Multicomponent elastic reflection full waveform inversion

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ABSTRACT

Full waveform inversion, as applied to reflection-mode, multicomponent land data is considered, in the context of the iterated recovery of parameters ranging from standard elastic λ, μ, and ρ through to petrophysically-relevant parameters such as the fluid term of Russell and Gray. Much of the effort lies in constructing a flexible framework for elastic multicomponent sensitivities. We approach the problem with a linearized scattering framework, employing a two-stage integration by parts procedure. Matrix forms for a multicomponent objective function, and a three-term gradient and nine-term Hessian are then determined, and the reductions necessary to invert PP, PS, SP, and SS modes independently or jointly are formed.

INTRODUCTION

CREWES is developing a type of full waveform inversion (FWI) we call IMMI, or *iterated modelling, migration and inversion* (Margrave et al., 2013; Pan et al., 2014), which appears to us to have the potential to determine subsurface elastic / geological properties from broadband, multicomponent land seismic surveys. The multicomponent aspect of the problem has some outstanding theoretical issues which need to be broached before we can tackle all of the practical implementation issues.

The precise parameterization of a full waveform inversion problem is a surprisingly subtle and important matter. This is already true in the scalar inverse problem, as was recently demonstrated by Anagaw (2014), in which significant differences in convergence between scalar FWI updates in velocity, slowness, and squared-slowness (with squared slowness appearing to win out) were reported. We must expect it to be doubly so for updates in multiple parameters, constructed with several components of elastic data. At present, we do not know how to predict in advance which of a candidate set of parameterizations will lead to the fastest convergence and the most useful model estimate. So, our approach is to frame FWI such that moving from one parameterization to another is as simple and seamless as possible.

The standard quantities of seismic full waveform inversion we require for this enterprise, namely the gradients, Hessian operators, sensitivity matrices and so on, have been discussed at great length in the geophysics literature (Lailly, 1983; Tarantola, 1984; Virieux and Operto, 2009), but although significant discussion can be found on the particulars of the multiparameter problem (Brossier et al., 2010; Plessix et al., 2013), the multicomponent nature of the general seismic inverse problem (Weglein et al., 2003; Zhang and Weglein, 2009), and applications involving multicomponent data (Operto et al., 2013), a flexible theory for land application, invoking in particular precritical reflection amplitudes, is noticeably lacking. In particular, an approach which is modular and straightforward to adapt viz. parameterization is sought. We might, for instance, find it important in one application to update in fluid-type parameters (Russell et al., 2011), while in another to update

V_P, V_S and ρ, and still another to update Lamé impedances $\lambda\rho$, $\mu\rho$ (Goodway, 2001). The history of AVO inversion and interpretation quite dramatically illustrates how varied the situationally-optimal seismic parameterization can be (Shuey, 1985; Castagna and Backus, 1993; Castagna et al., 1998; Russell et al., 2011; Lines et al., 2008; Innanen, 2011). Again, without being able to discern what if any single parameterization is optimal, our course is to remain as flexible as possible here.

We arrive at formulas for updating multiple elastic parameters using a range of multicomponent data modes. More importantly, in doing so we provide a definite procedure whereby updates in any desired parameterization can be formulated with any combination of PP, PS, SP and/or SS data.

Most of the legwork lies in the calculation of the sensitivities, which are the route by which elastic wave theory is introduced to the mathematics and computer science of FWI/IMMI. It is in the construction of sensitivities that most of the variability of re-parameterizing inversion, from, say, (γ, μ, ρ), to (V_P, V_S, ρ) is found. The sensitivities we will derive in this paper are entirely linearized (some of the wrinkles of moving to nonlinear updates are discussed in this year's report by Innanen, 2014a,b), and originate with the elastic Born approximation. We begin by reviewing a 2D P- and Sv-mode scattering formalism introduced by Weglein and Stolt in a set of unpublished notes in 1992, and recently extended to 3D P/Sv/Sh by Stolt and Weglein (2012).

Amongst the contributions of the Weglein-Stolt framework is the suggestion (and demonstration) that via integration by parts the Born integral can be transformed such that any differential operators within the scattering potential are moved to the Green's functions. This fact turns out to be crucial when we apply the Born formulation to the sensitivity calculation. We enact an altered two-stage version of it in our derivations. The first stage shifts the derivatives arising from the transformation from displacement to potential space. This occurs in all cases, independent of which sensitivity is being calculated at any later point. The second stage depends on the specific set of parameters being considered.

After the first stage of integration by parts, we can move to the calculation of the sensitivities. The parameters (γ, μ, ρ) are chosen as the basic set, and the procedure for jumping from (γ, μ, ρ) to any other set is detailed. Two classes of re-parameterization are identified, those involving addition and those involving multiplication of the elements of the basic set. The scattering potential operator is then written down in terms of perturbations in the chosen parameter set. The sensitivities are obtained by setting two of the three perturbations to zero, and sifting out the influence of the third at a fixed point in the Earth volume. This provides a 2×2 matrix of derivatives of wave components (PP, SP, PS, SS) calculated with respect to the parameter whose perturbation we kept.

Thereafter update formulas emerge pretty straightforwardly. We keep the gradient and Hessian operator constructions in a compact matrix form by using a quantity called the Frobenius product*. The Hessian is approximated such that the updates are of Gauss-

*It sounds worse than it is.

Newton type. Finally, we show how to restrict the updates to represent inversion of multicomponent data by the individual treatment of PP, PS, SP and/or SS modes, or by their simultaneous (joint) use.

The current developments are thus seen to be essentially generalizations of the multicomponent inversion methodologies which are longstanding CREWES deliverables. A key outcome of this paper is, thus, to demonstrate explicitly how in-place CREWES ideas and technology are a natural part of the CREWES IMMI / FWI package.

BACKGROUND

In this review section we discuss the 2D scattering formalism due to Weglein and Stolt (unpublished), which was described and extended significantly in a recent book (Stolt and Weglein, 2012). It has elsewhere been used as the basis for development of nonlinear inverse scattering series imaging and parameter estimation (Zhang and Weglein, 2009). Here we will use it as the starting point for multicomponent elastic sensitivity analysis.

2D elastic wave quantities

In a 2D elastic environment with one lateral coordinate x and one depth coordinate z, a displacement field $\mathbf{u} = (u_x, u_z)^T$ excited by the source $\mathbf{f} = (f_x, f_z)^T$ satisfies

$$\mathcal{L}\mathbf{u} = \mathbf{f}, \tag{1}$$

where

$$\mathcal{L} = \left[\rho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \partial_x\gamma\partial_x + \partial_z\mu\partial_z & \partial_x(\gamma - 2\mu)\partial_z + \partial_z\mu\partial_x \\ \partial_z(\gamma - 2\mu)\partial_x + \partial_x\mu\partial_z & \partial_z\gamma\partial_z + \partial_x\mu\partial_x \end{pmatrix}\right]. \tag{2}$$

Here $\rho = \rho(x, z)$ is the mass density, and $\gamma = \gamma(x, z)$ and $\mu = \mu(x, z)$ are the bulk and shear moduli respectively. The same force exciting a reference medium produces a reference field, $\mathbf{u}_0$, which satisfies $\mathcal{L}_0\mathbf{u}_0 = \mathbf{f}$, where

$$\mathcal{L}_0 = \left[\rho_0\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \partial_x\gamma_0\partial_x + \partial_z\mu_0\partial_z & \partial_x(\gamma_0 - 2\mu_0)\partial_z + \partial_z\mu_0\partial_x \\ \partial_z(\gamma_0 - 2\mu_0)\partial_x + \partial_x\mu_0\partial_z & \partial_z\gamma_0\partial_z + \partial_x\mu_0\partial_x \end{pmatrix}\right]$$

is the elastic wave operator characterized by the reference properties $\rho_0 = \rho_0(x, z)$, $\gamma_0 = \gamma_0(x, z)$ and $\mu_0 = \mu_0(x, z)$. The solutions $\mathbf{u}$ and $\mathbf{u}_0$ are obtained with the resolvents or Green's operators $\mathcal{G}$ and $\mathcal{G}_0$,

$$\mathcal{G} = \mathcal{L}^{-1} \;\text{ and }\; \mathcal{G}_0 = \mathcal{L}_0^{-1}, \tag{3}$$

whereby

$$\mathbf{u} = \mathcal{G}\mathbf{f} \;\text{ and }\; \mathbf{u}_0 = \mathcal{G}_0\mathbf{f}, \tag{4}$$

are determined.

P- and S-wave potential representation

Next, the elastic motions are transformed to the space of P- and S-wave potentials. Stolt and Weglein (2012) make use of the operator Π, which in 2D is

$$\Pi = \begin{bmatrix} \partial_x & \partial_z \\ -\partial_z & \partial_x \end{bmatrix}, \tag{5}$$

noting that when it is applied to the displacement vector $\mathbf{u}$ with a scaling operator Γ_0 it correctly produces the potentials ϕ_{p} and ϕ_{s}:

$$\begin{bmatrix} \phi_{\mathrm{p}} \\ \phi_{\mathrm{s}} \end{bmatrix} = \Gamma_0 \Pi \mathbf{u}, \tag{6}$$

where

$$\Gamma_0 = \begin{bmatrix} \gamma_0 & 0 \\ 0 & \mu_0 \end{bmatrix}. \tag{7}$$

An operator $\mathcal{O}$ is transformed to the same space via

$$O = \Pi \mathcal{O} \Pi^{-1} \Gamma_0^{-1}, \tag{8}$$

where

$$\Pi^{-1} = \Pi^T \nabla^{-2}. \tag{9}$$

In 2D, the operator ∇^{-2} is the formal inverse of the operator $\nabla^2 = \partial_x^2 + \partial_z^2$.

Elastic scattering of P- and S-wave potentials

The scattering picture involves the definition of the scattering operator $\mathcal{V}$, defined as the difference between two wave operators, a perturbed operator $\mathcal{L}$ and an unperturbed operator $\mathcal{L}_0$:

$$\mathcal{V} = \mathcal{L} - \mathcal{L}_0, \tag{10}$$

which forms the basis for the Lippmann-Schwinger, or Scattering equation, a relationship between the perturbed and unperturbed Green's operators:

$$\delta\mathcal{G} = \mathcal{G} - \mathcal{G}_0 = \mathcal{G}_0 \mathcal{V} \mathcal{G}. \tag{11}$$

These relationships lead rapidly to relationships between perturbed and unperturbed displacement fields. Because $\mathbf{u} = \mathcal{G}\mathbf{f}$, we have

$$\delta\mathbf{u} = \mathcal{G}_0 \mathcal{V} \mathbf{u}. \tag{12}$$

Transforming these operator relationships using equation (8) we find an expression for the change in the field $\delta\mathbf{G}$ in the presence of a general change in the properties of the medium (through $\mathcal{V}$):

$$\delta\mathbf{G} = \mathbf{G}_0 \mathbf{V} \mathbf{G}. \tag{13}$$

Our purpose in this paper is to produce clear, explicit formulas for calculating. The results so far quoted are largely formal; in the next section we will consider the explicit form of $\mathcal{V}$ and its transformation to $\mathbf{V}$, which underlie the sensitivity calculations.

ELASTIC BORN APPROXIMATION

We now begin to manipulate the scattering quantities reviewed in the previous section, such that full waveform inversion sensitivities emerge from it as straightforwardly as possible. The key step will be to implement a two-stage integration by parts regimen, a variation on the approach Stolt and Weglein (2012) found convenient for analyzing the direct inverse scattering problem. In our case it permits the denominator of the ratio we need to calculate, $\delta G/\delta s$, to be cleanly extracted from the integral so that the limit $\delta s \to 0$ can be taken. The first stage of the integration by parts, which we carry out once, in this section, is general and applies to all sensitivities. The second stage depends on the parameter whose sensitivity is being calculated, and so it must be deployed case by case.

Let the data comprise measurements of elastic motions on a surface $z = z_g$ of line receivers occupying positions x_g along the horizontal coordinate direction. Further, let these motions be due to line sources on a surface $z = z_s$ occupying positions x_s horizontally. These measured fields are Fourier-transformed with respect to time, x_g, and x_s, such that the results are functions of the respective conjugate variables ω, k_g and k_s. Under the Born approximation, equation (13),whose rightmost term becomes $\mathbf{G}_0$,

$$\delta\mathbf{G} \approx \mathbf{G}_0\mathbf{V}\mathbf{G}_0, \tag{14}$$

represents the difference between two such fields when the medium undergoes a small change $\mathbf{V}$. Writing this expression explicitly, and taking it to be a definition[†] and thus omitting the $\approx$ sign, and taking the dependence of all wave quantities on ω as implied, we have

$$\delta\mathbf{G}(k_g, z_g, k_s, z_s) = \int dx' \int dz' \mathbf{G}_0^{\text{out}}(k_g, z_g, x', z')\mathbf{V}(x', z')\mathbf{G}_0^{\text{in}}(x', z', k_s, z_s), \tag{15}$$

where

$$\delta\mathbf{G}(k_g, z_g, k_s, z_s) = \begin{bmatrix} \delta G_{\text{PP}}(k_g, z_g, k_s, z_s) & \delta G_{\text{PS}}(k_g, z_g, k_s, z_s) \\ \delta G_{\text{SP}}(k_g, z_g, k_s, z_s) & \delta G_{\text{SS}}(k_g, z_g, k_s, z_s) \end{bmatrix} \tag{16}$$

is a 2×2 matrix containing the first order changes in the fields for wave components scattering from P to P (top left), S to P (top right), P to S (bottom left), and S to S (bottom right)[‡] associated with the changes in the medium contained in $\mathbf{V}(x, z)$. If the reference (unperturbed) medium varies sufficiently slowly that P- and S-waves propagate approximately

[†] It is important to emphasize that although we adopt the Born approximation as a definition of the forward model in the sensitivity calculation, the resulting FWI updates are *not* consequently iterated linear inverse procedures. The first-order Fréchet kernel as used in FWI is built based on a linearization, but the forward modelling regimen by which the residuals are calculated is normally fully nonlinear, involving 2-way wave equations etc.

[‡] In scattering theory wave interactions occurring in sequence along the path between the source and the receiver appear in the mathematics going from right to left. Thus a C-wave converting from P to S is labelled "SP". Care should be taken that this backwards labelling does not cause confusion.

independently, the Green's operators $\mathbf{G}_0^{\text{out}}$ and $\mathbf{G}_0^{\text{in}}$ can be written

$$
\begin{aligned}
\mathbf{G}_0^{\text{out}}(k_g, z_g, x', z') &= \begin{bmatrix} G_{\text{P}_0}(k_g, z_g, x', z') & 0 \\ 0 & G_{\text{S}_0}(k_g, z_g, x', z') \end{bmatrix} \\
\mathbf{G}_0^{\text{in}}(x', z', k_s, z_s) &= \begin{bmatrix} G'_{\text{P}_0}(x', z', k_s, z_s) & 0 \\ 0 & G'_{\text{S}_0}(x', z', k_s, z_s) \end{bmatrix},
\end{aligned}
\tag{17}
$$

where $G'_{\text{P}_0} = -G_{\text{P}_0} V_{P_0}^2/\omega^2\gamma_0$ and $G'_{\text{S}_0} = -G_{\text{S}_0} V_{S_0}^2/\omega^2\mu_0$, and G_{P_0} and G_{S_0} are Green's functions which away from the source and in the space domain satisfy

$$
\left[\nabla^2 + \omega^2 V_{P_0}^{-2}(x,z)\right] G_{\text{P}_0}(x, z, x_s, z_s, \omega) = 0 \tag{18}
$$

and

$$
\left[\nabla^2 + \omega^2 V_{S_0}^{-2}(x,z)\right] G_{\text{S}_0}(x, z, x_s, z_s, \omega) = 0 \tag{19}
$$

respectively. Most importantly for our purposes is the transformed scattering potential

$$
\mathbf{V}(x', z') = \Pi \mathcal{V}(x', z') \Pi^T. \tag{20}
$$

The extra Γ_0^{-1} in equation (8) has been incorporated into $\mathbf{G}_0^{\text{in}}$ in equation (17), through the primed functions G'_{P_0} and G'_{S_0}. In the following developments we will spend most of our time manipulating the elements of the perturbation operator in displacement space, $\mathcal{V}$, because the parameters we would like to use are more conveniently expressed in that domain, at least initially. However we do want the end result to be in the P- and S-wave potential domain, as that domain is where our multicomponent ideas are most clearly exposed. To manage this, we analyze the entire operator $\Pi\mathcal{V}(x', z')\Pi^T$, but we move the Π operators off the perturbation $\mathcal{V}$ and onto the Green's functions on the left and the right of the Born integral.

Integration by parts

It is not difficult to show by construction that, provided there are no contributions to the scattering integral at infinity, the replacement

$$
\begin{aligned}
\delta\mathbf{G} &= \int dx' \int dz' \mathbf{G}_0^{\text{out}} \left[\Pi\mathcal{V}(x', z')\Pi^T\right] \mathbf{G}_0^{\text{in}} \\
&= -\int dx' \int dz' \left(\Pi^T \mathbf{G}_0^{\text{out}}\right)^T \mathcal{V}(x', z') \left(\Pi^T \mathbf{G}_0^{\text{in}}\right)
\end{aligned}
\tag{21}
$$

can be made. Then the derivatives in the Π operators act on the Green's operators rather than on the perturbations, freeing the latter up to be conveniently manipulated in the sensitivity calculation. Meanwhile the two effective Green's operators on the left and right side of the perturbation operator are

$$
\left(\Pi^T \mathbf{G}_0^{\text{out}}\right)^T = \begin{bmatrix} \partial_{x'} G_{\text{P}_0} & \partial_{z'} G_{\text{P}_0} \\ -\partial_{z'} G_{\text{S}_0} & \partial_{x'} G_{\text{S}_0} \end{bmatrix}, \tag{22}
$$

and

$$(\Pi^T \mathbf{G}_0^{\text{in}}) = \begin{bmatrix} \partial_{x'} G'_{\text{P}_0} & -\partial_{z'} G'_{\text{S}_0} \\ \partial_{z'} G'_{\text{P}_0} & \partial_{x'} G'_{\text{S}_0} \end{bmatrix}. \tag{23}$$

A further step we could now take would be to replace these derivatives with products of the individual (P- and S-) components of the Green's operator and horizontal and vertical wavenumbers. Since we have already assumed the media are smooth enough that we can approximate the elastic field as the independent propagation of P- and S-wave potentials, no generality would be lost in doing so. Stolt and Weglein (2012) take that approach. However, since the eventual use of the quantities we are constructing (i.e., practical and numerical implementation of FWI) is different from that conceived of by Stolt and Weglein (i.e., analysis and implementation in a plane-wave environment or within a ray-based coordinate system), we will leave the derivative forms in place. In numerical schemes we anticipate that these will be easier to calculate than local wavenumber vectors.

ELASTIC PARAMETERIZATIONS OF FWI UPDATES

We are now in a position to decide on the parameterization of interest (Figure 1). Each set of three parameters we could choose leads to a different sensitivity, and we will not calculate all of them straight through. Rather, we will show a general approach and exemplify it with a few cases. We begin with γ, μ and ρ (i.e., P-wave modulus, S-wave modulus and density) having been chosen as the basic set, from which all others are derived.

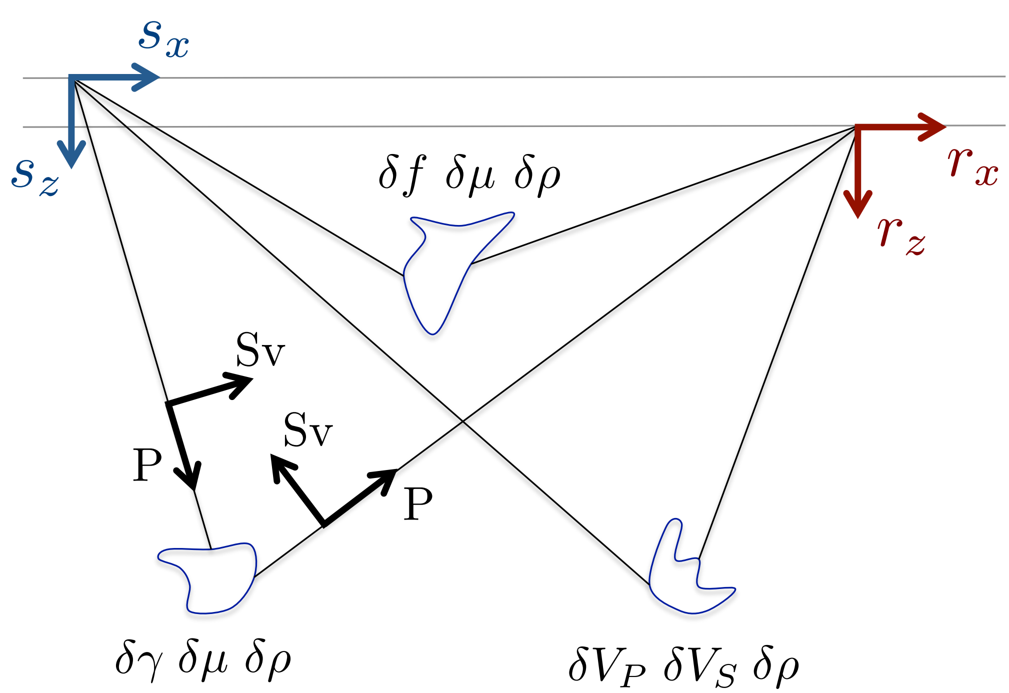

FIG. 1. 2D elastic/multicomponent inversion and sensitivity analysis for a variety of parameterizations: moduli $\gamma\ \mu\ \rho$, velocities $V_P\ V_S\ \rho$, or the fluid Russell-Gray form $f\ \mu\ \rho$ are examples.

Basic parameterization: γ, μ, ρ updates

The elastic equation of motion was written in equation (2) in terms of the γ, μ, ρ parameterization. Choosing these as the FWI model parameters, i.e.,

$$\begin{aligned} s_\gamma = \gamma, \quad s_{\gamma 0} = \gamma_0, \quad \delta s_\gamma = s_\gamma - s_{\gamma 0} \\ s_\mu = \mu, \quad s_{\mu 0} = \mu_0, \quad \delta s_\mu = s_\mu - s_{\mu 0} \\ s_\rho = \rho, \quad s_{\rho 0} = \rho_0, \quad \delta s_\rho = s_\rho - s_{\rho 0}, \end{aligned} \tag{24}$$

the scattering operator in displacement space $\mathcal{V}$ defined in equation (10) is found to be

$$\mathcal{V}_{\gamma\mu\rho} = \left[\begin{array}{cc} \omega^2 \delta s_\rho + \partial_{x'} \delta s_\gamma \partial_{x'} + \partial_{z'} \delta s_\mu \partial_{z'} & \partial_{x'} [\delta s_\gamma - 2\delta s_\mu] \partial_{z'} + \partial_{z'} \delta s_\mu \partial_{x'} \\ \partial_{z'} [\delta s_\gamma - 2\delta s_\mu] \partial_{x'} + \partial_{x'} \delta s_\mu \partial_{z'} & \omega^2 \delta s_\rho + \partial_{z'} \delta s_\gamma \partial_{z'} + \partial_{x'} \delta s_\mu \partial_{x'} \end{array} \right]. \tag{25}$$

Each of the perturbations δs_γ, δs_μ, and δs_ρ are general functions of space, but for neatness we will not always explicitly put the dependence into the formulas. $\mathcal{V}_{\gamma\mu\rho}$ can be substituted into equation (21) and analyzed for sensitivities in each of the three parameters in the subscript.

If a different parameterization is desired, the transformation should happen now, to equation (25). Let us consider some of the possible transformations we might consider employing, distinguishing between ones involving addition and ones involving multiplication of the base set.

Re-parameterization involving addition

If the transformation between parameters involves only addition, then the transformation should be done using the “bare” perturbations δs_X.

Example 1: λ, μ, ρ updates

The most straightforward mapping is from γ, μ, ρ to λ, μ ρ. Take the perturbation operator in equation (25) and replace the three update functions with

$$\begin{aligned} \delta s_\gamma &\rightarrow \delta s_\lambda + 2\delta s_\mu \\ \delta s_\mu &\rightarrow \delta s_\mu \\ \delta s_\rho &\rightarrow \delta s_\rho. \end{aligned} \tag{26}$$

These will correctly lead to updates in the parameter set $s_\lambda = \lambda$, $s_\mu = \mu$ and $s_\rho = \rho$.

Example 2: κ, μ, ρ updates

Similarly, to map from γ, μ, ρ to κ, μ, ρ, replace the steps in equation (25) as follows:

$$\begin{aligned} \delta s_\gamma &\rightarrow \delta s_\kappa + (4/3)\delta s_\mu \\ \delta s_\mu &\rightarrow \delta s_\mu \\ \delta s_\rho &\rightarrow \delta s_\rho. \end{aligned} \tag{27}$$

These will correctly lead to updates in the parameter set $s_\kappa = \kappa$, $s_\mu = \mu$ and $s_\rho = \rho$.

Re-parameterization involving multiplication

If the transformation from γ, μ, ρ to the desired parameter set involves multiplication, we proceed not with the bare updates δs_X, but instead using the dimensionless perturbations $(\delta s_X / s_{X_0})$. Thus, the transformations will involve background model parameters s_{X_0}, both in the base set and in the new set, sometimes alone and sometimes in ratios with other background parameters.

Example 3: V_P, V_S, ρ updates

To map from γ, μ, ρ to V_P, V_S and ρ, replace the steps in equation (25) as follows:

$$\begin{aligned}
\delta s_\gamma &\rightarrow \left(\frac{s_{\gamma 0}}{s_{\mathrm{p}_0}}\right)\delta s_{\mathrm{p}} + \left(\frac{s_{\gamma 0}}{s_{\rho 0}}\right)\delta s_\rho \\
\delta s_\mu &\rightarrow \left(\frac{s_{\mu 0}}{s_{\mathrm{s}_0}}\right)\delta s_{\mathrm{s}} + \left(\frac{s_{\mu 0}}{s_{\rho 0}}\right)\delta s_\rho \\
\delta s_\rho &\rightarrow \delta s_\rho .
\end{aligned} \tag{28}$$

These steps will correctly update the parameter set $s_{\mathrm{p}} = V_P^2$, $s_{\mathrm{s}} = V_S^2$ and $s_\rho = \rho$.

Example 4: Goodway updates $\lambda\rho$, $\mu\rho$, ρ

Meanwhile to map to the Lamé impedances as examined in AVO analysis by Goodway (2001) we first move from γ, μ, ρ to λ, μ, ρ, and thence make the replacements

$$\begin{aligned}
\delta s_\lambda &\rightarrow \left(\frac{s_{\lambda_0}}{s_{\lambda\rho_0}}\right)\delta s_{\lambda\rho} - \left(\frac{s_{\lambda_0}}{s_{\rho 0}}\right)\delta s_\rho \\
\delta s_\mu &\rightarrow \left(\frac{s_{\mu_0}}{s_{\mu\rho_0}}\right)\delta s_{\mu\rho} - \left(\frac{s_{\mu_0}}{s_{\rho_0}}\right)\delta s_\rho \\
\delta s_\rho &\rightarrow \delta s_\rho ,
\end{aligned} \tag{29}$$

which update the parameter set $s_{\lambda\rho} = \lambda\rho$, $s_{\mu\rho} = \mu\rho$ and $s_\rho = \rho$.

Example 5: Russell-Gray updates f, μ, ρ

Russell et al. (2011) employ a f, μ, ρ formulation, in which f is the *fluid term*, which is a perturbation upon a dry κ or λ value. We can accommodate it in FWI by replacing

$$s_\lambda = s_{\lambda_0} + \delta s_\lambda \tag{30}$$

with

$$s_f = s_{\lambda_0} + \delta s_f , \tag{31}$$

where the key difference is one of interpretation: the reference medium would be considered *dry*, and the task of updating would be to determine the updates δs_f which as iterations progress determine the saturated model. Thus the mathematical parameterization would simply be

$$
\begin{aligned}
\delta s_\lambda &\rightarrow \delta s_f \\
\delta s_\mu &\rightarrow \delta s_\mu \\
\delta s_\rho &\rightarrow \delta s_\rho,
\end{aligned}
\tag{32}
$$

with the main difference from the λ, μ, ρ updates being their interpretation in terms of poroelastic geology rather than straight elastic properties.

SENSITIVITIES FOR MULTICOMPONENT INVERSION

Next we will discuss the procedure for determining the sensitivities in any three elastic parameter sets. We will go through the procedure in detail for the base set of parameters γ, μ, and ρ; once this procedure is set, the nature of the variations needed to derive any of the sensitivities is evident, and a particular example of interest can be produced without introducing any new concepts.

Base parametrization: γ, μ, ρ

In equation (25) we had the displacement-space perturbation operator written in terms of γ, μ and ρ:

$$
\mathcal{V}_{\gamma\mu\rho}(x', z') = \begin{bmatrix} \omega^2 \delta s_\rho + \partial_{x'} \delta s_\gamma \partial_{x'} + \partial_{z'} \delta s_\mu \partial_{z'} & \partial_{x'} [\delta s_\gamma - 2\delta s_\mu] \partial_{z'} + \partial_{z'} \delta s_\mu \partial_{x'} \\ \partial_{z'} [\delta s_\gamma - 2\delta s_\mu] \partial_{x'} + \partial_{x'} \delta s_\mu \partial_{z'} & \omega^2 \delta s_\rho + \partial_{z'} \delta s_\gamma \partial_{z'} + \partial_{x'} \delta s_\mu \partial_{x'} \end{bmatrix}.
$$

To form the Born approximation, this quantity was sandwiched between Green's operators and the result was integrated over all space (see equation 21):

$$
\delta \mathbf{G} = -\int dx' \int dz' \begin{bmatrix} \partial_{x'} G_{\mathrm{P}_0} & \partial_{z'} G_{\mathrm{P}_0} \\ -\partial_{z'} G_{\mathrm{S}_0} & \partial_{x'} G_{\mathrm{S}_0} \end{bmatrix} \mathcal{V}(x', z') \begin{bmatrix} \partial_{x'} G'_{\mathrm{P}_0} & -\partial_{z'} G'_{\mathrm{S}_0} \\ \partial_{z'} G'_{\mathrm{P}_0} & \partial_{x'} G'_{\mathrm{S}_0} \end{bmatrix}. \tag{33}
$$

The *first-order Fréchet kernel*, or sensitivity, is the limit of the linearized ratio of the small change in the field arising from a small local change in the medium. Since $\delta\mathbf{G}$ is precisely a change in the field — the change associated with the general perturbation $\mathcal{V}$ — equation (33) is close to providing a linearized form of this quantity for us. What remains to do is

1. Isolate the effect of variation in *one* parameter within $\mathcal{V}$ (in its raw form it contains variations in all three), and

2. Isolate the *point location* at which this variation takes place (the integral currently sums up variations distributed over all space).

Task (1.) is where the second integration-by-parts procedure occurs.

The γ sensitivity and integration by parts

Let us carry these two tasks out in detail and solve for the γ sensitivity. First, we set $\delta s_\mu = \delta s_\rho = 0$, in which case the displacement-space perturbation operator simplifies to

$$\mathcal{V}^{\gamma}_{\gamma\mu\rho}(x', z') = \begin{bmatrix} \partial_{x'}\delta s_\gamma \partial_{x'} & \partial_{x'}\delta s_\gamma \partial_{z'} \\ \partial_{z'}\delta s_\gamma \partial_{x'} & \partial_{z'}\delta s_\gamma \partial_{z'} \end{bmatrix}, \tag{34}$$

which can be written

$$\mathcal{V}^{\gamma}_{\gamma\mu\rho}(x', z') = \begin{bmatrix} \partial_{x'} & 0 \\ 0 & \partial_{z'} \end{bmatrix} \begin{bmatrix} \delta s_\gamma & \delta s_\gamma \\ \delta s_\gamma & \delta s_\gamma \end{bmatrix} \begin{bmatrix} \partial_{x'} & 0 \\ 0 & \partial_{z'} \end{bmatrix}. \tag{35}$$

This isolates the influence of γ on the field. Now we do two things simultaneously. We integrate by parts, moving the left and right derivative operators in equation (35) onto the left and right operators in equation (33), and, we make the replacement $\delta s_\gamma(x', z') = \delta s_\gamma(x, z)\delta(x - x')\delta(z - z')$, in order to calculate the special $\delta\mathbf{G}$ due to a point variation in γ at the coordinates (x, z). We obtain

$$\begin{aligned} \delta\mathbf{G} = \int dx' \int dz' \delta s_\gamma(x, z)\delta(x - x')\delta(z - z') \\ \times \left(\mathbf{L}^{\gamma}_{\gamma\mu\rho}(x', z') \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{R}^{\gamma}_{\gamma\mu\rho}(x', z') \right), \end{aligned} \tag{36}$$

where

$$\begin{aligned} \mathbf{L}^{\gamma}_{\gamma\mu\rho}(x', z') &= \left(\begin{bmatrix} \partial_{x'} & 0 \\ 0 & \partial_{z'} \end{bmatrix} \begin{bmatrix} \partial_{x'} G_{\mathrm{P}_0} & \partial_{z'} G_{\mathrm{P}_0} \\ -\partial_{z'} G_{\mathrm{S}_0} & \partial_{x'} G_{\mathrm{S}_0} \end{bmatrix}^T \right)^T \\ &= \begin{bmatrix} \partial^2_{x'} G_{\mathrm{P}_0} & \partial^2_{z'} G_{\mathrm{P}_0} \\ -\partial_{x'}\partial_{z'} G_{\mathrm{S}_0} & \partial_{z'}\partial_{x'} G_{\mathrm{S}_0} \end{bmatrix}, \end{aligned} \tag{37}$$

and

$$\mathbf{R}^{\gamma}_{\gamma\mu\rho}(x', z') = \begin{bmatrix} \partial^2_{x'} G'_{\mathrm{P}_0} & -\partial_{x'}\partial_{z'} G'_{\mathrm{S}_0} \\ \partial^2_{z'} G'_{\mathrm{P}_0} & \partial_{z'}\partial_{x'} G'_{\mathrm{S}_0} \end{bmatrix} \tag{38}$$

are the left and right operators respectively. The delta functions sift out values at (x, z) from the integral, whereupon we divide by $\delta s_\gamma(x, z)$ and take the limit as this step goes to zero, which is the sensitivity we seek:

$$\frac{\partial\mathbf{G}(k_g, k_s)}{\partial s_\gamma(x, z)} = \mathbf{L}^{\gamma}_{\gamma\mu\rho}(x, z) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{R}^{\gamma}_{\gamma\mu\rho}(x, z). \tag{39}$$

Written out explicitly, the 2×2 matrix of sensitivities of the P-to-P, S-to-P, P-to-S and S-to-S wave field components to variations in γ is given by

$$\frac{\partial\mathbf{G}(k_g, k_s)}{\partial s_\gamma(x, z)} = \begin{bmatrix} \partial^2_{x} G_{\mathrm{P}_0} & \partial^2_{z} G_{\mathrm{P}_0} \\ -\partial_{x}\partial_{z} G_{\mathrm{S}_0} & \partial_{z}\partial_{x} G_{\mathrm{S}_0} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \partial^2_{x} G'_{\mathrm{P}_0} & -\partial_{x}\partial_{z} G'_{\mathrm{S}_0} \\ \partial^2_{z} G'_{\mathrm{P}_0} & \partial_{z}\partial_{x} G'_{\mathrm{S}_0} \end{bmatrix}. \tag{40}$$

Example: PP-γ sensitivity

To exemplify the process we have now undertaken, let us consider the (1,1) element of the matrix in equation (40):

$$\frac{\partial G_{\mathrm{PP}}(k_g,k_s)}{\partial s_\gamma(x,z)} = -\frac{\omega^2}{V_{P_0}^2}\left(\frac{1}{s_{\gamma 0}}\right) G_P(k_g,z_g,x,z,\omega)G_P(x,z,k_s,z_s,\omega). \tag{41}$$

This individual element is analogous to a scalar wave velocity sensitivity (e.g., Margrave et al., 2011) or the acoustic bulk modulus sensitivity (Innanen, 2014b), and can be seen to have essentially the same form as both/either of those two quantities.

The μ sensitivity

We next discuss the calculation of the μ sensitivity. We begin again with the γ, μ, ρ perturbation operator in equation (33), but this time set the γ and ρ perturbations to zero: $\delta s_\gamma = \delta s_\rho = 0$, from which we obtain

$$\mathcal{V}^{\mu}_{\gamma\mu\rho}(x',z') = \left[\begin{array}{cc} \partial_{z'}\delta s_\mu \partial_{z'} & -2\partial_{x'}\delta s_\mu \partial_{z'} + \partial_{z'}\delta s_\mu\partial_{x'} \\ -2\partial_{z'}\delta s_\mu\partial_{x'} + \partial_{x'}\delta s_\mu\partial_{z'} & \partial_{x'}\delta s_\mu\partial_{x'} \end{array}\right]. \tag{42}$$

This can be broken up into the sum

$$\mathcal{V}^{\mu}_{\gamma\mu\rho} = \mathcal{V}^{\mu(1)}_{\gamma\mu\rho} + \mathcal{V}^{\mu(2)}_{\gamma\mu\rho}, \tag{43}$$

where

$$\begin{aligned} \mathcal{V}^{\mu(1)}_{\gamma\mu\rho}(x',z') &= \left[\begin{array}{cc} \partial_{z'}\delta s_\mu\partial_{z'} & \partial_{z'}\delta s_\mu\partial_{x'} \\ \partial_{x'}\delta s_\mu\partial_{z'} & \partial_{x'}\delta s_\mu\partial_{x'} \end{array}\right] \\ &= \left[\begin{array}{cc} \partial_{z'} & 0 \\ 0 & \partial_{x'} \end{array}\right]\left[\begin{array}{cc} \delta s_\mu & \delta s_\mu \\ \delta s_\mu & \delta s_\mu \end{array}\right]\left[\begin{array}{cc} \partial_{z'} & 0 \\ 0 & \partial_{x'} \end{array}\right], \end{aligned} \tag{44}$$

and

$$\begin{aligned} \mathcal{V}^{\mu(2)}_{\gamma\mu\rho}(x',z') &= -2\left[\begin{array}{cc} 0 & \partial_{x'}\delta s_\mu\partial_{z'} \\ \partial_{x'}\delta s_\mu\partial_{z'} & 0 \end{array}\right] \\ &= -2\left[\begin{array}{cc} 0 & \partial_{x'} \\ \partial_{z'} & 0 \end{array}\right]\left[\begin{array}{cc} 0 & \delta s_\mu \\ \delta s_\mu & 0 \end{array}\right]\left[\begin{array}{cc} 0 & \partial_{z'} \\ \partial_{x'} & 0 \end{array}\right] \end{aligned} \tag{45}$$

are summands whose derivatives can be brought out into left and right matrix operators. With all of the derivative operators shifted off δs_μ, we then set $\delta s_\mu(x',z') = \delta s_\mu(x,z)\delta(x-x')\delta(z-z')$, substitute the two components of $\mathcal{V}^{\mu}_{\gamma\mu\rho}$ into equation (33), enact the second stage of integration by parts, and evaluate the new integral. We obtain the μ sensitivity:

$$\begin{aligned} \frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\mu(x,z)} = &\left[\begin{array}{cc} \partial_z\partial_x G_{\mathrm{P}_0} & \partial_x\partial_z G_{\mathrm{P}_0} \\ -\partial_z^2 G_{\mathrm{S}_0} & \partial_x^2 G_{\mathrm{S}_0} \end{array}\right]\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]\left[\begin{array}{cc} \partial_x\partial_z G'_{\mathrm{P}_0} & -\partial_z^2 G'_{\mathrm{S}_0} \\ \partial_x\partial_z G'_{\mathrm{P}_0} & \partial_x^2 G'_{\mathrm{S}_0} \end{array}\right] \\ &- 2\left[\begin{array}{cc} \partial_x^2 G_{\mathrm{P}_0} & \partial_z^2 G_{\mathrm{P}_0} \\ -\partial_x\partial_z G_{\mathrm{S}_0} & \partial_z\partial_x G_{\mathrm{S}_0} \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\left[\begin{array}{cc} \partial_z^2 G'_{\mathrm{P}_0} & \partial_z\partial_x G'_{\mathrm{S}_0} \\ \partial_x^2 G'_{\mathrm{P}_0} & -\partial_x\partial_z G'_{\mathrm{S}_0} \end{array}\right], \end{aligned} \tag{46}$$

the second of our base set of three sensitivity operators.

The ρ sensitivity

The density sensitivity is relatively straightforward. Setting $\delta s_\gamma = \delta s_\mu = 0$, the perturbation operator becomes

$$\mathcal{V}^{\rho}_{\gamma\mu\rho}(x', z') = \begin{bmatrix} \omega^2 \delta s_\rho & 0 \\ 0 & \omega^2 \delta s_\rho \end{bmatrix}, \tag{47}$$

from which, upon setting $\delta s_\rho(x', z') = \delta s_\rho(x, z)\delta(x-x')\delta(z-z')$ (no second integration by parts is necessary in this case) and integrating equation (33), we obtain the desired result:

$$\frac{\partial \mathbf{G}(k_g, k_s)}{\partial s_\rho(x, z)} = \begin{bmatrix} \omega\partial_x G_{\mathrm{P}_0} & \omega\partial_z G_{\mathrm{P}_0} \\ -\omega\partial_z G_{\mathrm{S}_0} & \omega\partial_x G_{\mathrm{S}_0} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \omega\partial_x G'_{\mathrm{P}_0} & -\omega\partial_z G'_{\mathrm{S}_0} \\ \omega\partial_z G'_{\mathrm{P}_0} & \omega\partial_x G'_{\mathrm{S}_0} \end{bmatrix}. \tag{48}$$

Other parametrizations

The above examples demonstrate three specific applications of the two-stage integration by parts procedure leading to multicomponent elastic sensitivities appropriate for reflection full waveform inversion. The same procedure can be applied to $(\mathcal{V}^1_{123}, \mathcal{V}^2_{123}, \mathcal{V}^3_{123})$ for any three parameters, for instance those in equations (26)–(32), to derive the appropriate sensitivities.

OBJECTIVE FUNCTION AND ITS DERIVATIVES

We will now construct a simple least-squares objective function whose derivatives make use of the sensitivities calculated in the previous section, and which brings in the matrix multicomponent framework in a natural way. Assuming we have access to some or all of four components of measured data, P-to-P, P-to-S, S-to-P and S-to-S, such that the residuals can be written

$$\delta\mathbf{P}(k_g, k_s) = \begin{bmatrix} \delta P_{\mathrm{PP}}(k_g, k_s) & \delta P_{\mathrm{SP}}(k_g, k_s) \\ \delta P_{\mathrm{PS}}(k_g, k_s) & \delta P_{\mathrm{SS}}(k_g, k_s) \end{bmatrix}, \tag{49}$$

the simplest least-squares objective function ϕ is of the form

$$\phi = \frac{1}{2}\int d\omega \sum_{k_g, k_s} \mathrm{tr}\left(\delta\mathbf{P}^H \delta\mathbf{P}\right), \tag{50}$$

where $\cdot^H$ is the Hermitian transpose, and "tr" indicates the trace, or sum of the diagonals, of a matrix. The quantity tr$A^T B$ is called the *Frobenius product* of A and B; it is a generalization of the scalar product between two vectors, generating the sum of the products of every element of two square matrices.

The gradient and Hessian functions are then produced by taking the first and second derivatives of ϕ with respect to each of the three model parameters. The gradient function associated with parameter X is given by

$$g_X(x, z) = -\int d\omega \sum_{k_g, k_s} \mathrm{tr}\left\{\left[\frac{\partial \mathbf{G}(k_g, k_s)}{\partial s_X(x, z)}\right]^T \delta\mathbf{P}^*\right\}, \tag{51}$$

and the residual-independent part of the Hessian function associated with the pair of parameters X and Y is given by

$$H_{XY}(x,z,x',z') = \int d\omega \sum_{k_g,k_s} \mathrm{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_X(x',z')}\right]^H \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_Y(x,z)}\right] \right\}. \tag{52}$$

We have again made use twice of the Frobenius product to keep the notation compact.

UPDATING

A Gauss-Newton update in three parameters X, Y, and Z, distributed in spatial coordinates x and z is

$$\begin{bmatrix} \delta s_X(x,z) \\ \delta s_Y(x,z) \\ \delta s_Z(x,z) \end{bmatrix} = \int dx' \int dz' \mathcal{H}^{-1}(x,z,x',z')\mathbf{g}(x',z'), \tag{53}$$

where

$$\mathbf{g}(x,z) = \begin{bmatrix} g_X \\ g_Y \\ g_Z \end{bmatrix}, \quad \mathcal{H}(x,z,x',z') = \begin{bmatrix} H_{XX} & H_{XY} & H_{XZ} \\ H_{YX} & H_{YY} & H_{YZ} \\ H_{ZX} & H_{ZY} & H_{ZZ} \end{bmatrix}. \tag{54}$$

Example: γ, μ, ρ update

To give one explicit example, consider our base parameters γ, μ, ρ. The Gauss-Newton update will be in our framework

$$\begin{bmatrix} \delta s_\gamma(x,z) \\ \delta s_\mu(x,z) \\ \delta s_\rho(x,z) \end{bmatrix} = \int dx' \int dz' \mathcal{H}^{-1}(x,z,x',z')\mathbf{g}(x',z'), \tag{55}$$

where

$$\mathbf{g}_{\gamma\mu\rho}(x,z) = \begin{bmatrix} g_\gamma(x,z) \\ g_\mu(x,z) \\ g_\rho(x,z) \end{bmatrix}, \tag{56}$$

and

$$\mathcal{H}_{\gamma\mu\rho}(x,z,x',z') = \begin{bmatrix} H_{\gamma\gamma}(x,z,x',z') & H_{\gamma\mu}(x,z,x',z') & H_{\gamma\rho}(x,z,x',z') \\ H_{\mu\gamma}(x,z,x',z') & H_{\mu\mu}(x,z,x',z') & H_{\mu\rho}(x,z,x',z') \\ H_{\rho\gamma}(x,z,x',z') & H_{\rho\mu}(x,z,x',z') & H_{\rho\rho}(x,z,x',z') \end{bmatrix}. \tag{57}$$

The individual gradient functions are

$$\begin{aligned} g_\gamma(x,z) &= -\int d\omega \sum_{k_g,k_s} \mathrm{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\gamma(x,z)}\right]^T \delta\mathbf{P}^* \right\}, \\ g_\mu(x,z) &= -\int d\omega \sum_{k_g,k_s} \mathrm{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\mu(x,z)}\right]^T \delta\mathbf{P}^* \right\}, \\ g_\rho(x,z) &= -\int d\omega \sum_{k_g,k_s} \mathrm{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\rho(x,z)}\right]^T \delta\mathbf{P}^* \right\}, \end{aligned} \tag{58}$$

and those for the Hessian are likewise

$$
\begin{aligned}
H_{\gamma\gamma}(x,z,x',z') &= \int d\omega \sum_{k_g,k_s} \text{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\gamma(x',z')}\right]^H \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\gamma(x,z)}\right] \right\} \\
H_{\gamma\mu}(x,z,x',z') &= \int d\omega \sum_{k_g,k_s} \text{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\mu(x',z')}\right]^H \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\gamma(x,z)}\right] \right\} \\
H_{\gamma\rho}(x,z,x',z') &= \int d\omega \sum_{k_g,k_s} \text{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\rho(x',z')}\right]^H \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\gamma(x,z)}\right] \right\},
\end{aligned}
\tag{59}
$$

and

$$
\begin{aligned}
H_{\mu\gamma}(x,z,x',z') &= \int d\omega \sum_{k_g,k_s} \text{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\gamma(x',z')}\right]^H \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\mu(x,z)}\right] \right\} \\
H_{\mu\mu}(x,z,x',z') &= \int d\omega \sum_{k_g,k_s} \text{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\mu(x',z')}\right]^H \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\mu(x,z)}\right] \right\} \\
H_{\mu\rho}(x,z,x',z') &= \int d\omega \sum_{k_g,k_s} \text{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\rho(x',z')}\right]^H \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\mu(x,z)}\right] \right\},
\end{aligned}
\tag{60}
$$

and

$$
\begin{aligned}
H_{\rho\gamma}(x,z,x',z') &= \int d\omega \sum_{k_g,k_s} \text{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\gamma(x',z')}\right]^H \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\rho(x,z)}\right] \right\} \\
H_{\rho\mu}(x,z,x',z') &= \int d\omega \sum_{k_g,k_s} \text{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\mu(x',z')}\right]^H \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\rho(x,z)}\right] \right\} \\
H_{\rho\rho}(x,z,x',z') &= \int d\omega \sum_{k_g,k_s} \text{tr}\left\{ \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\rho(x',z')}\right]^H \left[\frac{\partial \mathbf{G}(k_g,k_s)}{\partial s_\rho(x,z)}\right] \right\}.
\end{aligned}
\tag{61}
$$

The three sensitivities combined to produce these twelve functions are given in equations (40), (46) and (48).

Updating with individual PP, PS, SP, SS components

To update using *only* PP, PS, SP or SS data we can set the other three components of the residuals matrix to zero, that is, we reduce to an update with PP data only with the replacement

$$
\delta \mathbf{P}(k_g,k_s) \rightarrow \begin{bmatrix} \delta P_{\text{PP}}(k_g,k_s) & 0 \\ 0 & 0 \end{bmatrix},
\tag{62}
$$

with converted wave data only with the replacement

$$
\delta \mathbf{P}(k_g,k_s) \rightarrow \begin{bmatrix} 0 & \delta P_{\text{SP}}(k_g,k_s) \\ 0 & 0 \end{bmatrix},
\tag{63}
$$

with S to P conversions only with the replacement

$$\delta \mathbf{P}(k_g, k_s) \rightarrow \begin{bmatrix} 0 & 0 \\ \delta P_{\mathrm{PS}}(k_g, k_s) & 0 \end{bmatrix}, \tag{64}$$

and with pure S wave data only by the replacement

$$\delta \mathbf{P}(k_g, k_s) \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & \delta P_{\mathrm{SS}}(k_g, k_s) \end{bmatrix}. \tag{65}$$

Joint inversion

Multicomponent inversion has been posed in CREWES research as joint inversion[§], a framework most often employed to integrate data sets from different physical experiments (e.g., gravity and seismic). PP and PS data come from the same experiment, but their significant practical differences are such that implementing joint inversion makes sense. We must do some more work in order to incorporate practical differences between PP and PS inversion (e.g., data bandwidth etc.), but we can easily formulate a joint PP, PS inversion or SP, SS inversion.

P source

The full inverse problem given a P-wave source is set up with the replacement

$$\delta \mathbf{P}(k_g, k_s) \rightarrow \begin{bmatrix} \delta P_{\mathrm{PP}}(k_g, k_s) & \delta P_{\mathrm{SP}}(k_g, k_s) \\ 0 & 0 \end{bmatrix}. \tag{66}$$

S source

Meanwhile the full inverse problem given an S-wave source is set up with the replacement

$$\delta \mathbf{P}(k_g, k_s) \rightarrow \begin{bmatrix} 0 & 0 \\ \delta P_{\mathrm{PS}}(k_g, k_s) & \delta P_{\mathrm{SS}}(k_g, k_s) \end{bmatrix}. \tag{67}$$

CONCLUSIONS

Full waveform inversion applied to reflection mode, multicomponent land data is considered, in the context of the iterated recovery of sets of three parameters, ranging from standard elastic λ, μ, and ρ through to petrophysically relevant parameters such as the fluid term f of Russell and Gray. Much of the effort lies in constructing a flexible framework for elastic multicomponent sensitivities, which we address with a two-stage integration by parts regimen. Matrix forms for a multicomponent objective function and three term gradient and nine term Hessian are then determined, and the reductions necessary to invert PP, PS, SP, and SS modes independently or jointly are formed.

[§]See, for instance *http://www.crewes.org/ResearchLinks/JointInversion/*.

Two lines of research will be high priorities going forward. First, this theory can be integrated into ongoing CREWES research efforts in progressing IMMI (Margrave et al., 2013). The 2014 Priddis shoot (Hall et al., 2014) will likely be a key early-stage dataset to which this will apply. Second, analysis of the theoretical character of the possible parameterizations of FWI can begin. In this there are several dovetailing issues. It has been demonstrated in an acoustic setting that certain approximations of the inverse Hessian produce FWI updates which are consistent with linearized AVO inversion and linearized multiparameter inverse scattering (Innanen, 2014b). This analysis can now be rounded out to include more complete pictures of AVO, with the ability to include virtually any convenient AVO parameterization in the FWI framework.

Moreover, the linearized AVO results can now be extended to incorporate nonlinear amplitude phenomena. Recent theoretical and laboratory analysis (Innanen, 2013; Innanen and Mahmoudian, 2014; Kolb et al., 2014) has highlighted the importance of nonlinearity to real measurements and interpretations of AVO data. In a scalar setting, an incorporation of nonlinearity as far as second order in full waveform inversion reflected data amplitudes has been discussed (Innanen, 2014a); in principle this can now also be brought to bear on any three-parameter isotropic-elastic framing of AVO, through an integration of the current paper's results with the scalar nonlinear results.

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