Full viscoelastic waveform inversion: a mathematical framework

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ABSTRACT

In this report we study the multiparameter Full Waveform Inversion (FWI) in viscoelastic medium with five viscoelastic parameters. Parameters are, density, unrelaxed bulk modulus and shear modulus, and the differences between the relaxed and unrelaxed moduluses. Based on the Born approximation we derived the explicit form of the Fréchet kernels. We showed that how five viscoelastic parameters can be inverted in terms of the time and space partial derivatives of the forwarded and backwarded scattered wavefields. We also derived the Jacobian matrix that transform the three model of parametrization for inversion.

INTRODUTION

Full-waveform inversion (FWI) is a method to estimate the earth properties that affect the seismic wave-field, it is based on the minimizing the field data and synthetic seismogram generated from forward modeling (Virieux and Operto, 2009; Fichtner, 2010). An ultimate FWI technique should take attenuation and dispersions into account. Charara et al. (2000) study the FWI for a viscoelastic medium with nearly constant quality factor and examine their theoretical results for a 1D syntectic model. An appropriate choice of model parametrization is very important in Full Waveform Inversion. In an acoustic medium, the scattered wave field is described by density and P-wave velocity, however in more complex media more parameters are required. For example in a viscoelastic medium we need five parameters, three elastic properties, density, P- and S-wave velocities and two anelastic parameters related to the P- and S-wave quality factors.

VISCOELASTIC FULL WAVEFORM INVERSION

Gradient-based full-waveform inversion, is a method to estimate the subsurface parameters by iteratively minimizing the misfit function of the difference between recorded seismic data and modeled seismic data. Using the scattering potential obtained for viscoelastic medium we can compute the sensitivity of each scattered wave field to the perturbations. The gradient and the Hessian directly dependent on the radiation patterns. By including the attenuation in initial model the full waveform inversion will be more accurate. Due to frequency-dependent amplitude decrease and phase velocities, attenuation significantly affect amplitudes and phases of seismic signals (Causse et al., 1999).

The objective of full waveform inversion is to find an optimal Earth model, $\tilde{\mathbf{m}}$, that minimises the misfit functional, $(\mathbf{m})$, used to quantify the differences between the observed seismograms, $\mathbf{u}_0(\mathbf{x}, t)$, and the synthetic seismograms, $\mathbf{u}(\mathbf{m}; \mathbf{x}, t)$ (Fichtner, 2010). A model $\mathbf{m}$ is a functional of other quantities, the spatial distributions of the P wave velocity, $V_P(\mathbf{x})$, the S wave velocity, $V_S(\mathbf{x})$, density, $\rho(\mathbf{x})$ and Quality factors for P $Q_P(\mathbf{x})$ and S-waves $Q_S(\mathbf{x})$ that is

$$\mathbf{m}(\mathbf{x}) = [V_P(\mathbf{x}), V_S(\mathbf{x}), \rho(\mathbf{x}), Q_P(\mathbf{x}), Q_S(\mathbf{x})].$$

For a isotropic viscoelastic medium the displacement vector $\mathbf{u}(\mathbf{x},t)$ satisfies in the following equation

$$\rho(\mathbf{x})\frac{\partial^2 u^i}{\partial t^2}(\mathbf{x},t) - \frac{\partial \sigma^{ij}}{\partial x^j}(\mathbf{x},t) = 0, \tag{1}$$

where the linear relationship between stress tensor, $\sigma^{ij}(\mathbf{x},t)$ and strain tensor $\varepsilon^{ij}(\mathbf{x},t)$ can be expressed using the relaxation function $\Psi^{ijkl}(\mathbf{x},t)$ as

$$\sigma^{ij}(\mathbf{x},t) = \int_{-\infty}^{+\infty} dt'\Psi^{ijkl}(\mathbf{x},t-t')\varepsilon^{kl}(\mathbf{x},t'). \tag{2}$$

Time dependency of a material is characterized by the relaxation function, which is a fourth order tensor. The relaxation function for a standard linear model is given by (Carcione, 2014)

$$\Psi^{ijkl}(\mathbf{x},t) \equiv \Psi^{ijkl}(\mathbf{x},+\infty)\left[1 - \frac{1}{L}\sum_{l=1}^{L}\left(1 - \frac{\tau_l^{(\varepsilon)}(\mathbf{x})}{\tau_l^{(\sigma)}(\mathbf{x})}\right)\exp\left(-\frac{t}{\tau_l^{(\sigma)}(\mathbf{x})}\right)\right]H(t), \tag{3}$$

Here $\tau^{(\varepsilon)}$ and $\tau_l^{(\sigma)}$ are the stress and strain relaxation times of the lth mechanism. At time $t \to 0^+$, (3) reduces to

$$\Psi^{ijkl}(\mathbf{x},0^+) \equiv \Psi^{ijkl}(\mathbf{x},+\infty)\left[1 - \frac{1}{L}\sum_{l=1}^{L}\left(1 - \frac{\tau_l^{(\varepsilon)}(\mathbf{x})}{\tau_l^{(\sigma)}(\mathbf{x})}\right)\right].$$

So that we obtain the ratio of the relaxation times for stress and strain as a constant

$$\frac{\Psi^{ijkl}(\mathbf{x},0^+)}{\Psi^{ijkl}(\mathbf{x},+\infty)} = \frac{\tau_l^{(\varepsilon)}(\mathbf{x})}{\tau_l^{(\sigma)}(\mathbf{x})}, \tag{4}$$

this assumption is valid for seismic frequency bandwidth (Carcione, 2014). As a result, inserting the relaxation function (3) in constitutive equation (2)

$$\sigma^{ij}(\mathbf{x},t) = \Psi^{ijkl}(\mathbf{x},+\infty)\int_{-\infty}^{+\infty} dt'$$

$$\left[H(t-t')\varepsilon^{kl}(\mathbf{x},t') - \frac{1}{L}\sum_{\nu=1}^{L}\left(1 - \frac{\Psi^{ijkl}(\mathbf{x},0^+)}{\Psi^{ijkl}(\mathbf{x},+\infty)}\right)\exp\left(-\frac{t-t'}{\tau_\nu^{(\sigma)}(\mathbf{x})}\right)H(t-t')\varepsilon^{kl}(\mathbf{x},t')\right] =$$

$$\int_{-\infty}^{+\infty} dt' H(t-t')\varepsilon^{kl}(\mathbf{x},t')\Psi^{ijkl}(\mathbf{x},+\infty)$$

$$-\left(\Psi^{ijkl}(\mathbf{x},+\infty) - \Psi^{ijkl}(\mathbf{x},0^+)\right)\int_{-\infty}^{+\infty} dt'\frac{1}{L}\sum_{\nu=1}^{L}\exp\left(-\frac{t-t'}{\tau_\nu^{(\sigma)}(\mathbf{x})}\right)H(t-t')\varepsilon^{kl}(\mathbf{x},t').$$

So we can write the constitutive law as

$$\sigma^{ij}(\mathbf{x},t) = \Psi^{ijkl}(\mathbf{x},0^+)\varepsilon^{kl}(\mathbf{x},t) + \Delta\Psi^{ijkl}(\mathbf{x})\sum_{\nu=1}^{L}\epsilon_\nu^{kl}(\mathbf{x},t), \tag{5}$$

where the memory strain variables $\epsilon_\nu^{kl}(\mathbf{x},t)$ is defined as (Carcione et al., 1988)

$$\epsilon_\nu^{kl}(\mathbf{x},t) = \frac{1}{L}\int_{-\infty}^{+\infty} dt' \exp\left(-\frac{t-t'}{\tau_\nu^{(\sigma)}(\mathbf{x})}\right) H(t-t')\varepsilon^{kl}(\mathbf{x},t'). \tag{6}$$

In addition $\Delta\Psi^{ijkl}$ is the difference between relaxed and unrelaxed modulus.

$$\Delta\Psi^{ijkl} = \Psi^{ijkl}(\mathbf{x},+\infty) - \Psi^{ijkl}(\mathbf{x},0^+). \tag{7}$$

Next, we apply the single scattering approximation called Born approximation to the constitutive law (5) and wave equation (1). Perturbations in density, relaxation function, memory variable, stress and displacement are given by

$$\begin{aligned}
\rho(\mathbf{x}) &\longrightarrow \rho(\mathbf{x}) + \delta\rho(\mathbf{x}),\\
\Psi^{ijkl}(\mathbf{x},t) &\longrightarrow \Psi^{ijkl}(\mathbf{x},t) + \delta\Psi^{ijkl}(\mathbf{x},t),\\
\epsilon^{kl}(\mathbf{x},t) &\longrightarrow \epsilon^{kl}(\mathbf{x},t) + \delta\epsilon^{kl}(\mathbf{x},t),\\
\varepsilon^{kl}(\mathbf{x},t) &\longrightarrow \varepsilon^{kl}(\mathbf{x},t) + \delta\varepsilon^{kl}(\mathbf{x},t),\\
u^i(\mathbf{x},t) &\longrightarrow u^i(\mathbf{x},t) + \delta u^i(\mathbf{x},t).
\end{aligned} \tag{8}$$

Inserting (8) in equations (1) and (5)

$$(\rho(\mathbf{x}) + \delta\rho(\mathbf{x}))\frac{\partial^2(u^i+\delta u^i)}{\partial t^2}(\mathbf{x},t) - \frac{\partial(\sigma^{ij}+\delta\sigma^{ij})}{\partial x^j}(\mathbf{x},t) = 0, \tag{9}$$

$$\begin{aligned}
\sigma^{ij}(\mathbf{x},t) + \delta\sigma^{ij}(\mathbf{x},t) &= \left\{\Psi^{ijkl}(\mathbf{x},0^+) + \delta\Psi^{ijkl}(\mathbf{x},0^+)\right\}\left\{\varepsilon^{kl}(\mathbf{x},t) + \delta\varepsilon^{kl}(\mathbf{x},t)\right\}\\
&+ \left\{\Delta\Psi^{ijkl}(\mathbf{x}) + \delta\Delta\Psi^{ijkl}(\mathbf{x})\right\}\sum_{\nu=1}^{L}\left\{\epsilon_\nu^{kl}(\mathbf{x},t) + \delta\epsilon_\nu^{kl}(\mathbf{x},t)\right\},
\end{aligned} \tag{10}$$

keeping the first order of perturbations we arrive at the following inhomogeneous equations with new sources

$$\rho(\mathbf{x})\frac{\partial^2\delta u^i}{\partial t^2}(\mathbf{x},t) - \frac{\partial\delta\sigma^{ij}}{\partial x^j}(\mathbf{x},t) = -\delta\rho(\mathbf{x})\frac{\partial^2 u^i}{\partial t^2}(\mathbf{x},t), \tag{11}$$

$$\begin{aligned}
&\delta\sigma^{ij}(\mathbf{x},t) - \Psi^{ijkl}(\mathbf{x},0^+)\delta\varepsilon^{kl}(\mathbf{x}) - \Delta\Psi^{ijkl}(\mathbf{x})\sum_{\nu=1}^{L}\delta\epsilon^{kl}(\mathbf{x},t)\\
&= \delta\Psi^{ijkl}(\mathbf{x},0^+)\varepsilon^{kl}(\mathbf{x},t) + \delta\Delta\Psi^{ijkl}(\mathbf{x})\sum_{\nu=1}^{L}\epsilon_\nu^{kl}(\mathbf{x},t).
\end{aligned} \tag{12}$$

So the right hand side of equation (11) represents the force term and right hand side of (12) displays the moment source term. Now, using the integral solution of the wave equation and given the new source terms, scattered wave δu^i is given by (Tarantola, 1988)

$$\begin{aligned}
\delta u^i(\mathbf{x},t) = & \\
& -\int_V d\mathbf{x}' \int_{t_0}^{t_1} dt' G^{ij}(\mathbf{x},t;\mathbf{x}',t')\delta\rho(\mathbf{x}')\frac{\partial \delta u^j}{\partial t^2}(\mathbf{x}',t') \\
& -\int_V d\mathbf{x}' \int_{t_0}^{t_1} dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x},t;\mathbf{x}',t')\delta\Psi^{jklm}(\mathbf{x}',0^+)\varepsilon^{lm}(\mathbf{x}',t') \\
& -\int_V d\mathbf{x}' \int_{t_0}^{t_1} dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x},t;\mathbf{x}',t')\delta\Delta\Psi^{jklm}(\mathbf{x}')\sum_{\nu=1}^{L}\epsilon_\nu^{lm}(\mathbf{x}',t').
\end{aligned} \tag{13}$$

Where the retarded or causal Green's tensor $G^{ij}(\mathbf{x},t;\mathbf{x}',t')$, propagate the wave forward in time from $(\mathbf{x}',t') < (\mathbf{x},t)$ to $(\mathbf{x},t)$. We consider to the linear isotropic homogeneous case where the relaxation function Ψ^{ijkl} reduces to

$$\Psi^{jklm}(\mathbf{x}) = \delta^{jk}\delta^{lm}\kappa(\mathbf{x}) + \left(\delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl} - \frac{2}{3}\delta^{jk}\delta^{lm}\right)\mu(\mathbf{x}), \tag{14}$$

then we have

$$\begin{aligned}
&\delta u^i(\mathbf{x},t) = \\
&-\int_V d\mathbf{x}' \int_{t_0}^{t_1} dt' G^{ij}(\mathbf{x},t;\mathbf{x}',t')\frac{\partial \delta u^j}{\partial t^2}(\mathbf{x}',t')\delta\rho(\mathbf{x}') \\
&-\int_V d\mathbf{x}' \int_{t_0}^{t_1} dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x},t;\mathbf{x}',t')\delta^{jk}\delta^{lm}\varepsilon^{kl}(\mathbf{x}',t')\delta\kappa(\mathbf{x}',0^+) \\
&-\int_V d\mathbf{x}' \int_{t_0}^{t_1} dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x},t;\mathbf{x}',t')\left(\delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl} - \frac{2}{3}\delta^{jk}\delta^{lm}\right)\varepsilon^{lm}(\mathbf{x}',t')\delta\mu(\mathbf{x}',0^+) \\
&-\int_V d\mathbf{x}' \int_{t_0}^{t_1} dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x},t;\mathbf{x}',t')\delta^{jk}\delta^{lm}\sum_{\nu=1}^{L}\epsilon_\nu^{lm}(\mathbf{x}',t')\delta(\Delta\kappa(\mathbf{x}')) \\
&-\int_V d\mathbf{x}' \int_{t_0}^{t_1} dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x},t;\mathbf{x}',t')\left(\delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl} - \frac{2}{3}\delta^{jk}\delta^{lm}\right)\sum_{\nu=1}^{L}\epsilon_\nu^{lm}(\mathbf{x}',t')\delta(\Delta\mu(\mathbf{x}')),
\end{aligned} \tag{15}$$

considering to the perturbation in the displacement in terms of perturbations in the model parameters we have

$$\begin{aligned}
&\delta\mathbf{u} = \int_V d\mathbf{x}'\frac{\partial \mathbf{u}}{\partial \mathbf{m}}\delta\mathbf{m} = \\
&\int_V d\mathbf{x}'\left\{\frac{\partial \mathbf{u}}{\partial \rho}\delta\rho + \frac{\partial \mathbf{u}}{\partial \kappa}\delta\kappa + \frac{\partial \mathbf{u}}{\partial \mu}\delta\mu + \frac{\partial \mathbf{u}}{\partial \Delta\kappa}\delta\Delta\kappa + \frac{\partial \Delta\mathbf{u}}{\partial \Delta\mu}\delta\Delta\mu\right\}.
\end{aligned} \tag{16}$$

where the Fr$\acute{e}$chet kernels are given by

$$\begin{aligned}
\frac{\partial u_i}{\partial \rho} &= -\int_{t_0}^{t_1} dt' G^{ij}(\mathbf{x},t;\mathbf{x}',t') \frac{\partial u^j}{\partial t^2}(\mathbf{x}',t'), \\
\frac{\partial u_i}{\partial \kappa} &= -\int_{t_0}^{t_1} dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x},t;\mathbf{x}',t') \delta^{jk}\delta^{lm}\varepsilon^{lm}(\mathbf{x}',t'), \\
\frac{\partial u_i}{\partial \mu} &= -\int_{t_0}^{t_1} dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x},t;\mathbf{x}',t'), \left(\delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl} - \frac{2}{3}\delta^{jk}\delta^{lm}\right) \varepsilon^{lm}(\mathbf{x}',t'), \\
\frac{\partial u_i}{\partial \Delta\kappa} &= -\int_{t_0}^{t_1} dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x},t;\mathbf{x}',t') \delta^{jk}\delta^{lm} \sum_{\nu=1}^{L} \epsilon_\nu^{lm}(\mathbf{x}',t'), \\
\frac{\partial \Delta u_i}{\partial \Delta\mu} &= -\int_{t_0}^{t_1} dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x},t;\mathbf{x}',t') \left(\delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl} - \frac{2}{3}\delta^{jk}\delta^{lm}\right) \sum_{\nu=1}^{L} \epsilon_\nu^{lm}(\mathbf{x}',t').
\end{aligned} \tag{17}$$

To invert the properties we calculate the adjoint of equation (16)

$$\delta\hat{\mathbf{m}}(\mathbf{x}') = \int d\mathbf{x} \int dt \left[\frac{\partial \mathbf{u}}{\partial \mathbf{m}}(\mathbf{x},t;\mathbf{x}')\right]^* \delta\mathbf{u}(\mathbf{x},t),$$

so the inverted properties can be written as

$$\begin{aligned}
&\delta\rho(\mathbf{x}') = \\
&-\int d\mathbf{x} \int dt' \int dt G^{ij}(\mathbf{x}',t';\mathbf{x},t) \frac{\partial^2 u^j}{\partial t^2}(\mathbf{x}',t') \delta u_i(\mathbf{x},t), \\
&\delta\kappa(\mathbf{x}',0^+) = \\
&-\int d\mathbf{x} \int dt \int dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x}',t';\mathbf{x},t) \delta^{jk}\delta^{lm}\varepsilon^{lm}(\mathbf{x}',t') \delta u_i(\mathbf{x},t), \\
&\delta\mu(\mathbf{x}',0^+) = \\
&-\int d\mathbf{x} \int dt \int dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x}',t';\mathbf{x},t) \left(\delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl} - \frac{2}{3}\delta^{jk}\delta^{lm}\right) \varepsilon^{lm}(\mathbf{x}',t') \delta u_i(\mathbf{x},t), \\
&\delta\Delta\kappa(\mathbf{x}') = \\
&-\int d\mathbf{x} \int dt \int dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x}',t';\mathbf{x},t) \delta^{jk}\delta^{lm} \sum_{\nu=1}^{L} \epsilon_\nu^{lm}(\mathbf{x}',t') \delta u_i(\mathbf{x},t), \\
&\delta\Delta\mu(\mathbf{x}') = \\
&-\int d\mathbf{x} \int dt \int dt' \frac{\partial G^{ij}}{\partial x'^k}(\mathbf{x}',t';\mathbf{x},t) \left(\delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl} - \frac{2}{3}\delta^{jk}\delta^{lm}\right) \sum_{\nu=1}^{L} \epsilon_\nu^{lm}(\mathbf{x}',t') \delta u_i(\mathbf{x},t).
\end{aligned} \tag{18}$$

In above equations the Green's function is advanced green's function which is anti-causal propagator. Other words, it propagate the wavefield backward in time. As a result the backscattered wave in time is defined as

$$\delta u_j(\mathbf{x}',t') = \int d\mathbf{x} \int dt G^{ij}(\mathbf{x}',t';\mathbf{x},t) \delta u_i(\mathbf{x},t)$$

where the advanced Green's function $G^{ij}(\mathbf{x}', t'; \mathbf{x}, t)$ propagate the residual wave $\delta u_i(\mathbf{x}, t)$ back in time from $(\mathbf{x}, t) > (\mathbf{x}', t')$ to $(\mathbf{x}', t')$. Finally we have

$$\begin{aligned}
\delta\hat{\rho}(\mathbf{x}') &= -\int dt' \frac{\partial^2 u_j}{\partial t'^2}(\mathbf{x}', t')\delta u_j(\mathbf{x}, t) = -\int dt' \frac{\partial u_j}{\partial t'}(\mathbf{x}', t')\frac{\partial \delta u_j}{\partial t'}(\mathbf{x}', t'), \\
\delta\hat{\kappa}(\mathbf{x}', 0^+) &= -\int dt' \delta^{jk}\delta^{lm}\varepsilon^{lm}(\mathbf{x}', t')\frac{\partial \delta u_j}{\partial x^k}(\mathbf{x}', t'), \\
\delta\hat{\mu}(\mathbf{x}', 0^+) &= -\int dt' \left(\delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl} - \frac{2}{3}\delta^{jk}\delta^{lm}\right)\varepsilon^{lm}(\mathbf{x}', t')\frac{\partial \delta u_j}{\partial x^k}(\mathbf{x}', t'), \\
\delta\Delta\hat{\kappa}(\mathbf{x}') &= -\int dt' \delta^{jk}\delta^{lm}\sum_{\nu=1}^{L}\epsilon_\nu^{lm}(\mathbf{x}', t')\frac{\partial \delta u_j}{\partial x^k}(\mathbf{x}', t'), \\
\delta\Delta\hat{\mu}(\mathbf{x}') &= -\int dt' \left(\delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl} - \frac{2}{3}\delta^{jk}\delta^{lm}\right)\sum_{\nu=1}^{L}\epsilon_\nu^{lm}(\mathbf{x}', t')\frac{\partial \delta u_j}{\partial x^k}(\mathbf{x}', t').
\end{aligned} \tag{19}$$

By exchanging $(\mathbf{x}', t') \to (\mathbf{x}, t)$ we have

$$\begin{aligned}
\delta\hat{\rho}(\mathbf{x}) &= -\int dt \frac{\partial u_j}{\partial t}\frac{\partial \delta u_j}{\partial t}, \\
\delta\hat{\kappa}(\mathbf{x}', 0^+) &= -\int dt \sum_l \varepsilon^{ll}\sum_k \frac{\partial \delta u_k}{\partial x^k}, \\
\delta\hat{\mu}(\mathbf{x}', 0^+) &= -\int dt \sum_{lm}\varepsilon^{lm}\left(\frac{\partial \delta u_l}{\partial x^m} + \frac{\partial \delta u_m}{\partial x^l}\right) - \frac{2}{3}\int dt \sum_l \varepsilon^{ll}\sum_k \frac{\partial \delta u_k}{\partial x^k}, \\
\delta\Delta\hat{\kappa}(\mathbf{x}) &= -\int dt \sum_l \sum_{\nu=1}^{L}\epsilon_\nu^{ll}\sum_k \frac{\partial \delta u_k}{\partial x^k}, \\
\delta\Delta\hat{\mu}(\mathbf{x}) &= -\int dt \sum_{lk}\sum_{\nu=1}^{L}\epsilon_\nu^{lk}\left(\frac{\partial \delta u_l}{\partial x^k} + \frac{\partial \delta u_k}{\partial x^l}\right) - \frac{2}{3}\int dt \sum_l \sum_{\nu=1}^{L}\epsilon_\nu^{ll}\sum_k \frac{\partial \delta u_k}{\partial x^k}.
\end{aligned} \tag{20}$$

All stress terms are in terms of forwarded wave field $\mathbf{u} = \mathbf{U}^{\mathrm{F}}$, other wavefields are backward waves shown by $\delta\mathbf{u} = \mathbf{U}^{\mathrm{B}}$

$$\begin{aligned}
\delta\hat{\rho}(\mathbf{x}) &= -\int dt \frac{\partial U_j^{\mathrm{F}}}{\partial t}\frac{\partial U_j^{\mathrm{B}}}{\partial t}, \\
\delta\hat{\kappa}(\mathbf{x}', 0^+) &= -\int dt \sum_l \frac{\partial U_l^{\mathrm{F}}}{\partial x^l}\sum_k \frac{\partial U_k^{\mathrm{B}}}{\partial x^k}, \\
\delta\hat{\mu}(\mathbf{x}', 0^+) &= -\int dt \sum_{lm}\left(\frac{\partial U_l^{\mathrm{F}}}{\partial x^m} + \frac{\partial U_m^{\mathrm{F}}}{\partial x^l}\right)\left(\frac{\partial U_l^{\mathrm{B}}}{\partial x^m} + \frac{\partial U_m^{\mathrm{B}}}{\partial x^l}\right) - \frac{2}{3}\int dt \sum_l \frac{\partial U_l^{\mathrm{F}}}{\partial x^l}\sum_k \frac{\partial U_k^{\mathrm{B}}}{\partial x^k}, \\
\delta\Delta\hat{\kappa}(\mathbf{x}) &= -\int dt \sum_l \sum_{\nu=1}^{L}\epsilon_\nu^{ll}\sum_k \frac{\partial U_k^{\mathrm{B}}}{\partial x^k}, \\
\delta\Delta\hat{\mu}(\mathbf{x}) &= -\int dt \sum_{lk}\sum_{\nu=1}^{L}\epsilon_\nu^{lk}\left(\frac{\partial U_l^{\mathrm{B}}}{\partial x^m} + \frac{\partial U_m^{\mathrm{B}}}{\partial x^l}\right) - \frac{2}{3}\int dt \sum_l \sum_{\nu=1}^{L}\epsilon_\nu^{ll}\sum_k \frac{\partial U_k^{\mathrm{B}}}{\partial x^k}.
\end{aligned} \tag{21}$$

In above equations the memory variables are in terms of forwarded wavefields. Now we write the gradient for the new model parameters. An infinitesimal change in the wave field in terms of changes in the properties is given by

$$\delta \mathbf{u}(\mathbf{x},t) = \int dt' \left[\frac{\partial \mathbf{u}(\mathbf{x},t')}{\partial \mathbf{m}(\mathbf{x})}\right] \delta \mathbf{m}(\mathbf{x}). \tag{22}$$

The adjoint of this equation has the form

$$\delta \hat{\mathbf{m}}(\mathbf{x}) = \int dt' \left[\frac{\partial \mathbf{u}(\mathbf{x},t')}{\partial \mathbf{m}(\mathbf{x})}\right]^* \delta \mathbf{u}(\mathbf{x},t'). \tag{23}$$

Infinitesimal changes in new parameters $\mathbf{M}$ in terms of old parameters $\mathbf{m}$ can be written as

$$\delta \hat{\mathbf{M}}(\mathbf{x}) = \int dt' \left[\frac{\partial \mathbf{u}(\mathbf{x},t')}{\partial \mathbf{m}(\mathbf{x})}\frac{\partial \mathbf{m}(\mathbf{x})}{\partial \mathbf{M}(\mathbf{x})}\right]^* \delta \mathbf{u}(\mathbf{x},t') = \frac{\partial \mathbf{m}(\mathbf{x})}{\partial \mathbf{M}(\mathbf{x})} \int dt' \left[\frac{\partial \mathbf{u}(\mathbf{x},t')}{\partial \mathbf{m}(\mathbf{x})}\right]^* \delta \mathbf{u}(\mathbf{x},t').$$

We can simplify the above equation using relation (23) and write

$$\delta \hat{\mathbf{M}}(\mathbf{x}) = \frac{\partial \mathbf{m}(\mathbf{x})}{\partial \mathbf{M}(\mathbf{x})}\delta \hat{\mathbf{m}}(\mathbf{x})$$

Assuming that $\mathbf{m} = (\rho, \kappa, \mu, \Delta\kappa, \Delta\mu)$ and $\mathbf{M} = (\rho, V_P, V_S, Q_\kappa, Q_\mu)$ we can write

$$\begin{aligned}
\delta \hat{V}_P &= \frac{\partial \kappa}{\partial V_P}\delta\hat{\kappa} + \frac{\partial \mu}{\partial V_P}\delta\hat{\mu} + \frac{\partial \rho}{\partial V_P}\delta\hat{\rho} + \frac{\partial \Delta\mu}{\partial V_P}\delta\widehat{\Delta\mu} + \frac{\partial \Delta\kappa}{\partial V_P}\delta\widehat{\Delta\kappa},\\
\delta \hat{V}_S &= \frac{\partial \kappa}{\partial V_S}\delta\hat{\kappa} + \frac{\partial \mu}{\partial V_S}\delta\hat{\mu} + \frac{\partial \rho}{\partial V_S}\delta\hat{\rho} + \frac{\partial \Delta\mu}{\partial V_S}\delta\widehat{\Delta\mu} + \frac{\partial \Delta\kappa}{\partial V_S}\delta\widehat{\Delta\kappa},\\
\delta \hat{Q}_\mu &= \frac{\partial \kappa}{\partial Q_\mu}\delta\hat{\kappa} + \frac{\partial \mu}{\partial Q_\mu}\delta\hat{\mu} + \frac{\partial \rho}{\partial Q_\mu}\delta\hat{\rho} + \frac{\partial \Delta\mu}{\partial Q_\mu}\delta\widehat{\Delta\mu} + \frac{\partial \Delta\kappa}{\partial Q_\mu}\delta\widehat{\Delta\kappa},\\
\delta \hat{Q}_\kappa &= \frac{\partial \kappa}{\partial Q_\kappa}\delta\hat{\kappa} + \frac{\partial \mu}{\partial Q_\kappa}\delta\hat{\mu} + \frac{\partial \rho}{\partial Q_\kappa}\delta\hat{\rho} + \frac{\partial \Delta\mu}{\partial Q_\kappa}\delta\widehat{\Delta\mu} + \frac{\partial \Delta\kappa}{\partial Q_\kappa}\delta\widehat{\Delta\kappa},\\
\delta \hat{\rho}^* &= \frac{\partial \kappa}{\partial \rho}\delta\hat{\kappa} + \frac{\partial \mu}{\partial \rho}\delta\hat{\mu} + \frac{\partial \rho}{\partial \rho}\delta\hat{\rho} + \frac{\partial \Delta\mu}{\partial \rho}\delta\widehat{\Delta\mu} + \frac{\partial \Delta\kappa}{\partial \rho}\delta\widehat{\Delta\kappa}.
\end{aligned} \tag{24}$$

The relationship between the velocities and bulk and shear modulus are given by

$$V_P = \sqrt{\frac{\kappa + \frac{4}{3}\mu}{\rho}}, \quad \text{or} \quad \kappa = \rho V_P^2 - \frac{4}{3}\rho V_S^2, \tag{25}$$

$$V_S = \sqrt{\frac{\mu}{\rho}}, \quad \text{or} \quad \mu = \rho V_S^2, \tag{26}$$

if we assume that the relaxation times for stress and strain are the same, for a standard linear solid we have

$$\mu(t) = \mu_r \left[1 + \frac{1}{KQ_\mu}\sum_{\nu=1}^{L} \exp\left(-\frac{t}{\tau_\nu}\right)\right], \tag{27}$$

$$\kappa(t) = \kappa_r \left[1 + \frac{1}{KQ_\kappa}\sum_{\nu=1}^{L} \exp\left(-\frac{t}{\tau_\nu}\right)\right], \tag{28}$$

here K is a constant. We can write the unrelaxed quantities at $t \to 0$

$$\mu_u = \mu_r \left[1 + \frac{b}{Q_\mu}\right], \quad \text{or} \quad \Delta\mu = -\mu_r \frac{b}{Q_\mu}, \tag{29}$$

$$\kappa_u = \kappa_r \left[1 + \frac{b}{Q_\kappa}\right], \quad \text{or} \quad \Delta\kappa = -\kappa_r \frac{b}{Q_\kappa}, \tag{30}$$

where $b = LK^{-1}$. Using these equations we obtain

$$\begin{aligned}
\delta\hat{V}_P &= (2\rho V_P)\delta\hat{\kappa},\\
\delta\hat{V}_S &= \left(-\frac{8}{3}\rho V_S\right)\delta\hat{\kappa} + (2\rho V_S)\delta\hat{\mu} + \left(-2\rho V_S \frac{b}{Q_\mu}\right)\delta\widehat{\Delta\mu},\\
\delta\hat{Q}_\mu &= \left(\rho V_S^2 \frac{b}{Q_\mu^2}\right)\delta\widehat{\Delta\mu},\\
\delta\hat{Q}_\kappa &= \left(\rho\left[V_P^2 - \frac{4}{3}V_S^2\right]\frac{b}{Q_\kappa^2}\right)\delta\widehat{\Delta\kappa},\\
\delta\hat{\rho}^* &= \left(V_P^2 - \frac{4}{3}V_S^2\right)\delta\hat{\kappa} + V_S^2\delta\hat{\mu} + \delta\hat{\rho} + V_S^2\frac{b}{Q_\mu}\delta\widehat{\Delta\mu} + \left(V_P^2 - \frac{4}{3}V_S^2\right)\frac{b}{Q_\kappa}\delta\widehat{\Delta\kappa}.
\end{aligned} \tag{31}$$

In matrix form

$$\begin{pmatrix} \delta\hat{V}_P \\ \delta\hat{V}_S \\ \delta\hat{Q}_\mu \\ \delta\hat{Q}_\kappa \\ \delta\hat{\rho}^* \end{pmatrix} = \begin{pmatrix} 2\rho V_P & 0 & 0 & 0 & 0 \\ -\frac{8}{3}\rho V_S & 2\rho V_S & 0 & -2\rho V_S\frac{b}{Q_\mu} & 0 \\ 0 & 0 & 0 & \rho V_S^2\frac{b}{Q_\mu^2} & 0 \\ 0 & 0 & 0 & 0 & \rho\left(V_P^2 - \frac{4}{3}V_S^2\right)\frac{b}{Q_\kappa^2} \\ V_P^2 - \frac{4}{3}V_S^2 & V_S^2 & 1 & V_S^2\frac{b}{Q_\mu} & \left(V_P^2 - \frac{4}{3}V_S^2\right)\frac{b}{Q_\kappa} \end{pmatrix} \begin{pmatrix} \delta\hat{\kappa} \\ \delta\hat{\mu} \\ \delta\hat{\rho} \\ \delta\widehat{\Delta\mu} \\ \delta\widehat{\Delta\kappa} \end{pmatrix}. \tag{32}$$

It would be useful to write changes in model parameters in density-impedance parametrization. In terms of impedances

$$\delta\hat{Z}_P = \frac{\partial\kappa}{\partial Z_P}\delta\hat{\kappa} + \frac{\partial\mu}{\partial Z_P}\delta\hat{\mu} + \frac{\partial\rho}{\partial Z_P}\delta\hat{\rho} + \frac{\partial\Delta\mu}{\partial Z_P}\delta\widehat{\Delta\mu} + \frac{\partial\Delta\kappa}{\partial Z_P}\delta\widehat{\Delta\kappa}, \tag{33}$$

$$\delta\hat{Z}_S = \frac{\partial\kappa}{\partial Z_S}\delta\hat{\kappa} + \frac{\partial\mu}{\partial Z_S}\delta\hat{\mu} + \frac{\partial\rho}{\partial Z_S}\delta\hat{\rho} + \frac{\partial\Delta\mu}{\partial Z_S}\delta\widehat{\Delta\mu} + \frac{\partial\Delta\kappa}{\partial Z_S}\delta\widehat{\Delta\kappa}, \tag{34}$$

The relation between impedances and moduluses are

$$Z_P = \rho V_P, \quad \text{or} \quad \kappa = \frac{Z_P^2 - \frac{4}{3}\rho Z_S^2}{\rho}, \tag{35}$$

$$Z_S = \rho V_S, \quad \text{or} \quad \mu = \frac{Z_S^2}{\rho}, \tag{36}$$

In matrix form

$$\begin{pmatrix} \delta\hat{Z}_P \\ \delta\hat{Z}_S \\ \delta\hat{Q}_\mu \\ \delta\hat{Q}_\kappa \\ \delta\hat{\rho}^* \end{pmatrix} = \begin{pmatrix} 2V_P & 0 & 0 & 0 & 0 \\ -\frac{8}{3}V_S & 2V_S & 0 & -2V_S\frac{b}{Q_\mu} & 0 \\ 0 & 0 & 0 & \rho V_S^2\frac{b}{Q_\mu^2} & 0 \\ 0 & 0 & 0 & 0 & \rho\left(V_P^2 - \frac{4}{3}V_S^2\right)\frac{b}{Q_\kappa^2} \\ V_P^2 - \frac{4}{3}V_S^2 & V_S^2 & 1 & V_S^2\frac{b}{Q_\mu} & \left(V_P^2 - \frac{4}{3}V_S^2\right)\frac{b}{Q_\kappa} \end{pmatrix} \begin{pmatrix} \delta\hat{\kappa} \\ \delta\hat{\mu} \\ \delta\hat{\rho} \\ \delta\widehat{\Delta\mu} \\ \delta\widehat{\Delta\kappa} \end{pmatrix}. \tag{37}$$

So the Fréchet kernels for Q_κ and Q_μ are proportional to the Q_κ^{-2} and Q_μ^{-2} respectively. As a result for the large initial Q model the kernels are almost zero Fichtner (2010).

SUMMARY AND FUTURE DIRECTIONS

In summary, we analyzed the mathematical framework of full viscoelastic waveform Inversion in time domain. First we showed that how perturbations in density, relaxation function and stress generate the scattered wave. This is called the Born approximation, which is the single scattering assumption describing the relationship between the perturbations in reference medium with the scattered wave field. By inserting the changes in density, relation function and stress in wave equation, we arrive at a wave equation governed the scattered wave with new sources. Perturbations in density act as point force and perturbations in relaxation function and stress act as moment tensor source. By having the sources and using the retarded Green's tensor we can obtain the integral equation for the solution of the scattered wave. This integral equation can be expanded to extract the five Fréchet kernels for density, unrelaxed bulk modulus and shear modulus,and the differences between the relaxed and unrelaxed moduluses. By applying the adjoint operation we invert the aforementioned five parameters in terms of forwarded and backscattered wave field in time. Backscattered wave field is obtained by using the integral equation including the multiplication of advanced Green's tensor on the forward scattered wavefield. We also obtained the Jacobian transformations that relate the five viscoelastic parameters based on density and relaxation functions in terms of model parameters density-velocity-quality factor and density-impedance-quality factors.

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