Partial wave analysis of seismic wave scattering

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ABSTRACT

Developments in seismic full waveform inversion have brought a renewed interest in the scattering picture of wave propagation in recent years. Our experience has been that advances in our understanding of particular aspects of scattering has led rapidly to a concurrent advance in our understanding of how to pose and analyze practical seismic inversion methods. In this paper we study the partial wave analysis of elastic wave scattering in an isotropic radially heterogenous medium in the context of Born-approximation. We show that in the presence of a scatterer there is a phase shift in the outgoing scattered spherical elastic wave. We also obtain the scattering amplitudes for scattering of P- and S-wave in terms of phase shift for P-, SV- and SH-waves. We show that the phase shifts can be calculated using the Lippman-Schwinger integral equation. A clear and consistent theory of elastic partial wave scattering will lead to better sensitivity or Jacobian matrices, a critical matter for the success of elastic seismic full waveform inversion.

INTRODUCTION

In seismic forward problem it is supposed that the source term is known and has no dependency on the radiated field. The Born model of inverse scattering is a linearization of the transformation from perturbations in a medium to a scattered waves field. The basic idea in inverse scattering as its name indicates is the determination of physical properties of medium from a scattered wavefield as data (Beylkin and Burridge, 1990). In other words, if we have measurements of a field which is assumed to be the solution of the inhomogeneous wave equation, we can determine the perturbation that causes the scattering. In the Born approximation this perturbation appears as a source term in the inhomogeneous wave equation (Sato et al., 2012). In order to invert, we don't need to know the radiated field throughout the volume of the medium. Information about the field and its first derivative on the surface surrounding the volume is sufficient for inverting the physical properties of medium.

Partial wave analysis is a useful method to study the scattering for the case that the perturbation of the medium is radially symmetric (depending only on the magnitude of the distance) and is effective over a finite range (Zettili, 2001). In this method a plane wave is written as a superposition of an infinite number of components, each with a definite angular momentum. These are called *partial waves*. The process of decomposing a plane wave into the partial waves is referred to as partial wave analysis. The angular distribution of scattered particles in a particular process is described in terms of a differential cross section.

The key equation in scattering theory in the Lippmann-Schwinger integral equation for a scattered wave in which its asymptotic behaviour is

$$\psi_s(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta,\varphi)\frac{e^{ikr}}{r}. \tag{1}$$

Here the first term is an initial incident plane wave and the second term is the outgoing spherical wave modulated by the angle dependent function f, called the scattering amplitude. This contains all information about the scattered wave and is directly related to the differential scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{outgoing flux/solid angle}}{\text{incoming flux/area}} = |f(\theta, \varphi)|^2. \tag{2}$$

Taking the integral of (2) over solid angle results in the scattering cross section which can be used to calculate the attenuation of the scattered wave due to energy spreading over space, called the coda-attenuation factor (Sato et al., 2012). The differential cross section idea has had significant seismic application. Exact and approximate expressions for scattered and transmitted fields caused by an incident compressional or shear wave from a spherical inclusion in an infinitely isotropic elastic medium have been investigated by many authors (Hinders, 1991; Korneev and Johnson, 1993a,b, 1996; Politz, 1991).

With partial wave analysis, the scattering amplitude can be expressed as a function of a phase shift parameter. The advantage of this method is that a complex function f is expressed in terms of one parameter, phase shift, instead of phase shift and amplitude. Phase shift contains all information about the properties of the scatterer and is obtained by the boundary conditions on the wavefiled. In the partial wave analysis picture, the only difference between the radial wave function in the presence and absence of a scatterer is phase shift.

In this paper we generalize the partial wave analysis to the scattering of elastic waves. For an isotropic homogeneous medium, the elastic wave equation reduces to two vector Helmholtz equations, one for P-wave (compressional) and the other for S-wave (shear) with the solutions called Hanson vectors (Morse and Feshbach, 1953; Hill, 1996). The angular part of the solutions are expressed in terms of spherical harmonics and their derivatives with respect to angle parameters. The scattering amplitude for the P-wave is expressed in terms of spherical harmonics and for the S-wave is expressed as a derivative of the spherical harmonics. In the case of the incident P-wave, the partial wave analysis works for scattered P-wave. It means that scattering amplitude for a scattered P-wave can be expressed in terms of phase shift. On the other hand for a scattered S-wave we can not make a phase shift interpretation as in the P-wave case, because initially there is no S-wave to compare with the scattered wave. There is a similar explanation for initial incident S-wave.

The next step to obtain the scattering amplitude is calculation of the phase shift. In this part we use the Lippmann-Schwinger integral equation with the dyadic Green's functions and a source term obtained from the perturbed wave equation in a Born approximation context.

This report is organized as follows. In section 1 we review the scattering and partial wave analysis for a one dimensional scalar wave with examples, then we study one-dimensional anelastic scattering. We extend the analysis to 3D scalar wave field scattering in section 2. In section 3 we develop the partial wave analysis in an elastic medium. Two cases will be investigated, scattering of initially incident P-wave and S-wave. In section 4, using the Green's function approach we extract the phase shift determined in section 4 for scattering of P-wave. We sum up in section 5 with a conclusion.

ONE-DIMENSIONAL SCALAR WAVE SCATTERING

In this section we will study the problem for elastic and anelastic scattering of the one-dimensional scalar wave equation. For the scalar case, "elastic" means no energy loss in the whole scattering process and "anelastic" means a portion of energy is converted to heat in the medium. The scalar wave equation in one-dimension is given by

$$\psi''(x,t) - c^{-2}(x)\ddot{\psi}(x,t) = 0, \tag{3}$$

where $c(x)$ is the velocity of the scalar wave, prime refers to the spatial derivative and dot indicates the time derivative. Inserting $\psi(x,t) = e^{-i\omega t}\phi(x)$ the above equation reduces to

$$\phi''(x) + \omega^2 c^{-2}(x)\phi(x) = 0. \tag{4}$$

The essential assumption in perturbation theory is the definition of the reference medium with a constant velocity c_0 and an actual medium with spacially dependent velocity $c(x)$. The relationship between the velocity in actual and reference medium is expressed by

$$c(x) = c_0 + \delta c(x) = c_0(1 + \xi(x)), \tag{5}$$

where $|\xi(x)| \ll 1$ is the fractional velocity. In the entire paper we assume that $\xi(x)$ is none-zero in a specific range say $0 < x < a$, called the perturbation range. Inserting (5) in (4) and using the $(1+x)^{-1} \approx 1 - x$ for $x \ll 1$, we arrive at

$$\phi''(x) + [k^2 - 2k^2\xi(x)]\phi(x) = 0, \tag{6}$$

where we defined $k = \frac{\omega}{c_0}$. We assume that in $x < 0$ the wave field is zero. The solution of eq.(6) in the absence of perturbation ($\xi(x) = 0$) or in the reference medium is

$$\phi_0(x) = \sin(kx) = \frac{1}{2i}(e^{ikx} - e^{-ikx}). \tag{7}$$

The first term is related to the outgoing wave and the second term to the incoming wave. What happens if $\xi(x) \neq 0$? In this case we expect only the outgoing wave to be affected by the perturbation. If there is no absorbtion or attenuation in the medium the incoming and outgoing fluxes should be equal. As a result, the effect of perturbation appears as a phase multiplied by the outgoing field. So the total wave in the actual medium or in the presence of perturbation is

$$\phi_p(x) = \frac{1}{2i}\left\{e^{2i\delta_k}e^{ikx} - e^{-ikx}\right\}, \tag{8}$$

where δ_k is called phase shift and is related to the perturbation in the medium. We can write the equation (8) as a superposition of incident and scattered waves. Comparing (7) and (8) we conclude

$$\phi_p(x) = \phi_0(x) + \phi_{sc}(x), \tag{9}$$

where the scattered wavefield is

$$\phi_{sc}(x) = \phi_p(x) - \phi_0(x) = e^{i\delta_k}\sin\delta_k e^{ikx}. \tag{10}$$

The amplitude of the outgoing field e^{ikx} called the scattering amplitude and its modulus square is the scattering cross section which is equal to $\sin^2\delta_k$. Also we can write the total wave in the region $x > a$ as

$$\phi_p(x) = e^{i\delta_k}\frac{1}{2i}\left\{e^{i\delta_k}e^{ikx} - e^{-i\delta_k}e^{-ikx}\right\} = e^{i\delta_k}\sin(kx + \delta_k). \tag{11}$$

Now, let us evaluate the phase shift method for simple examples.

Delta function perturbation

Consider a localized perturbation represented by a delta function located in $x = a$

$$\xi(x) = \xi_0 \delta(x-a), \qquad \xi_0 > 0. \tag{12}$$

The solution for both regions $0 < x < a$ and $x > a$ is a sin function. We denote the solution in $0 < x < a$ by ϕ_- and the solution in $x > a$ by ϕ_+

$$\phi_-(x) = A_- \sin(kx), \tag{13}$$

$$\phi_+(x) = e^{i\delta_k} \sin(kx + \delta_k). \tag{14}$$

We use equation (11) to write the solution in $x > a$. The wave equation (6) for potential (12) is

$$\phi''(x) + k^2[1 - 2\xi_0 \delta(x-a)]\phi(x) = 0. \tag{15}$$

To obtain the phase shift we have to use the boundary condition applied to solutions on the left and right hand side of point $x = a$. Since in the wave equation there is a delta function defined in $x = a$, the first derivative of the solution at this point is not continuous. Therefore the boundary condition for this example is continuity of the wave and a prescribed discontinuity of the first derivative at $x = a$. To examine the discontinuity of the solution we integrate the equation (15)

$$\left\{ \int_{a-\epsilon}^{a+\epsilon} \phi''(x)dx + k^2 \int_{a-\epsilon}^{a+\epsilon} \phi(x)dx - 2\xi_0 k^2 \int_{a-\epsilon}^{a+\epsilon} \delta(x-a)\phi(x)dx \right\}_{\epsilon \to 0} = 0, \tag{16}$$

which after simplification reduces to

$$\phi'_+(a) - \phi'_-(a) + k^2 \left[\phi_+(a) - \phi_-(a)\right] - 2\xi_0 k^2 \phi_+(a) = 0. \tag{17}$$

The third term goes to zero because of the continuity of the wave equation. Finally, these boundary conditions result

$$\phi'_+(a) - \phi'_-(a) = 2\xi_0 k^2 \phi_+(a), \tag{18}$$

$$\phi_+(a) = \phi_-(a). \tag{19}$$

After simplification we arrive at

$$\tan(ka + \delta_k) = \left[\frac{1}{\tan(ka)} + \frac{2\xi_0}{k}\right]^{-1}. \tag{20}$$

In the case of low frequency $ka \ll 1$ we have

$$\tan(\delta_k) \approx \frac{ka}{1 + 2\xi_0 a}. \tag{21}$$

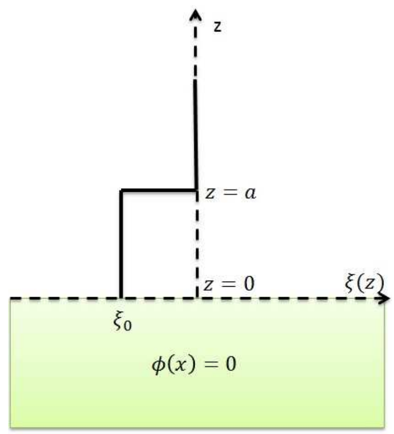

FIG. 1. Diagram illustrating the well perturbation in Eq.(22)

Well perturbation

Another example is scattering from the well perturbation

$$\xi(x) = \begin{cases} \xi_0, & 0 < x < a \\ 0, & x > a. \end{cases} \tag{22}$$

This perturbation represents an actual medium in which the velocity is higher than the reference medium in the perturbation interval $0 < x < a$. Similar to the first example, the solution in both regions is a sin function but with different amplitude and different phase. The wave equation in two regions are

$$\phi_-''(x) + (k^2 + 2\xi_0 k^2)\phi_-(x) = 0, \qquad 0 < x < a \tag{23}$$

$$\phi_+''(x) + k^2\phi_+(x) = 0, \qquad x > a \tag{24}$$

by definition $k_- = k\sqrt{1 + 2\xi_0}$, solutions are

$$\phi_-(x) = A_- \sin(k_- x), \tag{25}$$

$$\phi_+(x) = e^{i\delta_k} \sin(kx + \delta_k). \tag{26}$$

Finally using the continuity of the wave function and its first derivative we obtain the phase shift as

$$\delta_k = \tan^{-1}\left[\frac{\tan(ka\sqrt{1 + 2\xi_0})}{\sqrt{1 + 2\xi_0}}\right] - ka. \tag{27}$$

Also the amplitude of the wave in the interior region is

$$|A_-| = \frac{\sin(ka + \delta_k)}{\sin(ka)}. \tag{28}$$

In fig.2 we plot the phase shift as a function of $\kappa = ka$ for perturbation factor $\xi_0 = 0.1$. We can see that the shape of potential and phase shift are almost the same.

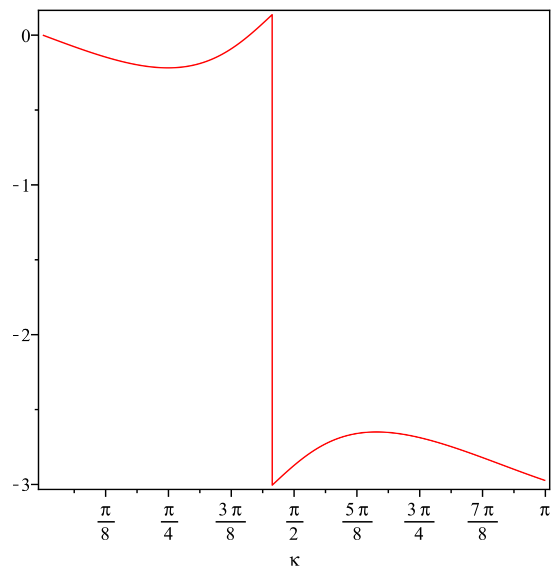

FIG. 2. Phase shift versus wave number for well perturbation model.

Anelastic scattering

Now let us consider anelastic scattering. In this case the total wave field is given by (Zettili, 2001)

$$\phi_p^a(x) = \frac{1}{2i}\left\{\gamma_k e^{2i\delta_k} e^{ikx} - e^{-ikx}\right\}, \tag{29}$$

where γ_k is a real number such that $0 < \gamma_k < 1$. Since there is energy loss due to absorbtion, the incoming and outgoing fluxes are not equal. Similar to the elastic case, the scattered wave is modulated by δ_k. The scattered wave field is

$$\phi_{sc}^a(x) = \phi_p^a(x) - \sin(kx) = \frac{1}{2i}(\gamma_k e^{i\delta_k} - 1)e^{ikx}. \tag{30}$$

The amplitude of the outgoing field e^{ikx} is called scattering amplitude and its modulus square is the scattering cross section which is equal to $\sin^2\delta_k$. Also we can write the wave in the region $x > a$ as

$$\phi_p^a(x) = \frac{1}{2}e^{i\delta_k}(\gamma_k + 1)\sin(kx + \delta_k) + \frac{1}{2i}e^{i\delta_k}(\gamma_k - 1)\cos(kx + \delta_k). \tag{31}$$

In the case that $\gamma_k = 1$, the anelastic scattered wave reduces to the elastic one.

SCATTERING OF 3D SCALAR WAVE

In this section we generalized the previous section results to the 3D case. The Helmholtz equation in 3D is given by

$$\psi''(\mathbf{x}, t) - c^{-2}(\mathbf{x})\ddot{\psi}(\mathbf{x}, t) = 0. \tag{32}$$

Inserting $\psi(\mathbf{x}) = \phi(\mathbf{x})e^{-i\omega t}$ in above equation leads to

$$\phi''(\mathbf{x}, t) + c^{-2}\omega^2\phi(\mathbf{x}) = 0. \tag{33}$$

Separation of variables for this equation results in the following solution

$$\phi_{lm}(\mathbf{x}) = f_l(r) Y_{lm}(\theta, \phi), \tag{34}$$

where f_l satisfies in the following equation

$$(r f_l(r))'' + \left(c^{-2}(\mathbf{x})\omega^2 - \frac{l(l+1)}{r^2} \right) (r f_l(r)) = 0. \tag{35}$$

Similar to the one dimension case, we write the perturbation in velocity

$$c(\mathbf{x}) = c_0 + \delta c(r) = c_0(1 + \xi(r)), \tag{36}$$

where $|\xi(r)| \ll 1$ is the fractional velocity. Then the following equation reduces to

$$f_l''(kr) + \frac{2}{kr} f_l'(kr) + \left(1 - 2\xi(r) - \frac{l(l+1)}{(kr)^2} \right) f_l(r) = 0. \tag{37}$$

Here prime is differentiation with respect to kr. We assume that at large distances $r \gg a$, the perturbation $\xi(r)$ falls off faster than r^{-2}, in this case

$$f_l''(kr) + \left(1 - \frac{l(l+1)}{(kr)^2} \right) f_l(kr) \approx 0. \tag{38}$$

The general solution of this equation is a combination of the spherical Hankel functions

$$f_l(kr) = A_l j_l(kr) + B_l n_l(kr), \tag{39}$$

where j_l is Bessel and n_l is Neumann function. For values near the origin

$$j_l(kr) \simeq \frac{2^l l!}{(2l+1)!}(kr)^l, \qquad n_l(kr) \simeq -\frac{(2l-1)!}{2^l l!}(kr)^{-l-1} \tag{40}$$

as $k \to 0$, all the partial waves go rapidly to zero except for the $l = 0$ wave. For large values of kr

$$j_l(kr) \simeq \frac{1}{kr} \sin\left(kr - \frac{l\pi}{2} \right), \qquad n_l(kr) \simeq -\frac{1}{kr} \cos\left(kr - \frac{l\pi}{2} \right). \tag{41}$$

Since the Neumann function diverges at the origin the solutions for the region near the origin are Bessel function. For large distances (39) reduces to

$$f_l(kr) = \frac{A_l}{kr} \sin\left(kr - \frac{l\pi}{2} \right) - \frac{B_l}{kr} \cos\left(kr - \frac{l\pi}{2} \right), \tag{42}$$

we can write this solution as follows

$$f_l(kr) \simeq \frac{C_l}{kr} \sin\left(kr - \frac{l\pi}{2} + \delta_l \right), \tag{43}$$

where

$$A_l = C_l \cos\delta_l, \qquad B_l = -C_l \sin\delta_l, \qquad C_l = \sqrt{A_l^2 + B_l^2}. \tag{44}$$

or

$$\delta_l = -\tan^{-1}\left(\frac{B_l}{A_l}\right). \tag{45}$$

Expansion of the incident plane wave is given by

$$e^{ikz} = e^{ikr\cos\theta} = \sum_l (2l+1) i^l j_l(kr) P_l(\cos\theta), \tag{46}$$

at large distances

$$e^{ikz} \simeq \sum_l (2l+1) i^l \frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr} P_l(\cos\theta). \tag{47}$$

Comparing to the asymptotic form of the total wave

$$\psi(\mathbf{x}) \simeq e^{ikz} + \frac{e^{ikr}}{r} f_l(\theta), \tag{48}$$

with

$$f_l(\theta) = \sum_l a_l \frac{1}{2k} i(2l+1) P_l(\cos\theta), \tag{49}$$

we arrive at

$$\psi(\mathbf{x}) \simeq \frac{1}{2ikr} \sum_l (2l+1)\left[(1-a_l)e^{ikr} - (-1)^l e^{-ikr}\right] P_l(\cos\theta). \tag{50}$$

On the other side we have

$$\psi(\mathbf{x}) = \sum_l \frac{(2l+1)}{2ikr} C_l \left[\exp(ikr + i\delta_l) - (-1)^l \exp(-ikr - i\delta_l)\right] P_l(\cos\theta), \tag{51}$$

comparing (50) and (51) we arrive at

$$a_l = [1 - e^{2i\delta_l}], \qquad C_l = e^{i\delta_l}. \tag{52}$$

The final form of the scattering amplitude is

$$f_l(\theta) = \frac{1}{k} \sum_l e^{i\delta_l} \sin\delta_l (2l+1) P_l(\cos\theta) \tag{53}$$

For an anelastic scattering, similar to the one-dimension case the scattering potential is

$$f_l^a(\theta) = \frac{1}{2ik} \sum_l (\gamma_k e^{2i\delta_l} - 1)(2l+1) P_l(\cos\theta) = \tag{54}$$

$$\frac{1}{k} \sum_l (\gamma_k \sin 2\delta_k + i(1 - \gamma_k \cos 2\delta_k)) P_l(\cos\theta). \tag{55}$$

It can be seen from (55) that scattering amplitude is a complex function which can be determined by having the phase shift δ_l. The spherical outgoing field is multiplied by the scattering amplitude that displays the pattern of scattering. Using the scattering amplitude we can also obtain the total cross section of scattering and we can see how energy is distributed over space after scattering.

SCATTERING IN HETEROGENOUS ELASTIC MEDIUM

Consider the wave propagation in an infinite heterogeneous, isotropic and non dispersive, non attenuating medium. The displacement field $\mathbf{U}$ satisfies in the elastic wave equation (e.g., Aki and Richards, 2002)

$$\boldsymbol{\nabla} \cdot \mathbf{T}_{\mathbf{U}} = \rho \frac{\partial^2 \mathbf{U}}{\partial t^2}, \tag{56}$$

where ρ is the spatial dependent density and $\mathbf{T}_{\mathbf{U}}$ is the stress tensor corresponding to the displacement field

$$\mathbf{T} = \lambda \boldsymbol{\nabla} \cdot \mathbf{U}\mathbb{I} + \mu(\boldsymbol{\nabla}\mathbf{U} + \mathbf{U}\boldsymbol{\nabla}). \tag{57}$$

The coefficients λ and μ are known as Lamé parameters. In addition the complete dyadic in spherical coordinate is given by

$$\mathbb{I} = \hat{\mathbf{r}}\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}. \tag{58}$$

For a heterogenous medium, physical properties of the medium like λ, μ and ρ vary between two or more points which means these elastic properties are spatially dependent. The displacement field thus satisfies

$$\mu \boldsymbol{\nabla}^2 \mathbf{U} + (\lambda + \mu)\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{U} + (\boldsymbol{\nabla}\lambda)\boldsymbol{\nabla} \cdot \mathbf{U} + \boldsymbol{\nabla}\mu \cdot (\boldsymbol{\nabla}\mathbf{U} + \mathbf{U}\boldsymbol{\nabla}) = \rho \frac{\partial^2 \mathbf{U}}{\partial t^2}. \tag{59}$$

for a radially heterogeneous medium $\rho(\mathbf{x}) = \rho(r)$,$\lambda(\mathbf{x}) = \lambda(r)$, and $\mu(\mathbf{x}) = \mu(r)$ the elastic wave equation reduces to (Ben-Menahem and Singh, 1981)

$$\mu \boldsymbol{\nabla}^2 \mathbf{U} + (\lambda + \mu)\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{U} + \hat{\mathbf{r}}\frac{d\lambda}{dr}\boldsymbol{\nabla} \cdot \mathbf{U} + \frac{d\mu}{dr}\left(2\frac{\partial \mathbf{U}}{\partial r} + \hat{\mathbf{r}} \times \boldsymbol{\nabla} \times \mathbf{U}\right) = \rho \frac{\partial^2 \mathbf{U}}{\partial t^2}. \tag{60}$$

In a homogeneous medium, where elastic properties are constant, equation (60) reduces to the Navier-Stokes equation

$$\alpha^2 \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{U}) - \beta^2 \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{U} + \omega^2 \mathbf{U} = 0' \tag{61}$$

where

$$\alpha^2 = \frac{\lambda + 2\mu}{\rho}, \qquad \beta^2 = \frac{\mu}{\rho}. \tag{62}$$

Using the Helmholtz decomposition

$$\mathbf{U} = \mathbf{U}_\alpha + \mathbf{U}_\beta, \qquad \boldsymbol{\nabla} \times \mathbf{U}_\alpha = 0, \qquad \boldsymbol{\nabla} \cdot \mathbf{U}_\beta = 0, \tag{63}$$

equation (61) reduces to

$$(\boldsymbol{\nabla}^2 + \alpha^2)\mathbf{U}_\alpha = 0, \qquad (\boldsymbol{\nabla}^2 + \beta^2)\mathbf{U}_\beta = 0 \tag{64}$$

So the elastic wave equation for a homogeneous medium reduces to the two Helmholtz vector equation, one for P-wave and the other for S-wave. Since the problem we investigate is scattering from an inclusion with spherical symmetry, we are interested in the solution of the above equation in a spherical coordinate system. The solutions of the vector wave equation called Hanson vectors are given by(Morse and Feshbach, 1953; Hill, 1996)

$$\mathbf{M}_{lm}^{f}=f_l(k_S r)\boldsymbol{\Lambda}_{lm}^{3} \tag{65}$$

$$\mathbf{N}_{lm}^{f}=\frac{1}{2l+1}\left\{(l+1)f_{l-1}(k_S r)\boldsymbol{\Lambda}_{lm}^{1}-lf_{l+1}(k_S r)\boldsymbol{\Lambda}_{lm}^{2}\right\} \tag{66}$$

$$\mathbf{L}_{lm}^{f}=\frac{1}{2l+1}\left\{f_{l-1}(k_P r)\boldsymbol{\Lambda}_{lm}^{1}+f_{l+1}(k_P r)\boldsymbol{\Lambda}_{lm}^{2}\right\} \tag{67}$$

where f_l is the solution of the spherical Bessel equation and a set of orthogonal vector functions $\boldsymbol{\Lambda}_{lm}^{i}, i=1,2,3$, are defined as

$$\boldsymbol{\Lambda}_{lm}^{1}=l\mathbf{P}_{lm}+\sqrt{l(l+1)}\mathbf{B}_{lm}, \tag{68}$$

$$\boldsymbol{\Lambda}_{lm}^{2}=-(l+1)\mathbf{P}_{lm}+\sqrt{l(l+1)}\mathbf{B}_{lm}, \tag{69}$$

$$\boldsymbol{\Lambda}_{lm}^{3}=\sqrt{l(l+1)}\mathbf{C}_{lm}, \tag{70}$$

where

$$\mathbf{P}_{lm}=\hat{\mathbf{r}}Y_{lm}(\theta,\varphi), \tag{71}$$

$$\sqrt{l(l+1)}\mathbf{B}_{lm}=\left(\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta}+\hat{\boldsymbol{\varphi}}\frac{im}{\sin\theta}\right)Y_{lm}(\theta,\varphi), \tag{72}$$

$$\sqrt{l(l+1)}\mathbf{C}_{lm}=\left(\hat{\boldsymbol{\theta}}\frac{im}{\sin\theta}-\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\theta}\right)Y_{lm}(\theta,\varphi). \tag{73}$$

Orthogonality relations are given as (Ben-Menahem and Singh, 1981)

$$\int\mathbf{P}_{lm}\cdot\mathbf{P}_{l'm'}^{*}d\Omega=\int\mathbf{B}_{lm}\cdot\mathbf{B}_{l'm'}^{*}d\Omega=\int\mathbf{C}_{lm}\cdot\mathbf{C}_{l'm'}^{*}d\Omega=\delta_{mm'}\delta_{ll'}\Gamma_{lm}, \tag{74}$$

also

$$\int\boldsymbol{\Lambda}_{l'm'}^{k}\cdot\boldsymbol{\Lambda}_{l'm'}^{*k'}d\Omega=\delta_{mm'}\delta_{ll'}\delta_{kk'}\Gamma_{lm}\gamma_k, \tag{75}$$

where

$$\gamma_1=l(2l+1), \tag{76}$$

$$\gamma_2=(l+1)(2l+1), \tag{77}$$

$$\gamma_3=l(l+1), \tag{78}$$

$$\Gamma_{lm}=\frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!}. \tag{79}$$

Assume that the incident wave is a P-wave in the z-direction. In this case the polarization also is in the z-direction, and the initial P-wave is given by $\mathbf{z}e^{ik_P z}$. To expand this vector function in terms of Hansson vectors we need to first expand the scalar plane wave as follows

$$e^{ik_P z}=e^{ik_P r\cos\theta}=\sum_l i^l(2l+1)j_l(k_P)P_l(\cos\theta). \tag{80}$$

Writing the unit vector $\mathbf{z}$ in spherical coordinates and doing some algebra (Appendix:A) we arrive at

$$\mathbf{z}e^{ikz} = \sum i^{l-1}(2l+1)\mathbf{L}_{l0}(k_P r, \theta). \tag{81}$$

This is the longitudinal wave because $\nabla \times (\mathbf{z}e^{ikz}) = 0$. In the case that the polarization is in the z-direction the incident wave is a superposition of the longitudinal Hansen vector $\mathbf{L}_{lm}$ with zero azimuth number $m = 0$. For an isotropic radially heterogenous medium, since the properties of medium is independent of direction, a scattered P-wave with zero m scattered to the waves with $m = 0$. Eq. (81) is the key equation for partial wave analysis when the incident wave is a P-wave.

We also need to obtain the partial wave expansion of the initial S-wave for partial wave analysis of scattering of S-waves. A shear wave can be represented by either $\mathbf{x}e^{ikz}$ or $\mathbf{y}e^{ikz}$, because both are transverse vectors

$$\nabla \cdot (\mathbf{x}e^{ikz}) = \nabla \cdot (\mathbf{y}e^{ikz}) = 0. \tag{82}$$

The expansions of S-waves in terms of transverse Hanson function $\mathbf{M}_{lm}$ and $\mathbf{N}_{lm}$ are (Appendix A)

$$\hat{\mathbf{x}}e^{ikz} = \sum_{lm} C^{+}_{lm}\mathbf{M}^{j}_{lm}(k_S r, \theta, \varphi) + C^{-}_{lm}\mathbf{N}^{j}_{lm}(k_S r, \theta, \varphi), \tag{83}$$

$$\hat{\mathbf{y}}e^{ikz} = i\sum_{lm} C^{-}_{lm}\mathbf{M}^{j}_{lm}(k_S r, \theta, \varphi) + C^{+}_{lm}\mathbf{N}^{j}_{lm}(k_S r, \theta, \varphi), \tag{84}$$

where we defined

$$C^{\pm}_{lm} = i^{l-1}\frac{2l+1}{2l(l+1)}\left[\delta_{m,1} \pm l(l+1)\delta_{m,-1}\right]. \tag{85}$$

In what follows for simplicity we skip writing the arguments for Hanson vectors. We show that the S-wave decomposed to the two divergenceless parts. The first part is related to the SH-wave and the second one to the SV-wave. The reason that we designate $\mathbf{M}_{lm}$ as an SH-wave is that it is orthogonal to the two other Hanson vectors

$$\int \mathbf{N}_{lm} \cdot \mathbf{M}^{*}_{lm} d\Omega = 0, \tag{86}$$

$$\int \mathbf{L}_{lm} \cdot \mathbf{M}^{*}_{lm} d\Omega = 0, \tag{87}$$

In consequence, horizontal and vertical S-waves are defined as follows

$$\mathbf{SV}_l = c_l\left[\mathbf{N}_{l1} - l(l+1)\mathbf{N}_{l,-1}\right], \tag{88}$$

$$\mathbf{SH}_l = c_l\left[\mathbf{M}_{l1} + l(l+1)\mathbf{M}_{l,-1}\right], \tag{89}$$

where

$$c_l = i^{l-1}\frac{2l+1}{2l(l+1)}. \tag{90}$$

Because of the orthogonality relation (86), we can easily show that the SH- and SV waves are orthogonal

$$\int \mathbf{SH}_l \cdot \mathbf{SV}^{*}_l d\Omega = 0. \tag{91}$$

Since the P-wave has components in three direction the SH wave can not be scattered to a P-wave. On the other hand a SV-wave is constructed of vectors in three direction; as a result a P-wave can be scattered to SV-wave and vice versa.

The first step is studying the seismic scattering in the far field is obtaining the incident waves in the far distance. First consider the SH-wave, for large distance

$$\mathbf{M}_{l\pm}^{j} = \frac{\sqrt{l(l+1)}}{k_S r} \sin\left(k_S r - \frac{l\pi}{2}\right) \mathbf{C}_{l\pm}, \tag{92}$$

so we have

$$\mathbf{SH} = \frac{1}{k_S r} \sum_l i^{l-2} c_l \left[i^{-l} e^{ik_S r} - i^l e^{-ik_S r}\right] \sqrt{l(l+1)} \left(\mathbf{C}_{l1} + l(l+1)\mathbf{C}_{l,-1}\right), \tag{93}$$

The asymptotic form of $\mathbf{N}$ is

$$\mathbf{N}_{l\pm}^{j} = \frac{\sqrt{l(l+1)}}{k_S r} \cos\left(k_S r - \frac{l\pi}{2}\right) \mathbf{B}_{l\pm}, \tag{94}$$

so for SV-wave we find

$$\mathbf{SV} = \frac{1}{k_S r} \sum_l i^{l-1} c_l \left[i^{-l} e^{ik_S r} + i^l e^{-ik_S r}\right] \sqrt{l(l+1)} \left(\mathbf{B}_{l1} - l(l+1)\mathbf{B}_{l,-1}\right), \tag{95}$$

on the other hand we have

$$\sqrt{l(l+1)} \left(\mathbf{B}_{l1} - l(l+1)\mathbf{B}_{l,-1}\right) = 2\left(\cos\varphi \frac{dP_{l1}}{d\theta} \hat{\boldsymbol{\theta}} - \sin\varphi \frac{P_{l1}}{\sin\theta} \hat{\boldsymbol{\varphi}}\right),$$

$$\sqrt{l(l+1)} \left(\mathbf{C}_{l1} + l(l+1)\mathbf{C}_{l,-1}\right) = 2i\left(\cos\varphi \frac{P_{l1}}{\sin\theta} \hat{\boldsymbol{\theta}} - \sin\varphi \frac{dP_{l1}}{d\theta} \hat{\boldsymbol{\varphi}}\right).$$

Therefore (93) and (95) lead to

$$\mathbf{SH} = \frac{2}{k_S r} \sum_l i^{l-1} c_l \left[i^{-l} e^{ik_S r} - i^l e^{-ik_S r}\right] \left(\cos\varphi \frac{P_{l1}}{\sin\theta} \hat{\boldsymbol{\theta}} - \sin\varphi \frac{dP_{l1}}{d\theta} \hat{\boldsymbol{\varphi}}\right)' \tag{96}$$

$$\mathbf{SV} = \frac{2}{k_S r} \sum_l i^{l-1} c_l \left[i^{-l} e^{ik_S r} + i^l e^{-ik_S r}\right] \left(\cos\varphi \frac{dP_{l1}}{d\theta} \hat{\boldsymbol{\theta}} - \sin\varphi \frac{P_{l1}}{\sin\theta} \hat{\boldsymbol{\varphi}}\right). \tag{97}$$

The partial wave expansion we obtained for SH- and SV-waves contain both waves converging to and diverging from the origin. As we will show later the effect of a perturbation in the medium appears as a phase shift in scattered waves diverging from origin.

SCATTERING OF P-WAVE

Let us analyze the scattering of the different types of waves. For an incident P-wave, we show in Eq.(81) that the azimuth number m is zero. In an isotropic medium the physical properties don't change in the different directions, so for the initially incident P-wave, scattered waves have zero m values.

The scattered wave field contains the P- and SV wave vector fields with $m = 0$. Since initially near the origin we have only the P-wave the radial part of the total wave for P-wave includes both Bessel and Neuman spherical functions. On the other hand for the SV-wave, we have just first order Hankel functions in the radial part of the wave

$$\mathbf{U} = \sum_l A_l^P \mathbf{L}_{l0}^j + B_l^P \mathbf{L}_{l0}^n + C_l^S \mathbf{N}_{l0}^h. \tag{98}$$

Here j refers to the spherical Bessel, n to the Neuman and h to the Hankel functions. To extract the partial wave analysis we need to write the total wave field in the asymptotic region. In this region as $kr \to \infty$ we have

$$j_{l\pm1}(k_P r) \approx \mp \frac{1}{k_P r} \cos\left(k_P r - \frac{l\pi}{2}\right), \tag{99}$$

$$n_{l\pm1}(k_P r) \approx \mp \frac{1}{k_P r} \sin\left(k_P r - \frac{l\pi}{2}\right), \tag{100}$$

$$h_{l\pm1}^{(1)}(k_S r) \approx \mp \frac{1}{k_S r} (-i)^l e^{ik_S r}. \tag{101}$$

Inserting the asymptotic form of the above functions in (65) to (67)

$$\mathbf{L}_{l0}^j(k_P r, \theta) = \frac{1}{k_P r} \cos\left(k_P r - \frac{l\pi}{2}\right) P_l(\theta)\hat{\mathbf{r}}, \tag{102}$$

$$\mathbf{L}_{l0}^n(k_P r, \theta) = \frac{1}{k_P r} \sin\left(k_P r - \frac{l\pi}{2}\right) P_l(\theta)\hat{\mathbf{r}}, \tag{103}$$

$$\mathbf{N}_{l0}^h(k_S r, \theta) = \frac{i^l}{k_S r} e^{ik_S r} \frac{dP_l(\theta)}{d\theta} \hat{\boldsymbol{\theta}}. \tag{104}$$

We note that from (102) and (103) the angle-dependent part of the PP-wave can be effectively described by the Legender function $P_l(\theta)$. Furthermore for the PS-wave the angle dependent part is the first derivative of the Legender function. As we will see later the same dependency upon the Legender function will be established for PP- and PS-scattering amplitudes. Inserting (102) to (104) to (98) results

$$\mathbf{U} = \sum_l \frac{1}{k_P r} \left[A_l^P \cos\left(k_P r - \frac{l\pi}{2}\right) + B_l^P \sin\left(k_P r - \frac{l\pi}{2}\right)\right] P_l(\theta)\hat{\mathbf{r}} + C_l^S \frac{i^l}{k_S r} e^{ik_S r} \frac{dP_l(\theta)}{d\theta} \hat{\boldsymbol{\theta}}. \tag{105}$$

Now we define the amplitudes for the incident and scattered P-waves in terms of a phase factor δ_l^P as follows

$$A_l^P = D_l^P \cos \delta_l^P, \qquad B_l^P = D_l^P \sin \delta_l^P, \qquad \tan \delta_l^P = \frac{B_l^P}{A_l^P}. \tag{106}$$

As a result we may try to rewrite the wavefield in terms of phase shift

$$\mathbf{U} = \frac{e^{-ik_P r}}{2k_P r}\sum_l (D_l^P e^{-i\delta_l^P} i^l) P_l(\theta)\hat{\mathbf{r}} +$$

$$\frac{e^{ik_P r}}{r}\sum_l \frac{1}{2k_P}\left[D_l^P e^{i\delta_l^P} e^{-i\frac{l\pi}{2}}\right] P_l(\theta)\hat{\mathbf{r}} + \frac{e^{ik_S r}}{r}\sum_l \frac{i^l C_l^S}{k_S}\frac{dP_l(\theta)}{d\theta}\hat{\boldsymbol{\theta}}. \quad (107)$$

The scattering amplitude of the P-wave is a complex function that has amplitude and phase. The main idea behind the partial wave analysis is that we can determine the scattering amplitude in terms of a real phase factor. To do this, first we write the wave field in terms of scattering amplitudes

$$\mathbf{U} = \frac{1}{k_P r}\sum_l i^{l-1}\cos\left(k_P r - \frac{l\pi}{2}\right) P_l(\theta)\hat{\mathbf{r}} + \frac{e^{ik_P r}}{r}\sum_l \mathcal{F}_l^{PP}(\theta)\hat{\mathbf{r}} + \frac{e^{ik_S r}}{r}\sum_l \mathcal{F}_l^{PS}(\theta)\hat{\boldsymbol{\theta}}. \quad (108)$$

Where the first term is the initial P-wave in the z-direction, the second term is the scattered PP-wave and the third term is scattered PS-wave. In addition $\mathcal{F}_l^{PP}(\theta)$ and $\mathcal{F}_l^{PS}(\theta)$ are the scattering patterns for scattered PP and PS-wave modes. Separating the spherical out-going waves in (108) takes the form

$$\mathbf{U} = \frac{e^{-ik_P r}}{2k_P r}\sum_l i^{2l-1} P_l(\theta)\hat{\mathbf{r}} + \frac{e^{ik_P r}}{r}\left\{\sum_l \mathcal{F}_l^{PP}(\theta) - i\frac{P_l(\theta)}{2k_P}\right\}\hat{\mathbf{r}} + \frac{e^{ik_S r}}{r}\sum_l \mathcal{F}_l^{PS}(\theta)\hat{\boldsymbol{\theta}}. \quad (109)$$

Comparing (107) and (109) we obtain

$$D_l^P = i^{l-1} e^{i\delta_l^P}. \quad (110)$$

In addition, the scattering amplitudes for PP- and PS-waves can be expressed in terms of phase shift

$$\mathcal{F}_l^{PP}(\theta) = \frac{P_l(\theta)}{k_P} e^{i\delta_l^P}\sin\delta_l^P, \quad (111)$$

$$\mathcal{F}_l^{PS}(\theta) = i^l \frac{C_l^S}{k_S}\frac{dP_l(\theta)}{d\theta}. \quad (112)$$

Therefore the PP-scattering amplitude can be determined by having the phase shift δ_l^P. If the phase shift goes to zero we don't have the scattered P-wave, namely the perturbation in the medium appears as a phase shift in scattering potential. The advantages of partial wave analysis is that determination of a complex amplitude with a real and imaginary parts reduces to determination of a real phase number. However the above analysis does not hold for the scattered PS wave because the incident wave is a P-wave. Let us insert (110) in (113) and see the difference between the wave field in a reference medium without perturbation and an actual medium with perturbation.

$$\mathbf{U} = \frac{e^{-ik_P r}}{r}\sum_l \frac{(-1)^l}{2ik_P} P_l(\theta)\hat{\mathbf{r}} + \frac{e^{ik_P r}}{r}\sum_l \frac{e^{2i\delta_l^P}}{2ik_P} P_l(\theta)\hat{\mathbf{r}} + \frac{e^{ik_S r}}{r}\sum_l C_l^S \frac{i^l}{k_S}\frac{dP_l(\theta)}{d\theta}\hat{\boldsymbol{\theta}}. \quad (113)$$

Comparing this to the case with no perturbation, we observe that for P-wave, spherical wave emerging from scatterer there is a an additional phase shift $e^{2i\delta_l^P}$. In a word the effects of perturbation in the medium appear as a phase shift in outgoing spherical P-wave. For the S-wave we don't have such a phase shift because the initial incident wave is P-wave.

The scattering amplitude contains the perturbation in density and elastic properties, namely

$$\mathcal{F}_l^{PP}(\theta, \delta\rho, \delta\lambda, \delta\mu) = P_l(\theta)\Delta_l^{PP}(\delta\rho, \delta\lambda, \delta\mu), \tag{114}$$

$$\mathcal{F}_l^{PS}(\theta, \delta\rho, \delta\lambda, \delta\mu) = \frac{dP_l(\theta)}{d\theta}\Delta_l^{PS}(\delta\rho, \delta\lambda, \delta\mu), \tag{115}$$

where we defined the perturbation-dependent term as

$$\Delta_l^{PP}(\delta\rho, \delta\lambda, \delta\mu) = \frac{1}{k_P}e^{i\delta_l^P}\sin\delta_l^P, \tag{116}$$

$$\Delta_l^{PS}(\delta\rho, \delta\lambda, \delta\mu) = \frac{1}{k_S}i^l C_l^S. \tag{117}$$

As a result the scattering pattern can be written as product of an angular term and a term containing the perturbation of properties of the medium. Let us consider the three first terms of the scattering pattern, in this case

$$\mathcal{F}^{PP} = \Delta_0^{PP} - \frac{1}{2}\Delta_2^{PP} + \cos\theta\Delta_1^{PP} + \frac{3}{2}\cos^2\theta\Delta_2^{PP}, \tag{118}$$

also for PS-scattering potential we have

$$\mathcal{F}^{PS} = -\sin\theta\Delta_1^{PS} - 3\sin\theta\cos\theta\Delta_2^{PS}. \tag{119}$$

Note that in the absence of any perturbation in medium, phase shift δ_l^P for all values of l is zero. This means that the phase shift measures how the wave in the reference medium differs from the wave in the actual medium. Now let us derive the Δ_l^{PP} in terms of medium properties. The scattering potential for PP-wave mode is given by

$$\mathcal{F}^{PP} = \rho_0\left[\left(-1 + \cos\theta + \frac{2}{\gamma_0^2}\sin^2\theta\right)\delta\tilde{\rho} - 2\delta\tilde{\alpha} + \frac{4}{\gamma_0^2}\sin^2\theta\delta\tilde{\beta}\right]. \tag{120}$$

Collecting the same powers of $\cos$ we arrive at

$$\mathcal{F}^{PP} = \rho_0\left[\left(\frac{2}{\gamma_0^2} - 1\right)\delta\tilde{\rho} - 2\delta\tilde{\alpha} + \frac{4}{\gamma_0^2}\delta\tilde{\beta} + \cos\theta\delta\tilde{\rho} - \cos^2\theta\frac{2}{\gamma_0^2}(\delta\tilde{\rho} + 2\delta\tilde{\beta})\right]. \tag{121}$$

comparing (116) and (121) we obtain

$$\Delta_0^{PP} = \rho_0\left[\frac{1}{\gamma_0^2}\left(\delta\tilde{\rho} + 2\delta\tilde{\beta}\right) - (\delta\tilde{\rho} + 2\delta\tilde{\alpha})\right],$$

$$\Delta_1^{PP} = \rho_0\delta\tilde{\rho},$$

$$\Delta_2^{PP} = \rho_0\left[-\frac{2}{\gamma_0^2}(\delta\tilde{\rho} + 2\delta\tilde{\beta})\right].$$

Let us consider to an example: scattering from a hard sphere elastic medium where inside the sphere the wavefiled is zero. In this case the boundary condition for the wavefield is given by

$$A_l^P \mathbf{L}_{l0}^j(k_P a,\theta) + B_l^P \mathbf{L}_{l0}^n(k_P a,\theta) + C_l^S \mathbf{N}_{l0}^h(k_S a,\theta) = 0. \tag{122}$$

The above vector identity splits to two scalar equation with three unknown as

$$A_l^P j_{l-1}(k_P a) + B_l^P n_{l-1}(k_P a) + C_l(l+1) h_{l-1}(k_P a) = 0, \tag{123}$$

$$A_l^P j_{l+1}(k_P a) + B_l^P n_{l+1}(k_P a) - C_l l h_{l+1}(k_P a) = 0, \tag{124}$$

and solving these equation for A_l^P and B_l^P gives

$$\tan\delta_l^P = \frac{B_l^P}{A_l^P} = -\frac{j_{l-1}(k_P a) + j_{l+1}\frac{(l+1)h_{l-1}(k_P a)}{l h_{l+1}(k_P a)}}{n_{l-1}(k_P a) + n_{l+1}\frac{(l+1)h_{l-1}(k_P a)}{l h_{l+1}(k_P a)}}. \tag{125}$$

SCATTERING OF S-WAVE

Assume that the incident wave is an S-wave in the x-direction. Since the S-wave depends to the azimuthal number $m = \pm 1$, the scattered wave also has the same dependency upon m. This happens in a radially heterogenous medium, because the perturbations are independent of azimuth angle ϕ. In this case the total wavefield is a function of the basis vectors with $m = \pm 1$. At large distance we have P-, and SH- and SV waves, so the total wavefield is a function of $\mathbf{L}_{lm}$, $\mathbf{N}_{lm}$ and $\mathbf{M}_{lm}$.

$$\begin{aligned}\mathbf{U} = \sum_l C_l^P \left[\mathbf{L}_{l1}^h - l(l+1)\mathbf{L}_{l,-1}^h\right] + \\ A_l^{SV}\left[\mathbf{N}_{l1}^j - l(l+1)\mathbf{N}_{l,-1}^j\right] + B_l^{SV}\left[\mathbf{N}_{l1}^n - l(l+1)\mathbf{N}_{l,-1}^n\right] + \\ A_l^{SH}\left[\mathbf{M}_{l1}^j + l(l+1)\mathbf{M}_{l,-1}^j\right] + B_l^{SH}\left[\mathbf{M}_{l1}^n + l(l+1)\mathbf{M}_{l,-1}^n\right].\end{aligned} \tag{126}$$

In partial wave analysis we are interested in the asymptotic form of the total wave field in the far distance. This would help us to express the scattering amplitudes in terms of one real parameter called phase shift instead of a complex number which has real and imaginary parts. It follows from (102) to (104) for large values of kr

$$\mathbf{L}_{l\pm}^h = \frac{i^l e^{ik_P r}}{k_P r}\mathbf{P}_{l\pm}, \tag{127}$$

$$\begin{aligned}\mathbf{N}_{l\pm}^j &= \frac{\sqrt{l(l+1)}}{k_S r}\cos\left(k_S r - \frac{l\pi}{2}\right)\mathbf{B}_{l\pm}, \\ \mathbf{N}_{l\pm}^n &= \frac{\sqrt{l(l+1)}}{k_S r}\sin\left(k_S r - \frac{l\pi}{2}\right)\mathbf{B}_{l\pm}, \\ \mathbf{M}_{l\pm}^j &= \frac{\sqrt{l(l+1)}}{k_S r}\sin\left(k_S r - \frac{l\pi}{2}\right)\mathbf{C}_{l\pm},\end{aligned} \tag{128}$$

$$\mathbf{M}^n_{l\pm} = -\frac{\sqrt{l(l+1)}}{k_S r}\cos\left(k_S r - \frac{l\pi}{2}\right)\mathbf{C}_{l\pm}. \tag{129}$$

With this we can write (133) as

$$\mathbf{U} = \frac{e^{ik_P r}}{k_P r}\sum_l C^P_l i^l P_{l1}\hat{\mathbf{r}} +$$
$$\frac{2}{k_S r}\sum_l \left\{A^{SV}_l \cos\left(k_S r - \frac{l\pi}{2}\right) + B^{SV}_l \sin\left(k_P r - \frac{l\pi}{2}\right)\right\}\left(\cos\varphi\frac{dP_{l1}}{d\theta}\hat{\boldsymbol{\theta}} - \sin\varphi\frac{P_{l1}}{\sin\theta}\hat{\boldsymbol{\varphi}}\right)$$
$$\frac{2i}{k_S r}\sum_l \left\{A^{SH}_l \sin\left(k_S r - \frac{l\pi}{2}\right) - B^{SH}_l \cos\left(k_S r - \frac{l\pi}{2}\right)\right\}\left(\cos\varphi\frac{P_{l1}}{\sin\theta}\hat{\boldsymbol{\theta}} - \sin\varphi\frac{dP_{l1}}{d\theta}\hat{\boldsymbol{\varphi}}\right). \tag{130}$$

Now let us define two phase factors for SV- and SH-waves such that the amplitudes take the forms.

$$A^{SV}_l = C^{SV}_l \cos\delta^{SV}_l, \qquad B^{SV}_l = C^{SV}_l \sin\delta^{SV}_l, \tag{131}$$
$$A^{SH}_l = C^{SH}_l \cos\delta^{SH}_l, \qquad B^{SH}_l = C^{SH}_l \sin\delta^{SH}_l. \tag{132}$$

Therefore we can identify the wavefield in terms of phase shifts

$$\mathbf{U} = \frac{e^{ik_P r}}{k_P r}\sum_l A^P_l i^{-l}\cos\varphi P_{l1}\hat{\mathbf{r}} +$$
$$\frac{1}{k_S r}\sum_l \left\{\left(C^{SV}_l i^{-l} e^{-i\delta^{SV}_l}\right)e^{ik_S r} + \left(C^{SV}_l i^{l} e^{i\delta^{SV}_l}\right)e^{-ik_S r}\right\}\left(\cos\varphi\frac{dP_{l1}}{d\theta}\hat{\boldsymbol{\theta}} - \sin\varphi\frac{P_{l1}}{\sin\theta}\hat{\boldsymbol{\varphi}}\right) +$$
$$\frac{1}{k_S r}\sum_l \left\{\left(C^{SH}_l i^{-l} e^{-i\delta^{SH}_l}\right)e^{ik_S r} - \left(C^{SH}_l i^{l} e^{i\delta^{SH}_l}\right)e^{-ik_S r}\right\}\left(\cos\varphi\frac{P_{l1}}{\sin\theta}\hat{\boldsymbol{\theta}} - \sin\varphi\frac{dP_{l1}}{d\theta}\hat{\boldsymbol{\varphi}}\right). \tag{133}$$

where we split the spherical incident and outgoing wave field. On the other hand we can write the wave field in terms of scattering amplitudes for P-, SH- and SV-waves. For an incident S-wave propagating in the z-direction with polarization in the x-direction, the incident plus scattered wave to be

$$\mathbf{U} = \mathbf{x}e^{ikz} + \frac{e^{ik_P r}}{k_P r}\sum_l \mathcal{F}^P_l\hat{\mathbf{r}} + \frac{e^{ik_S r}}{k_S r}\sum_l \mathcal{F}^{SV}_l\hat{\boldsymbol{\theta}} + \frac{e^{ik_S r}}{k_S r}\sum_l \mathcal{F}^{SH}_l\hat{\boldsymbol{\varphi}}. \tag{134}$$

Using the expansion of the incident wave the result is

$$\mathbf{U} = \frac{e^{ik_P r}}{k_P r}\sum_l \mathcal{F}^P_l\hat{\mathbf{r}} + \frac{e^{ik_S r}}{k_S r}\sum_l \mathcal{F}^{SV}_l\hat{\boldsymbol{\theta}} + \frac{e^{ik_S r}}{k_S r}\sum_l \mathcal{F}^{SH}_l\hat{\boldsymbol{\varphi}} +$$
$$\cos\varphi\frac{e^{-ik_S r}}{k_S r}\sum_l i^{2l-1}c_l\left(\frac{dP_{l1}}{d\theta} - \frac{P_{l1}}{\sin\theta}\right)\hat{\boldsymbol{\theta}} + \frac{e^{ik_S r}}{k_S r}\sum_l \frac{c_l}{i}\left(\frac{dP_{l1}}{d\theta} + \frac{P_{l1}}{\sin\theta}\right)\cos\varphi\hat{\boldsymbol{\theta}}$$
$$\sin\varphi\frac{e^{-ik_S r}}{k_S r}\sum_l i^{2l-1}c_l\left(\frac{dP_{l1}}{d\theta} - \frac{P_{l1}}{\sin\theta}\right)\hat{\boldsymbol{\varphi}} - \frac{e^{ik_S r}}{k_S r}\sum_l \frac{c_l}{i}\left(\frac{dP_{l1}}{d\theta} + \frac{P_{l1}}{\sin\theta}\right)\sin\varphi\hat{\boldsymbol{\varphi}}. \tag{135}$$

Comparing with the Eq. (133) we arrive at

$$\mathcal{F}_l^P = A_l^P i^{-l} \cos\varphi P_{l1}, \tag{136}$$

$$C_l^{SV} = i^{l-1} c_l e^{-i\delta_l^{SV}}, \tag{137}$$

$$C_l^{SH} = i^{l-1} c_l e^{-i\delta_l^{SH}}, \tag{138}$$

and

$$\mathcal{F}_l^{SV} = -2c_l \left[e^{-i\delta_l^{SV}} \sin\delta_l^{SV} \frac{dP_{l1}}{d\theta} + e^{-i\delta_l^{SH}} \sin\delta_l^{SH} \frac{P_{l1}}{\sin\theta} \right] \cos\varphi, \tag{139}$$

$$\mathcal{F}_l^{SH} = 2c_l \left[e^{-i\delta_l^{SH}} \sin\delta_l^{SH} \frac{dP_{l1}}{d\theta} + e^{-i\delta_l^{SV}} \sin\delta_l^{SV} \frac{P_{l1}}{\sin\theta} \right] \sin\varphi. \tag{140}$$

Next we examine how the phase shift can be extracted using the green function.

GREEN'S FUNCTION

The equation of motion for elastic wave is given by

$$\left\{ \delta_{jk}\omega^2 \rho(\mathbf{r}) + \partial_i c_{ijkl}(\mathbf{r}) \partial_l \right\} G_{km}(\mathbf{r}, \mathbf{r}') = \delta_{jm} \delta(\mathbf{r} - \mathbf{r}'). \tag{141}$$

The Green's dyadic, $G_{km}(\mathbf{r}, \mathbf{r}')$, is the response at location $\mathbf{r}$ in the k-direction due to an impulsive force applied at $\mathbf{r}'$ in the mth direction and c_{ijkl} is stiffness tensor. The Green's function for unbounded media is given by(Ben-Menahem and Singh, 1981)

$$\mathrm{G}(\mathbf{r}, \mathbf{r}') = -i\frac{i}{\mu} k_\beta \sum_{l,m} \frac{1}{\Gamma_{lm}} \left[\frac{1}{l(l+1)} \mathbf{M}_{lm}^{(j)}(k_\beta r_<) \mathbf{M}_{lm}^{*(h)}(k_\beta r_>) + \right.$$

$$\left. \frac{1}{l(l+1)} \mathbf{N}_{lm}^{(j)}(k_\beta r_<) \mathbf{N}_{lm}^{*(h)}(k_\beta r_>) + \left(\frac{\alpha}{\beta}\right)^3 \mathbf{L}_{lm}^{(j)}(k_\alpha r_<) \mathbf{L}_{lm}^{*(h)}(k_\alpha r_>) \right]$$

where $r_> = \min(r, r')$ and $r_> = \max(r, r')$. In the case that the incident wave is a P-wave, the scattered waves are PP- and PS-wave modes, and as a result the Green's function reduces to the propagation of P- and SV waves as follows

$$\mathrm{G}(\mathbf{r}, \mathbf{r}') = -\frac{i}{\mu} k_\beta \sum_{l} \frac{1}{\Gamma_{l0}} \left[\frac{1}{l(l+1)} \mathbf{N}_{l0}^{(j)}(k_\beta r_<) \mathbf{N}_{l0}^{*(h)}(k_\beta r_>) + \left(\frac{\alpha}{\beta}\right)^3 \mathbf{L}_{l0}^{(j)}(k_\alpha r_<) \mathbf{L}_{l0}^{*(h)}(k_\alpha r_>) \right].$$

To extract the scattered wave from the incident wave, we have to split the wave equation into the inhomogeneous wave equation, namely a wave equation for a scattered wave in a reference medium with a source term constructed from incident wavefield and perturbation parameters. First the elastic wave equation for radially heterogenous medium is given by

$$\mu \boldsymbol{\nabla}^2 \mathbf{U} + (\lambda + \mu) \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{U} + \hat{\mathbf{r}} \frac{d\lambda}{dr} \boldsymbol{\nabla} \cdot \mathbf{U} + \frac{d\mu}{dr} \left(2\frac{\partial \mathbf{U}}{\partial r} + \hat{\mathbf{r}} \times \boldsymbol{\nabla} \times \mathbf{U} \right) + \rho\omega^2 \mathbf{U} = 0. \tag{142}$$

The Born approximation is given by

$$\mathbf{U} = \mathbf{U}^i + \mathbf{U}^S, \tag{143}$$

where $\mathbf{U}^i$ is incident wave and $\mathbf{U}^s$ is the scattered wave with $\parallel \mathbf{U}^s \parallel \ll \parallel \mathbf{U}^i \parallel$. In addition the perturbation in the properties of medium is given by

$$\lambda(r) = \lambda_0 + \delta\lambda(r),$$

$$\mu(r) = \mu_0 + \delta\mu(r),$$

$$\rho(r) = \rho_0 + \delta\rho(r),$$

where $\delta\lambda(r), \delta\mu(r), \delta\rho(r) \ll 1$. Inserting the Born approximation (143) and perturbation terms in above equation (142)

$$\mu_0\boldsymbol{\nabla}^2\mathbf{U}^i + (\lambda_0+\mu_0)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{U}^i + \rho_0\omega^2\mathbf{U}^i +$$
$$\mu_0\boldsymbol{\nabla}^2\mathbf{U}^s + (\lambda_0+\mu_0)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{U}^s + \rho_0\omega^2\mathbf{U}^s +$$
$$\delta\mu(r)\boldsymbol{\nabla}^2\mathbf{U}^i + (\delta\lambda(r)+\delta\mu(r))\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{U}^i + \delta\rho(r)\omega^2\mathbf{U}^i +$$
$$\delta\mu(r)\boldsymbol{\nabla}^2\mathbf{U}^S + (\delta\lambda(r)+\delta\mu(r))\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{U}^S + \delta\rho(r)\omega^2\mathbf{U}^S +$$
$$\hat{\mathbf{r}}\frac{d\delta\lambda}{dr}\boldsymbol{\nabla}\cdot(\mathbf{U}^i+\mathbf{U}^S) + \frac{d\delta\mu}{dr}\left(2\frac{\partial(\mathbf{U}^i+\mathbf{U}^S)}{\partial r} + \hat{\mathbf{r}}\times\boldsymbol{\nabla}\times(\mathbf{U}^i+\mathbf{U}^S)\right) = 0.$$

The first term is zero because it is the homogeneous wave equation for the incident wave. The second term is the same equation for the scattered wave, but it is not zero because the scattered wave does not satisfy the homogeneous wave equation. Furthermore with factors of the all terms with factors of the scattered wave and perturbation terms are zero. Finally the equation reduces to

$$\mu_0\boldsymbol{\nabla}^2\mathbf{U}^s + (\lambda_0+\mu_0)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{U}^s + \rho_0\omega^2\mathbf{U}^s + \mathbf{V}^B(\mathbf{U}^i) = 0, \tag{144}$$

where the vector potential term is given by

$$\mathbf{V}^B = \delta\mu\boldsymbol{\nabla}^2\mathbf{U}^i + (\delta\lambda+\delta\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{U}^i + \delta\rho\omega^2\mathbf{U}^i - \hat{\mathbf{r}}\delta\lambda\frac{d}{dr}\boldsymbol{\nabla}\cdot\mathbf{U}^i$$
$$-\delta\mu\frac{d}{dr}\left(2\frac{\partial\mathbf{U}^i}{\partial r} + \hat{\mathbf{r}}\times\boldsymbol{\nabla}\times\mathbf{U}^i\right). \tag{145}$$

To extract the above equation we used partial integration. Using the following relations

$$\boldsymbol{\nabla}^2\mathbf{U}^i = -k_\alpha^2\mathbf{U}^i, \tag{146}$$

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{U}^i = -k_\alpha^2\mathbf{U}^i, \tag{147}$$

$$\boldsymbol{\nabla}\cdot\mathbf{U}^i = -k_\alpha f_l(k_\alpha r)P_l, \tag{148}$$

the vector potential (145) can be written as

$$\mathbf{V}^B = i^{l-1}(2l+1)\left\{-k_\alpha^2(\delta\lambda+2\delta\mu)\mathbf{L}_{l0} + \delta\rho\omega^2\mathbf{L}_{l0} + k_\alpha\hat{\mathbf{r}}\delta\lambda\frac{\partial f_l(k_\alpha r)}{\partial r}P_l - 2\delta\mu\frac{\partial^2\mathbf{L}_{l0}}{\partial r^2}\right\}.$$

In addition using the following relations

$$k_\alpha^2\delta\mu = \rho_0\omega^2\frac{1}{\gamma_0^2}\left(\frac{\delta\rho}{\rho_0} + 2\frac{\delta\beta}{\beta_0}\right).$$

$$k_\alpha^2 \delta\lambda = \rho_0 \omega^2 \left(\frac{\delta\rho}{\rho_0} \left\{ 1 - \frac{2}{\gamma_0^2} \right\} + 2 \left\{ \frac{\delta\alpha}{\alpha_0} - \frac{2}{\gamma_0^2} \frac{\delta\beta}{\beta_0} \right\} \right),$$

we can write

$$\mathbf{V}^B(\mathbf{r}') = v_r(r') P_l(\cos\theta') \hat{\mathbf{r}}' + v_\theta(r') \frac{dP_l(\cos\theta')}{d\theta'} \hat{\boldsymbol{\theta}}'. \tag{149}$$

where

$$v_r(r) = \rho_0 \omega^2 \left[\frac{\delta\rho}{\rho_0} \left\{ 1 - \frac{2}{\gamma_0^2} \right\} - \frac{4}{\gamma_0^2} \frac{\delta\beta}{\beta_0} - 2 \frac{1}{\gamma_0^2} \left(\frac{\delta\rho}{\rho_0} + 2 \frac{\delta\beta}{\beta_0} \right) \frac{\partial^2}{\partial(k_\alpha r)^2} \right] \frac{\partial f_l(k_\alpha r)}{\partial k_\alpha r}, \tag{150}$$

and

$$v_\theta(r) = -2\rho_0 \omega^2 \left[\frac{\delta\alpha}{\alpha_0} + \frac{1}{\gamma_0^2} \left(\frac{\delta\rho}{\rho_0} + 2 \frac{\delta\beta}{\beta_0} \right) \frac{\partial^2}{\partial(k_\alpha r)^2} \right] \frac{f_l(k_\alpha r)}{k_\alpha r}. \tag{151}$$

The Lippmann-Schwinger integral equation expresses the scattered wave field in terms of a retarded Green's function and a Born potential term, and is given by

$$\mathbf{U}^s = -\int d\Omega' \int r'^2 dr' \mathbf{V}^B(\mathbf{r}') \cdot \mathrm{G}_>(\mathbf{r}, \mathbf{r}'), \tag{152}$$

where the retarded dydic Green's function is

$$\mathrm{G}_>(\mathbf{r}, \mathbf{r}') = \mathrm{G}_{rr}(r, r', \theta') \hat{\mathbf{r}}' \hat{\mathbf{r}} + \mathrm{G}_{r\theta}(r, r', \theta') \hat{\mathbf{r}}' \hat{\boldsymbol{\theta}} + \mathrm{G}_{\theta r}(r, r', \theta') \hat{\boldsymbol{\theta}}' \hat{\mathbf{r}} + \mathrm{G}_{\theta\theta}(r, r', \theta') \hat{\boldsymbol{\theta}}' \hat{\boldsymbol{\theta}}. \tag{153}$$

Components of the dydic Green's function are given by

$$\mathrm{G}_{rr}(r, r', \theta') = g_{rr}(r, r') P_l(\cos\theta') P_l(\cos\theta), \tag{154}$$

$$\mathrm{G}_{r\theta}(r, r', \theta') = g_{r\theta}(r, r') P_l(\cos\theta') \frac{dP_l(\cos\theta)}{d\theta}, \tag{155}$$

$$\mathrm{G}_{\theta r}(r, r', \theta') = g_{\theta r}(r, r') \frac{dP_l(\cos\theta')}{d\theta'} P_l(\theta), \tag{156}$$

$$\mathrm{G}_{\theta\theta}(r, r', \theta') = g_{\theta\theta}(r, r') \frac{dP_l(\cos\theta')}{d\theta'} \frac{dP_l(\cos\theta)}{d\theta}. \tag{157}$$

In addition, the radial parts of the green function components are

$$g_{rr}(r, r') = i \left(\frac{\alpha}{\beta} \right)^3 h_{l-1}^{(1)}(k_\alpha r) \frac{dj_l(k_\alpha r')}{k_\alpha dr'}, \tag{158}$$

$$g_{r\theta}(r, r') = i h_{l-1}^{(1)}(k_\beta r) \frac{j_l(k_\beta r')}{(k_\beta r')}, \tag{159}$$

$$g_{\theta r}(r, r') = i \left(\frac{\alpha}{\beta} \right)^3 h_{l-1}^{(1)}(k_\alpha r) \frac{j_l(k_\alpha r')}{k_\alpha r'}, \tag{160}$$

$$g_{\theta\theta}(r, r') = i \frac{h_{l-1}^{(1)}(k_\beta r)}{l(l+1)} \left(\frac{dj_l(k_\beta r')}{k_\beta dr'} + \frac{j_l(k_\beta r')}{k_\beta r'} \right), \tag{161}$$

Finally the scattered wave reduces to

$$\mathbf{U}^s = \frac{4\pi}{2l+1}\times$$
$$P_l(\theta)\left\{\int r'^2 dr' \left[g_{rr}(r,r')v_r(r') + l(l+1)g_{r\theta}(r,r')v_\theta(r')\right]\right\}\hat{\mathbf{r}}+$$
$$\frac{dP_l(\theta)}{d\theta}\left\{\int r'^2 dr' \left[g_{\theta r}(r,r')v_r(r') + l(l+1)g_{\theta\theta}(r,r')v_\theta(r')\right]\right\}\hat{\boldsymbol{\theta}}. \tag{162}$$

In the far distance we have

$$g_{rr}(r,r') \approx \frac{e^{ik_P r}}{r}\left[(-i)^{l+1}\left(\frac{\alpha}{\beta}\right)^3 \frac{dj_l(k_\alpha r')}{k_\alpha^2 dr'}\right] = \frac{e^{ik_P r}}{r} g_{rr}(r'),$$
$$g_{r\theta}(r,r') \approx \frac{e^{ik_S r}}{r}\left[(-i)^{l+1}\frac{j_l(k_\beta r')}{(k_\beta^2 r')}\right] = \frac{e^{ik_S r}}{r} g_{r\theta}(r'),$$
$$g_{\theta r}(r,r') \approx \frac{e^{ik_P r}}{r}\left[(-i)^{l+1}\left(\frac{\alpha}{\beta}\right)^3 \frac{j_l(k_\alpha r')}{k_\alpha^2 r'}\right] = \frac{e^{ik_P r}}{r} g_{\theta r}(r'),$$
$$g_{\theta\theta}(r,r') \approx \frac{e^{ik_S r}}{r}\left[\frac{(-i)^{l+1}}{l(l+1)}\left(\frac{dj_l(k_\beta r')}{k_\beta dr'} + \frac{j_l(k_\beta r')}{k_\beta r'}\right)\right] = \frac{e^{ik_S r}}{r} g_{\theta\theta}(r').$$

Now let us consider the scattered wave

$$\mathbf{U}^s = \frac{e^{ik_P r}}{r}\sum_l \mathcal{F}_l^P(\theta)\hat{\mathbf{r}} + \frac{e^{ik_S r}}{r}\sum_l \mathcal{F}_l^S(\theta)\hat{\boldsymbol{\theta}}. \tag{163}$$

Comparing (162) and (163) we get

$$\mathcal{F}_l^P(\theta) = \frac{4\pi}{2l+1}P_l(\theta)\int r'^2 dr' \left[g_{rr}(r')v_r(r') + l(l+1)g_{r\theta}(r')v_\theta(r')\right], \tag{164}$$

and

$$\mathcal{F}_l^S(\theta) = \frac{4\pi}{2l+1}\frac{dP_l(\theta)}{d\theta}\int r'^2 dr' \left[g_{\theta r}(r')v_r(r') + l(l+1)g_{\theta\theta}(r')v_\theta(r')\right]. \tag{165}$$

If we compare the above scattering potential with

$$\mathcal{F}_l^P(\theta) = \frac{P_l(\cos\theta)}{k_P}e^{i\delta_l^P}\sin\delta_l^P, \tag{166}$$
$$\mathcal{F}_l^S(\theta) = i^l\frac{C_l^S}{k_S}\frac{dP_l(\cos\theta)}{d\theta}, \tag{167}$$

we arrive at

$$e^{i\delta_l^P}\sin\delta_l^P = \frac{4\pi k_\alpha}{2l+1}\int r'^2 dr' \left[g_{rr}(r')v_r(r') + l(l+1)g_{\theta r}(r')v_\theta(r')\right], \tag{168}$$
$$C_l^S = \frac{4\pi k_\beta i^{-l}}{2l+1}\int r'^2 dr' \left[g_{r\theta}(r')v_r(r') + l(l+1)g_{\theta\theta}(r')v_\theta(r')\right]. \tag{169}$$

Equations (168) and (169) indicate that the phase shift for P-wave and the scattering amplitude for the PS-wave can be obtained using the components of the Green's function and the Born vector potential.

CONCLUSION

The Born approximation is based on two major assumptions. The first one is that the scattered wavefield is comparatively small relative to the incident wavefield. The second one is that the fractional perturbations in physical properties are small enough that we need keep only the first-order terms. Using the Born approximation and taking into account the far distance approximation the scattered wave field is a spherical wave function. This outgoing field in the presence of a perturbation in medium undergoes a phase shift relative to the case with no perturbation. This is the central idea of Partial wave analysis. Analysis for scalar waves(quantum problems) and electromagnetic waves are well studied. However for elastic waves the case is rarely investigated, especially the phase shift interpretation of the scattering. In this research we applied the conventional partial wave analysis used in quantum theory and electrodynamics to elastic wave scattering. We demonstrated that in scattering from a perturbation in elastic properties of a medium the scattered outgoing wave is a spherical wave modulated by a scattering amplitude. We also showed that the scattering amplitude can be be expressed by one parameter called phase shift. The only difference between the wave in the absence and presence of the perturbation is this phase shift. It describes all perturbations in the medium. In other words, by knowing the phase shift we can determine the physical properties in the medium that caused the scattering.

Partial wave analysis reduces the problem of scattering to a phase shift that has all information about the target. In the case of the Born approximation it is a function of the perturbations in the the medium. Phase shift itself is obtained using the Lippmann-Schwinger equation in the Born approximation context, where we assume that the scattered wave field is small comparing to the initial incident wave. In this approximation the medium is considered as a unperturbed or reference medium plus a perturbation term in the elastic properties. In this case elastic wave equation reduces to a wave equation in reference medium with scattered wavefield on one side, and a source term constructed of the perturbations and incident wave in other side. Thus, the scattering amplitude for elastic waves is summarized by determining the of phase shift which contains all the information about the medium properties. The most important feature of this method from the physics point of view is that the only effect of a perturbation is a phase shift in the scattered wavefield. From a mathematical perspective, it reduces the problem of finding a complex function (two real numbers) to a single real number, the phase shift.

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APPENDIX A: EXPANSION OF P- AND S-WAVES

Assume that the incident wave field is a compressional wavefield propagating in the z-direction and polarized in the z-direction

$$\mathbf{U}_0 = \hat{\mathbf{z}} e^{ikz}. \tag{170}$$

To expand (170) in terms of spherical harmonics, first we write the vector $\mathbf{z}$ in terms of spherical unit vectors

$$\mathbf{z}e^{ikz} = \mathbf{r}\cos\theta e^{ikz} - \boldsymbol{\theta}\sin\theta e^{ikz}, \tag{171}$$

using the expansion of the plane wave e^{ikz} the first term reduces to

$$\cos\theta e^{ikz} = \sum i^l(2l+1)j_l(kr)\cos\theta P_l(\cos\theta). \tag{172}$$

Using the following relation for the Legendre function

$$(2l+1)xP_l = (l+1)P_{l+1} + lP_{l-1}, \tag{173}$$

(172) reduces to

$$\cos\theta e^{ikz} = \sum i^{l-1}[lj_{l-1}(kr) - (l+1)j_{l+1}(kr)]P_l. \tag{174}$$

The second term uses the following

$$(2l+1)P_l = \frac{d}{dx}[P_{l+1} - P_{l-1}], \tag{175}$$

the second term of (171) becomes

$$-\sin\theta e^{ikz} = \sum i^{l-1}\left\{j_{l-1}(kr) + j_{l+1}(kr)\right\}\frac{dP_l}{d\theta}. \tag{176}$$

Inserting (174) and (176) in (171) we arrive at

$$\mathbf{z}e^{ikz} = \sum i^{l-1}\left[\left(l\mathbf{r}P_l + \boldsymbol{\theta}\frac{dP_l}{d\theta}\right)j_{l-1} + \left(-(l+1)\mathbf{r}P_l + \boldsymbol{\theta}\frac{dP_l}{d\theta}\right)j_{l+1}\right]. \tag{177}$$

Finally the incident P-wave is expressed in terms of Hansson vector $\mathbf{L}_{lm}$

$$\mathbf{z}e^{ikz} = \sum i^{l-1}\delta_{m0}\left[\boldsymbol{\Lambda}^1_{lm}j_{l-1} + \boldsymbol{\Lambda}^2_{lm}j_{l+1}\right] = \sum i^{l-1}(2l+1)\delta_{m0}\mathbf{L}_{lm} \tag{178}$$

Now let us consider the divergenceless wave field, called the S-wave. We assume the wave traveling in the z-direction with the polarization in the x-direction

$$\hat{\mathbf{x}}e^{ikz} = (\mathbf{r}\sin\theta\cos\varphi + \boldsymbol{\theta}\cos\theta\cos\varphi - \boldsymbol{\varphi}\sin\varphi)\sum_l i^l(2l+1)j_l(kr)P_l(\cos\theta). \tag{179}$$

Since the left-hand side is a divegenceless vector, the right-hand side should be, as well. As a result we can expand the right-hand side in terms of $\mathbf{M}$ and $\mathbf{N}$ vectors

$$\hat{\mathbf{x}}e^{ikz} = \sum_{lm} c_{lm}\mathbf{M}_{lm} + b_{lm}\mathbf{N}_{lm}. \tag{180}$$

Multiplication both sides by $\sum_{l'm'}\mathbf{M}^*_{l'm'}$ and integration over $d\Omega$ the right hand side reduces to

$$\int\left[\sum_{lm,l'm'} c_{lm}\mathbf{M}_{lm}\cdot\mathbf{M}^*_{l'm'} + b_{lm}\mathbf{N}_{lm}\cdot\mathbf{M}^*_{l'm'}\right]d\Omega = \sum_{lm} c_{lm}l(l+1)j_l^2\frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!}, \tag{181}$$

where we used the orthogonality of the Hanson vectors. The left hand side reduces to (we eliminate j_l^2 from both sides)

$$\sum_{lm}\int(\boldsymbol{\theta}\cos\theta\cos\varphi-\boldsymbol{\varphi}\sin\varphi)\cdot\boldsymbol{\Lambda}^{*3}_{lm}i^l(2l+1)P_l(\cos\theta)\sin\theta d\theta d\varphi, \tag{182}$$

where the integration over φ gives

$$\int(\boldsymbol{\theta}\cos\theta\cos\varphi-\boldsymbol{\varphi}\sin\varphi)\cdot\boldsymbol{\Lambda}^{*3}_{lm}d\varphi=\int\left\{\left[-im\frac{Y^*_{lm}}{\sin\theta}\right]\cos\theta\cos\varphi+\left[\frac{\partial Y^*_{lm}}{\partial\theta}\right]\sin\varphi\right\}d\varphi. \tag{183}$$

Regarding to the integrations over φ

$$\int d\varphi e^{-im\varphi}\cos\varphi=\int d\varphi\frac{e^{-im\varphi}}{2}(e^{i\varphi}+e^{-i\varphi})=\pi(\delta_{m,1}+\delta_{m,-1}), \tag{184}$$

$$\int d\varphi e^{-im\varphi}\sin\varphi=\int d\varphi\frac{e^{-im\varphi}}{2i}(e^{i\varphi}-e^{-i\varphi})=-i\pi(\delta_{m,1}-\delta_{m,-1}), \tag{185}$$

The right hand side of (180) reduces to

$$-i\delta_{m,1}\pi\left(\frac{mP_{lm}}{\sin\theta}\cos\theta+\frac{dP_{lm}}{d\theta}\right)-i\pi\delta_{m,-1}\left(\frac{mP_{lm}}{\sin\theta}\cos\theta-\frac{dP_{lm}}{d\theta}\right). \tag{186}$$

Using the following integral

$$\int_0^\pi\left(\frac{P_{l1}}{\sin\theta}\cos\theta+\frac{dP_{l1}}{d\theta}\right)P_l\sin\theta d\theta=\int_0^\pi P_{l1}P_{l1}\sin\theta d\theta=\frac{2l(l+1)}{2l+1} \tag{187}$$

we find that

$$c_{l1}=c_l=-i^{l+1}\frac{2l+1}{2l(l+1)}. \tag{188}$$

In the same manner for $m=-1$ considering to the following property of Pl

$$P_{l,-1}=-\frac{(l-1)!}{(l+1)!}=-\frac{1}{l(l+1)}, \tag{189}$$

we get

$$c_{l-1}=-\frac{i^{l+1}}{2}(2l+1)=l(l+1)c_l. \tag{190}$$

or

$$c_{lm}=\left[\delta_{m,1}+l(l+1)\delta_{m,-1}\right]c_l, \tag{191}$$

To obtain the b_{lm} coefficients, let us consider to the case of waves traveling in the z-direction with the polarization in the y-direction

$$\hat{\mathbf{y}}e^{ikz}=(\mathbf{r}\sin\theta\sin\varphi+\boldsymbol{\theta}\cos\theta\sin\varphi+\boldsymbol{\varphi}\cos\varphi)\sum_l i^l(2l+1)j_l(kr)P_l(\cos\theta). \tag{192}$$

$\hat{\mathbf{y}}e^{ikz}$, can be evaluated from $\hat{\mathbf{x}}e^{ikz}$ by

$$\hat{\mathbf{y}}e^{ikz}=\frac{1}{ik}\boldsymbol{\nabla}\times(\hat{\mathbf{x}}e^{ikz})=\sum_{lm}c_{lm}\frac{1}{ik}\boldsymbol{\nabla}\times\mathbf{M}_{lm}+b_{lm}\frac{1}{ik}\boldsymbol{\nabla}\times\mathbf{N}_{lm}. \tag{193}$$

we have

$$\mathbf{M} = \frac{1}{k}\boldsymbol{\nabla} \times \mathbf{N}, \qquad \mathbf{N} = \frac{1}{k}\boldsymbol{\nabla} \times \mathbf{M} \tag{194}$$

we have

$$\hat{\mathbf{y}}e^{ikz} = -i\sum_{lm} c_{lm}\mathbf{N}_{lm} + b_{lm}\mathbf{M}_{lm}, \tag{195}$$

In a similar manner to that in which we extracted c_{lm}, we obtain

$$b_{lm} = \left[\delta_{m,1} - l(l+1)\delta_{m,-1})\right] c_l, \tag{196}$$

inserting (191) and () in () and () we arrive at

$$\hat{\mathbf{x}}e^{ikz} = -\sum_{lm} i^{l+1}\frac{2l+1}{2l(l+1)}\left\{\left[\delta_{m,1} + l(l+1)\delta_{m,-1}\right]\mathbf{M}_{lm} + \left[\delta_{m,1} - l(l+1)\delta_{m,-1}\right]\mathbf{N}_{lm}\right\}, \tag{197}$$

$$\hat{\mathbf{y}}e^{ikz} = -i\sum_{lm} i^{l+1}\frac{2l+1}{2l(l+1)}\left\{\left[\delta_{m,1} + l(l+1)\delta_{m,-1}\right]\mathbf{N}_{lm} + \left[\delta_{m,1} - l(l+1)\delta_{m,-1}\right]\mathbf{M}_{lm}\right\}, \tag{198}$$

or

$$\hat{\mathbf{x}}e^{ikz} = \sum_{lm} c_l\left\{\left[M_{l1} + N_{l1}\right] + l(l+1)\left[M_{l,-1} - N_{l,-1}\right]\right\} \tag{199}$$

$$\hat{\mathbf{y}}e^{ikz} = i\sum_{lm} c_l\left\{\left[M_{l1} + N_{l1}\right] - l(l+1)\left[M_{l,-1} - N_{l,-1}\right]\right\} \tag{200}$$

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