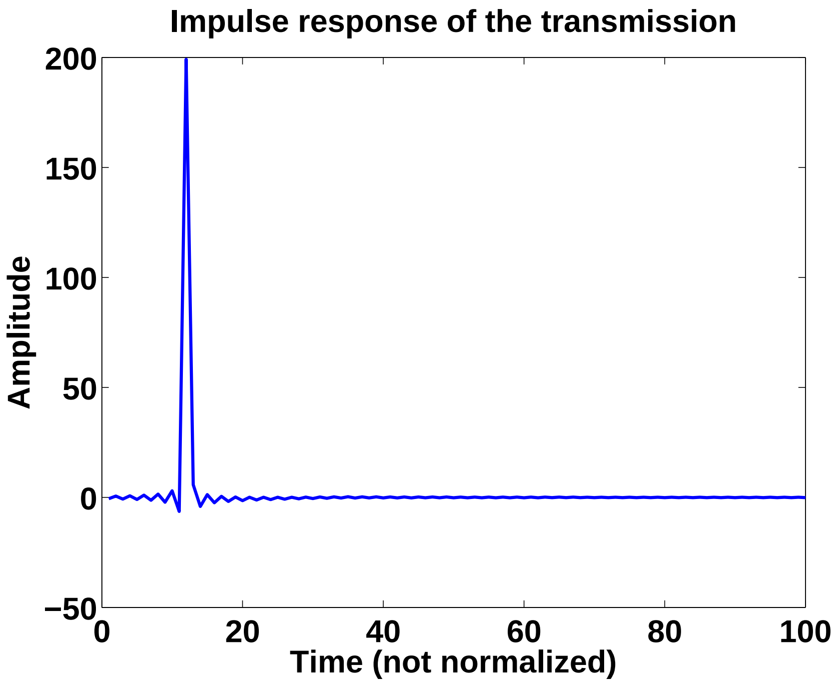

FIG. 9. Impulse response of a delta spike, transmission.

It would also be useful to extend this to 2D and 3D models, where the transition zone varies linearly in only one dimension.

CONCLUSIONS

We have demonstrated an exact analytic solution to the 1D wave equation for a physical medium with a linear velocity ramp sandwiched between two constant velocity regions. From this, we computed reflection and transmission coefficients for the ramp, both numerically and analytically, showing that these coefficients are frequency dependent. Asymptotically, they behave like the coefficients for a jump discontinuity in velocity. The results for varying density, and varying modulus of elasticity are different – reflecting the important physical observation that it is not enough to know only the velocity field. One must know the density and elasticity in the non-constant case. Finally, we numerically computed the transmitted and reflected waveforms that result when a delta spike impinges on the velocity ramp, observed that the reflected waveform is actually broadened into a boxcar.

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