CREWES Consortium for Research in Elastic Wave Exploration Seismology

Deconvolution With Multigrid

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Purpose

- Introduce multigrid method for linear systems
- Demonstrate application of multigrid to deconvolution and surface consistent statics

Outline

- Iterative methods and convergence
- Spectral performance
- Multigrid methods
- Deconvolution
- Surface Consistent Statics
- Conclusions

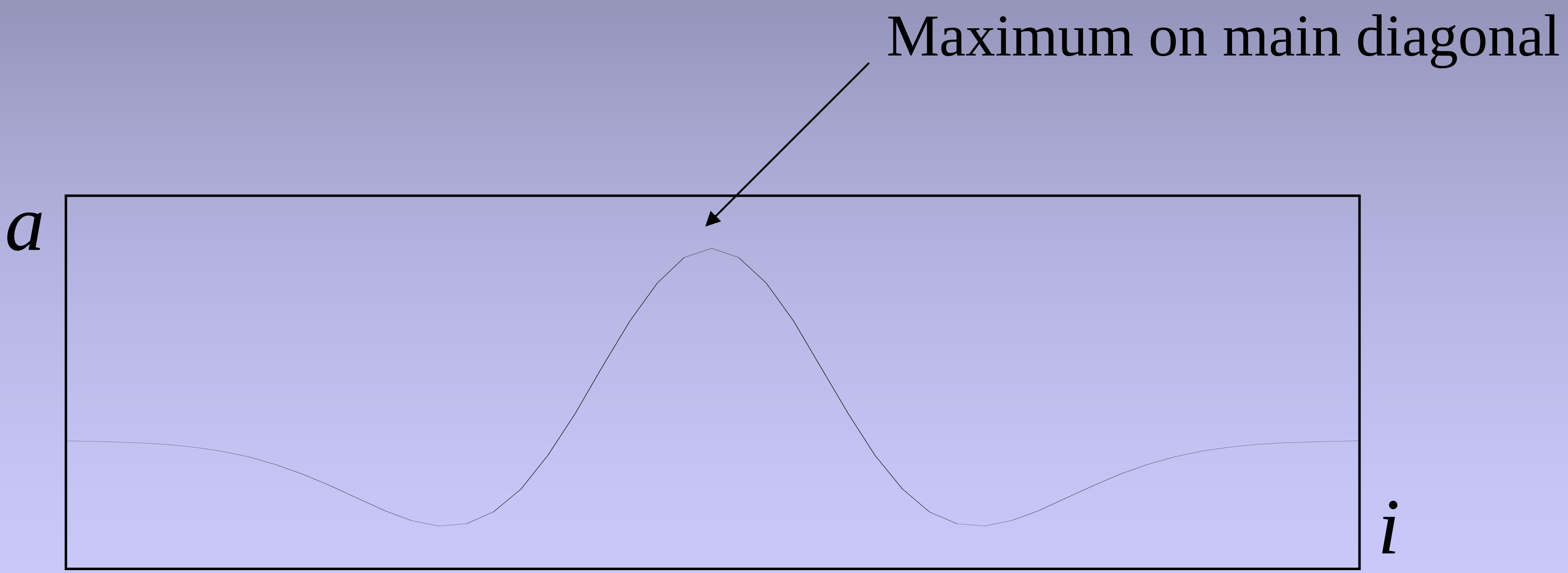
Linear Systems

$$\mathbf{Ax} = \mathbf{b}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

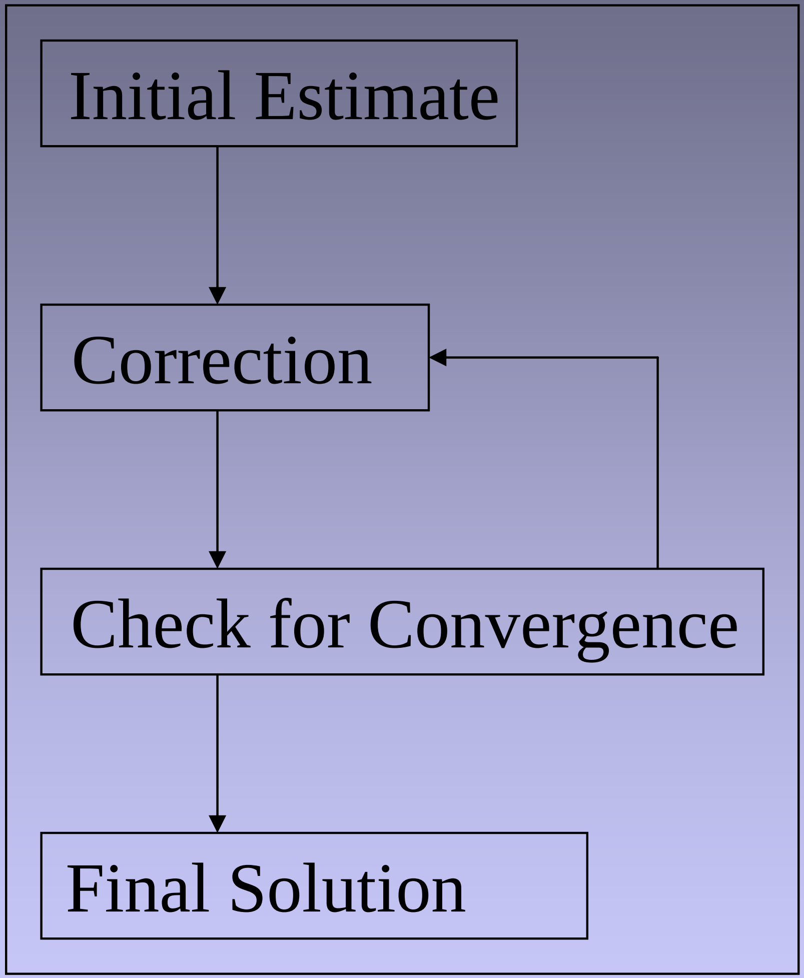
Diagonal Dominance

- Most of the entries in the matrix are on or near the main diagonal
- Each point in solution is dependant on other local points
- Condition is usually met by physical systems and discrete PDE’s



$$\cdots + a_{i,i-2}x_{i-2} + a_{i,i-1}x_{i-1} + a_{i,i}x_i + a_{i,i+1}x_{i+1} + a_{i,i+2}x_{i+2} + \cdots = b_i$$

Iterative methods



- Jacobi
- Gauss-Seidel
- Conjugate Gradient
- ILU decomposition

Gauss-Seidel Method

Write out i^{th} equation in system

$$
\begin{bmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & a_{i,i-2} & a_{i,i-1} & a_{i,i} & a_{i,i+1} & a_{i,i+1} & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{bmatrix}
\begin{bmatrix} \vdots \\ x_{i-2} \\ x_{i-1} \\ x_i \\ x_{i+1} \\ x_{i+2} \\ \vdots \end{bmatrix}
=
\begin{bmatrix} \vdots \\ b_{i-1} \\ b_i \\ b_{i+1} \\ \vdots \end{bmatrix}
$$

$$\cdots + a_{i,i-2}x_{i-2} + a_{i,i-1}x_{i-1} + a_{i,i}x_i + a_{i,i+1}x_{i+1} + a_{i,i+2}x_{i+2} + \cdots = b_i$$

Solve for x_i

$$\frac{b_i - \cdots + a_{i,i-2}x_{i-2} + a_{i,i-1}x_{i-1} + a_{i,i+1}x_{i+1} + a_{i,i+2}x_{i+2} + \cdots}{a_{i,i}} = x_i$$

Gauss-Seidel

Move down the vector correcting each entry in sequence

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \cdots & x_{N-1} & x_N \end{bmatrix}$$

Uses updated values as soon as they are accessible

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots & x_{N-1} & x_N \end{bmatrix}$$

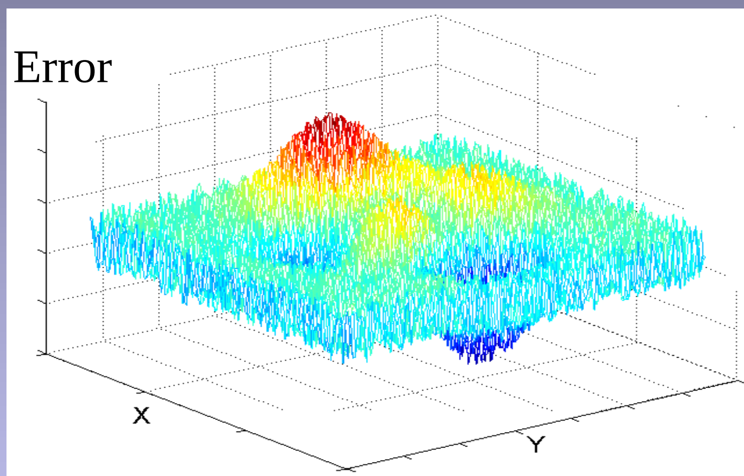
updated

Not updated yet

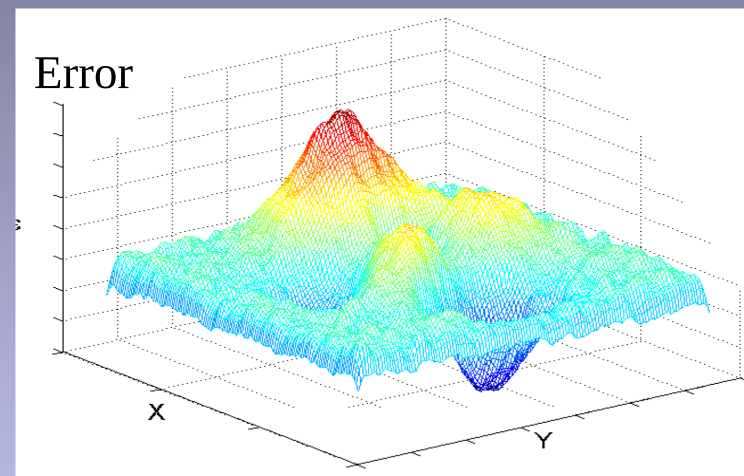
Being Corrected

Solving Laplace’s Equation with Gauss-Seidel $\nabla^2 x = 0$

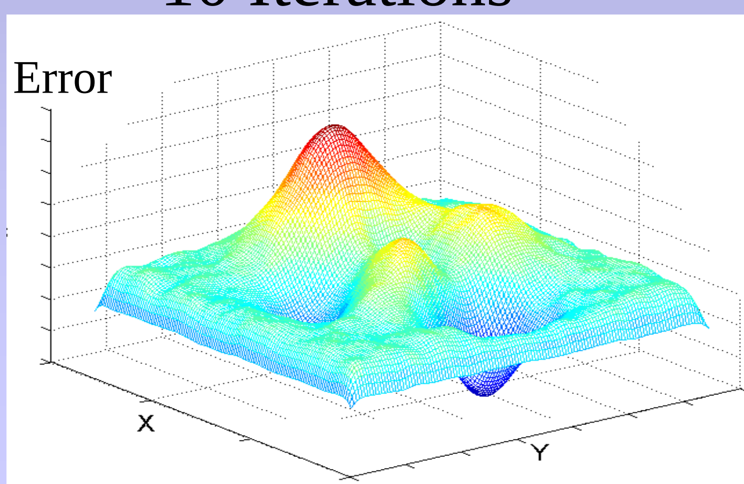
Initial Estimate



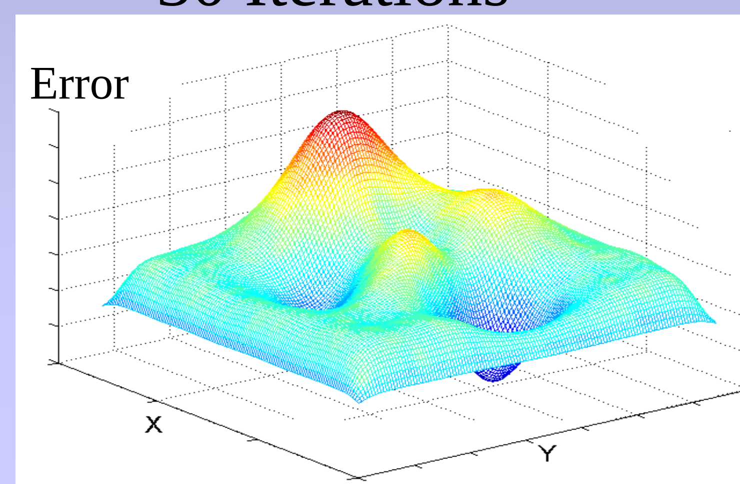
3 Iterations



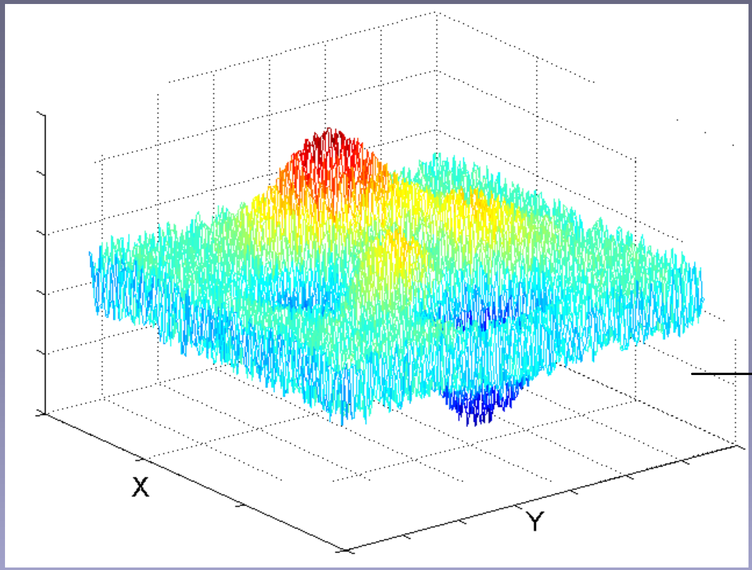
10 Iterations

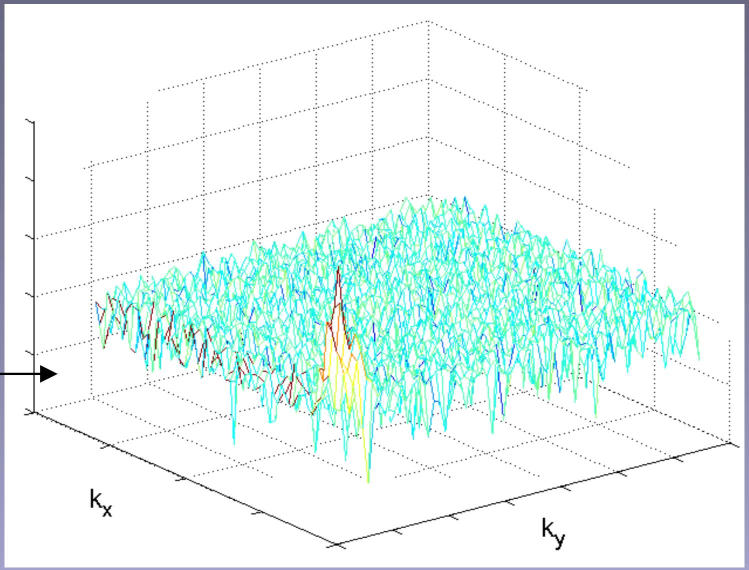


30 Iterations



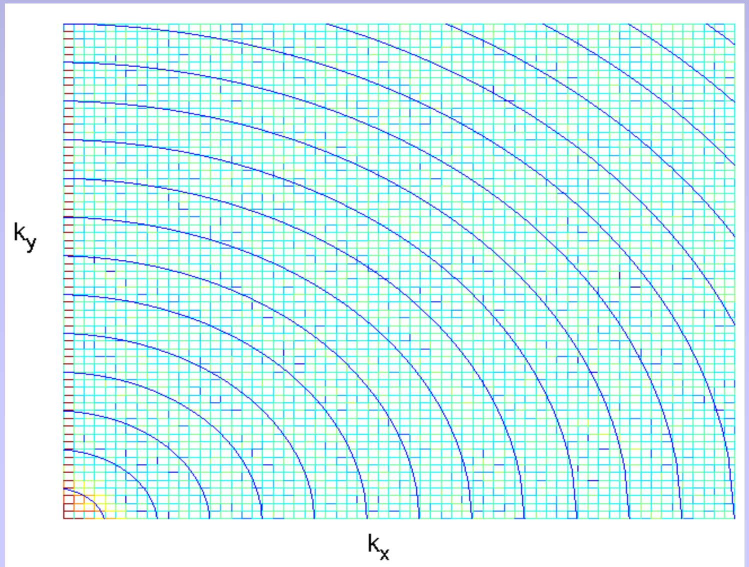
Spectral performance

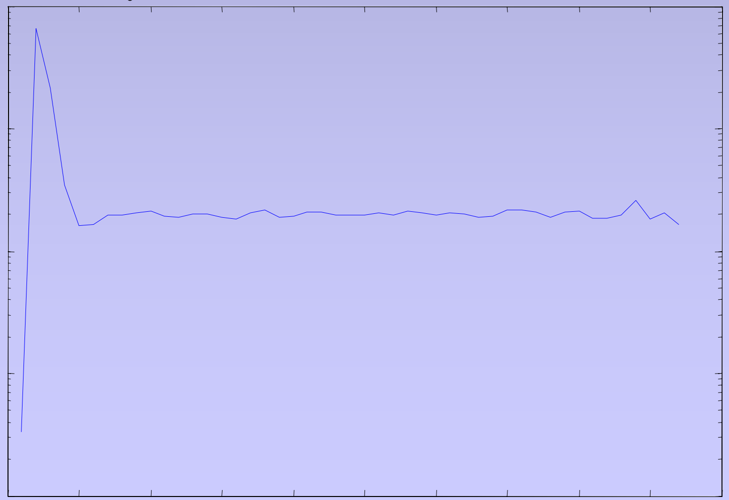




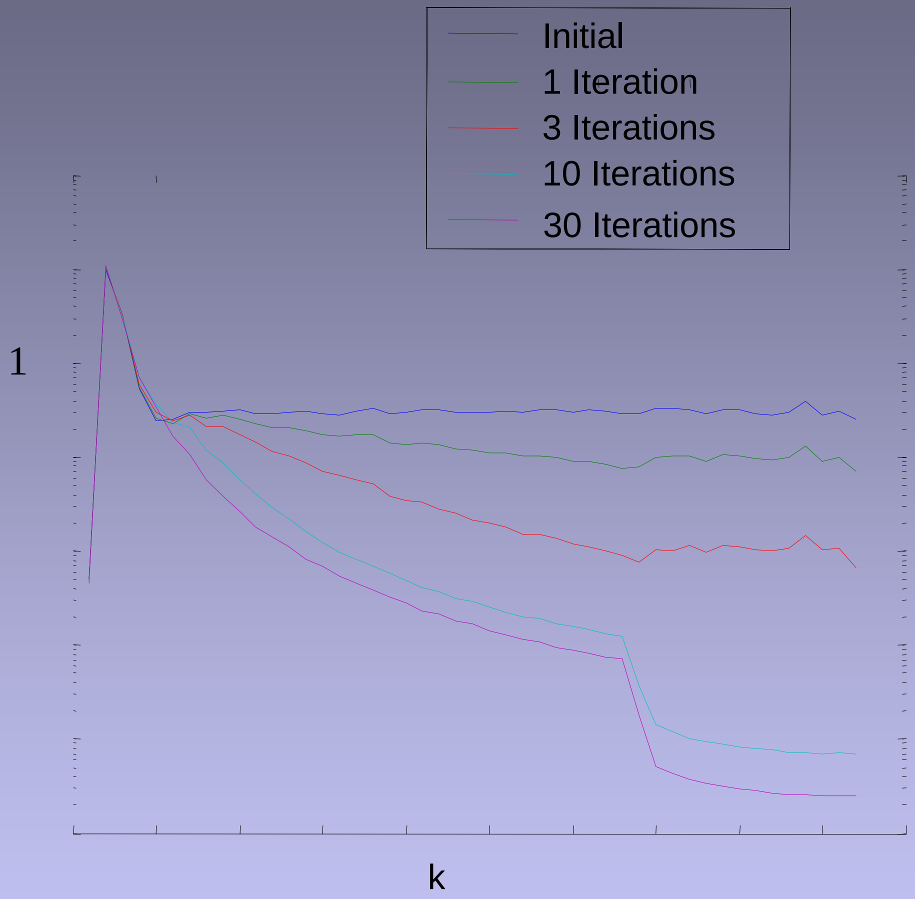
Take a 2D Fourier transform of the surface

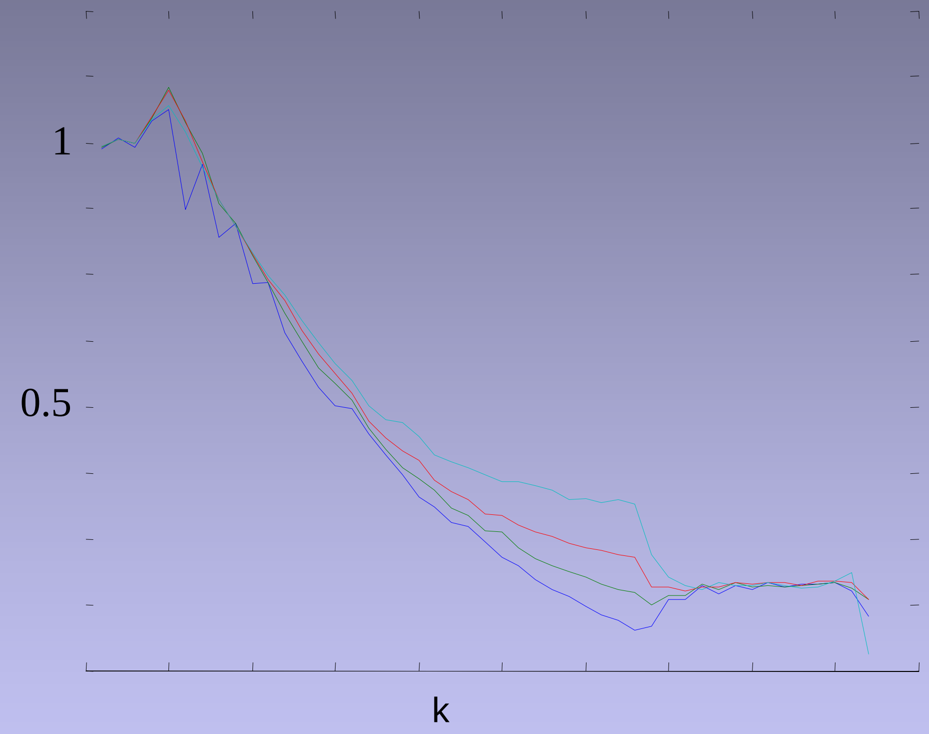
Average spectrum across $k = \sqrt{k_x^2 + k_y^2}$





Spectral Performance





Gauss-Seidel most effective on error components at or near Nyquist, reduces high frequency error by a factor of approximately 0.3 with each pass.

Theoretical Performance

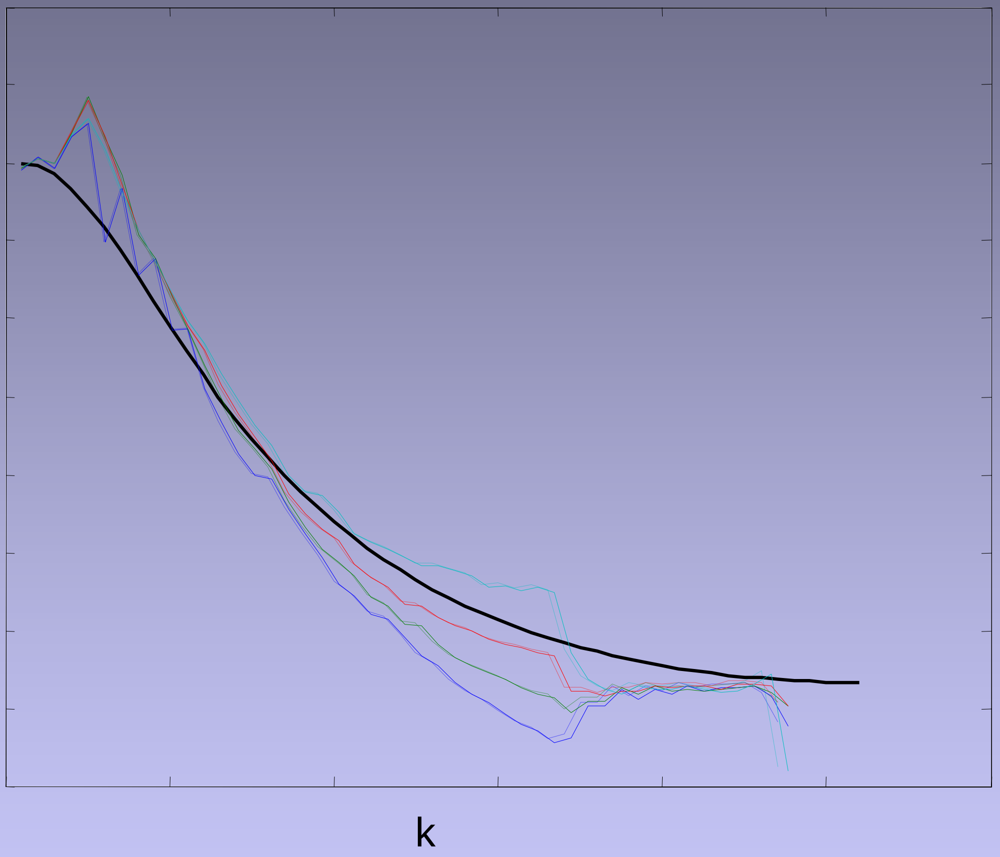
Details in Shewchuk (2002)

$$Ax = b$$

$$A = (L + D + U)$$

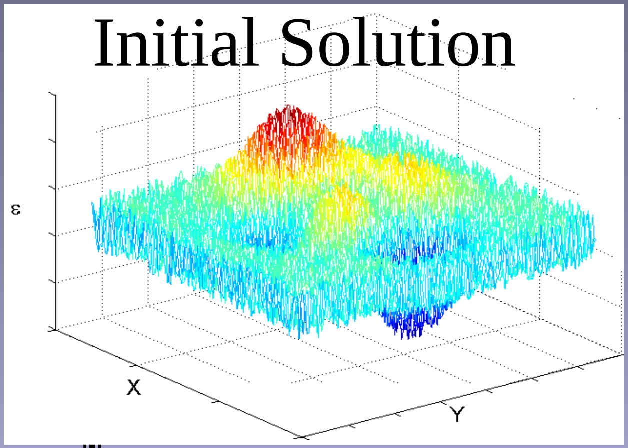
$$A \Rightarrow \begin{bmatrix} \ddots & & & & \\ & \ddots & & U & \\ & & D & & \\ & L & & \ddots & \\ & & & & \ddots \end{bmatrix}$$

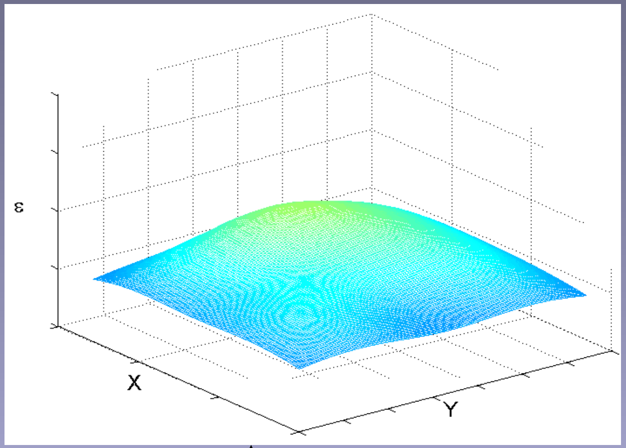
$$(D + L)^{-1} U \varepsilon_{old} = \varepsilon_{new}$$



Coarse Grid Correction

129x129

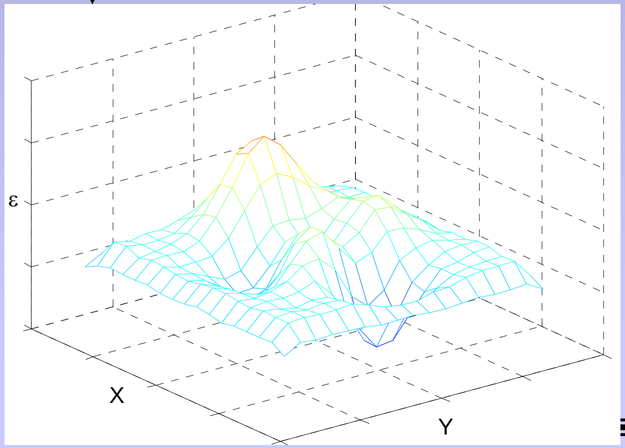


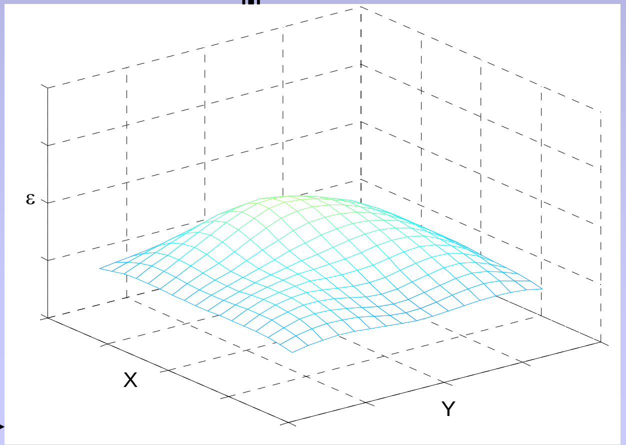


Restriction

Interpolation

17x17

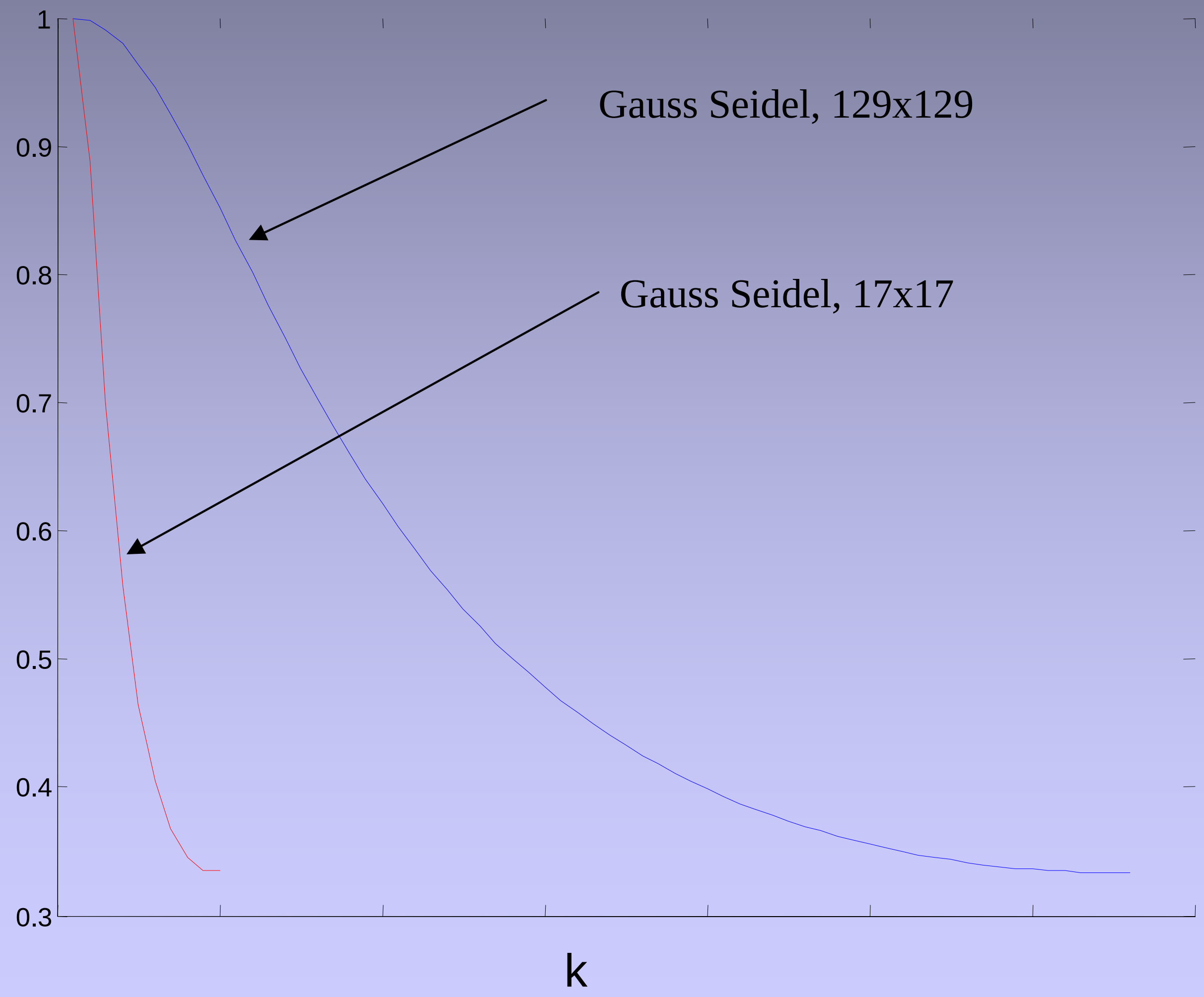




3x Gauss-Seidel

Spectral Performance

On coarse grid, Gauss-Seidel will attenuate longer wavelength error

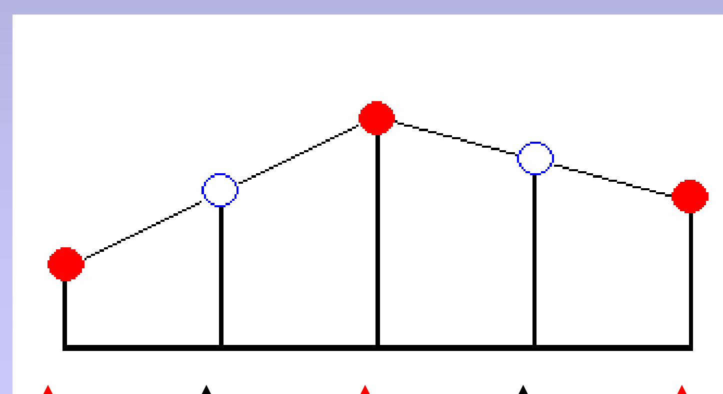


Improved Initial Estimate

Speed of convergence of iterative methods depends on the quality of the initial estimate

$$\begin{bmatrix} \ast & \bullet & \ast & \bullet & \ast \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \ast & \bullet & \ast & \bullet & \ast \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \ast & \bullet & \ast & \bullet & \ast \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \Rightarrow \begin{bmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_3 \\ \hat{b}_5 \end{bmatrix}$$

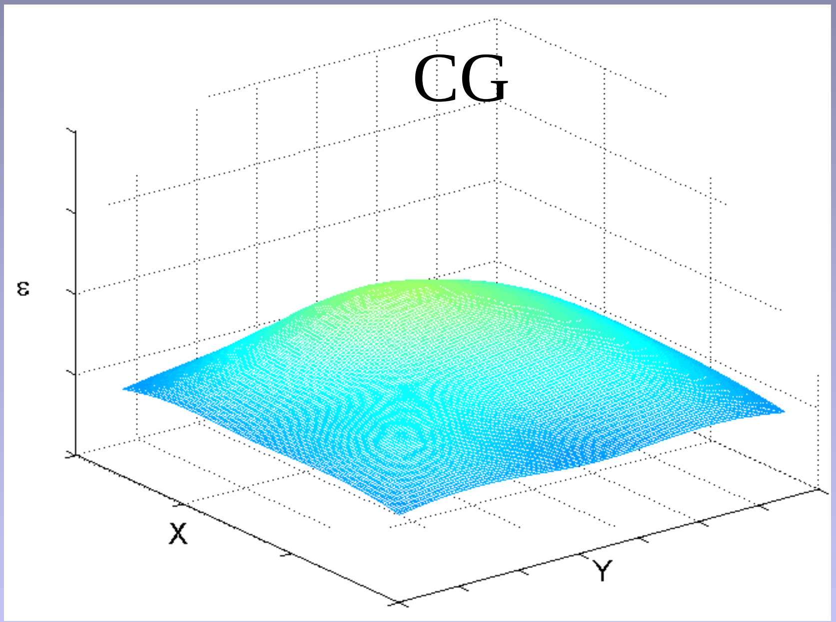
(Anti-alias filtered)

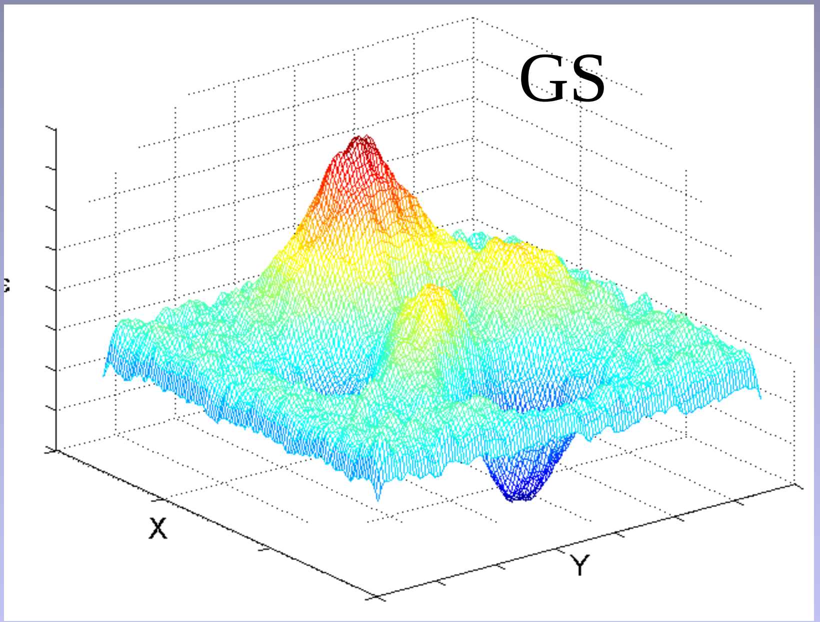


Interpolated

Solved for on coarser grid

Coarse Grid Correction vs Gauss-Seidel





Both results used similar number of floating point operations

Deconvolution

$$\mathbf{Wr} = \mathbf{s}$$ Convolutional Model

$$\mathbf{W}^T\mathbf{Wr} = \mathbf{W}^T\mathbf{s}$$ Ensure Diagonal Dominance

Index Notation

$$\sum_j [W^T W](i,j) r(j) = [W^T s](i), \qquad j = 1, N$$

Solve for r(i)

$$\frac{\sum_j W^T W(i,j) r(j) - W^T s(i)}{W^T W(i,i)} = r(i), \qquad j = 1, N,\ j \neq i$$

Gauss-Seidel Deconvolution

Cycle through each entry of *r*, updating it as you go

$$A = W^T W$$

$$\vdots$$

$$r(9) = \frac{\cdots A(9,7)r(7) + A(9,8)r(8) + A(9,10)r(10) + A(9,11)r(11) \cdots - W^T s(9)}{A(9,9)}$$

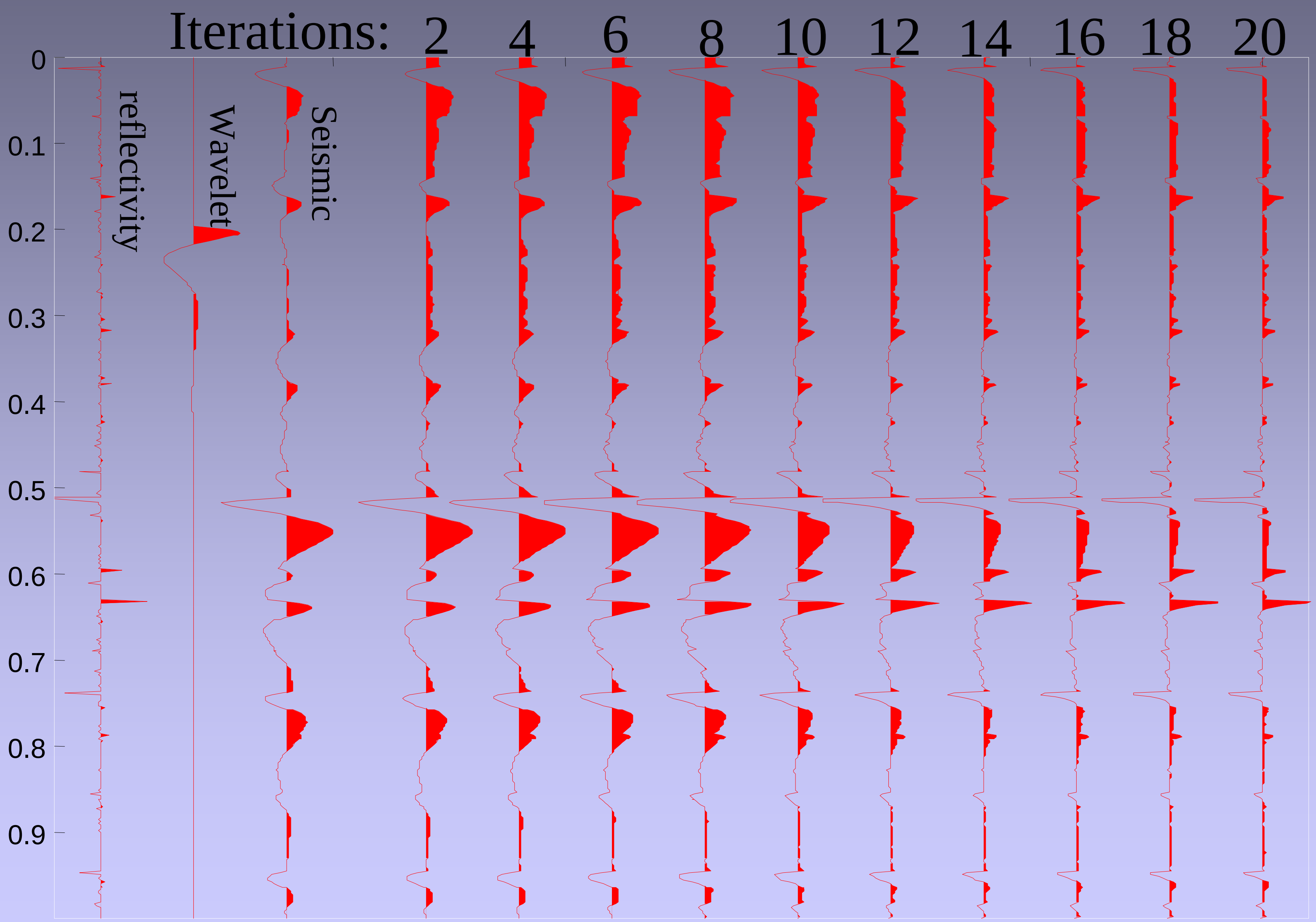
$$r(10) = \frac{\cdots A(10,8)r(8) + A(10,9)r(9) + A(10,11)r(11) + A(10,12)r(12) \cdots - W^T s(10)}{A(10,10)}$$

$$r(11) = \frac{\cdots A(11,9)r(9) + A(11,10)r(10) + A(11,12)r(12) + A(11,13)r(13) \cdots - W^T s(11)}{A(11,11)}$$

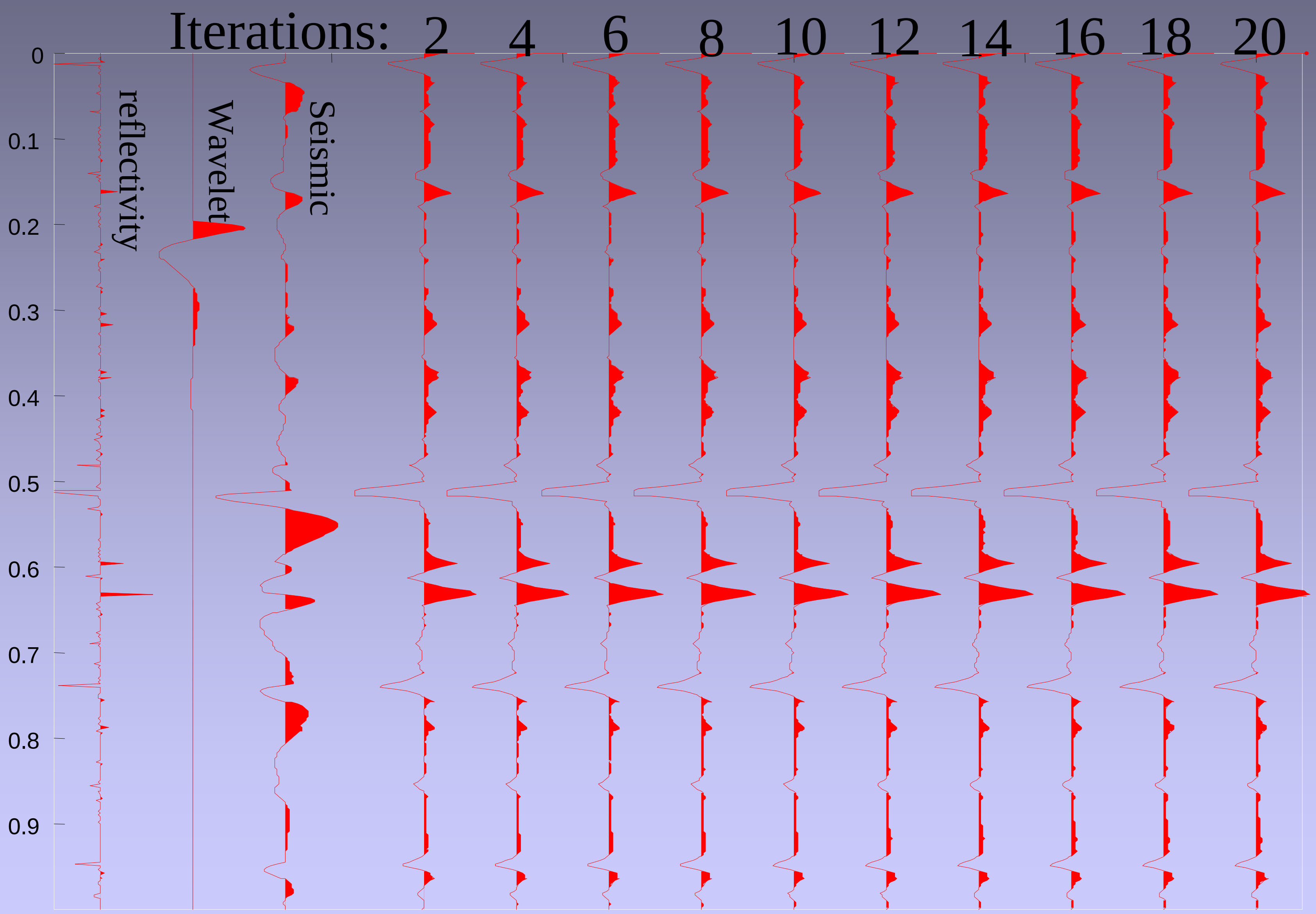
$$\vdots$$

Because operator is biased, convergence is improved by alternating directions

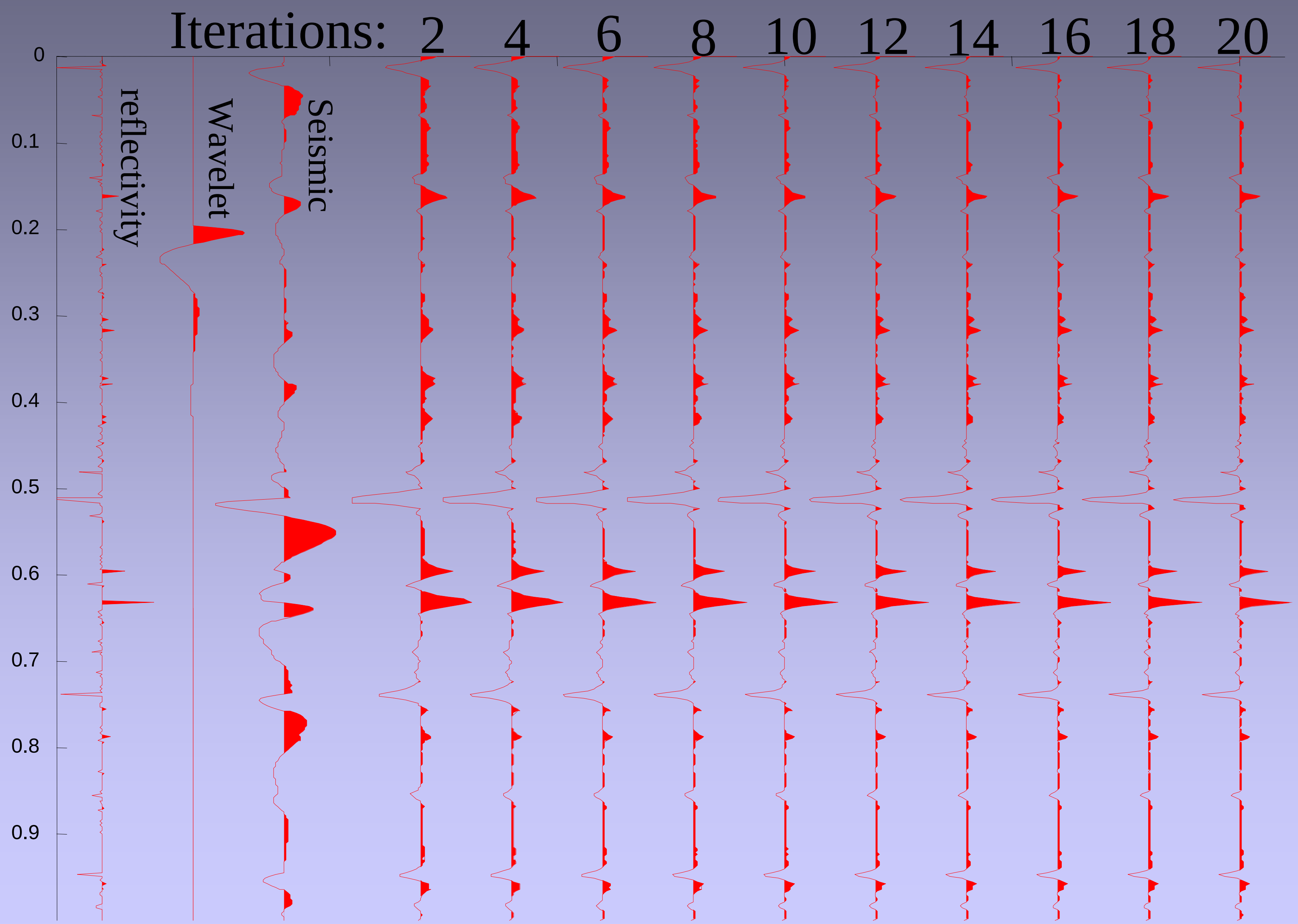
Gauss-Seidel Results (Inefficient, but converges)



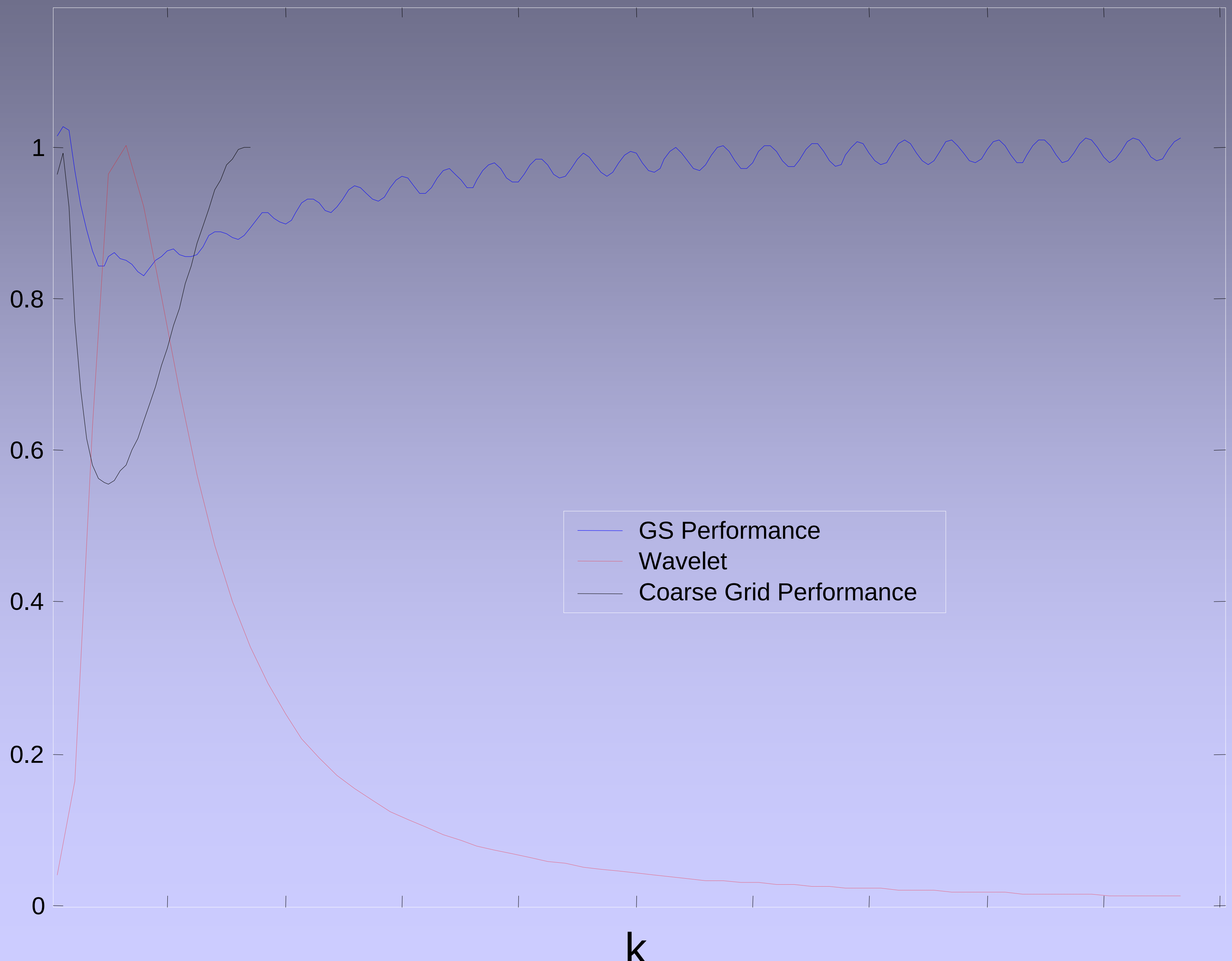
Coarse grid Correction



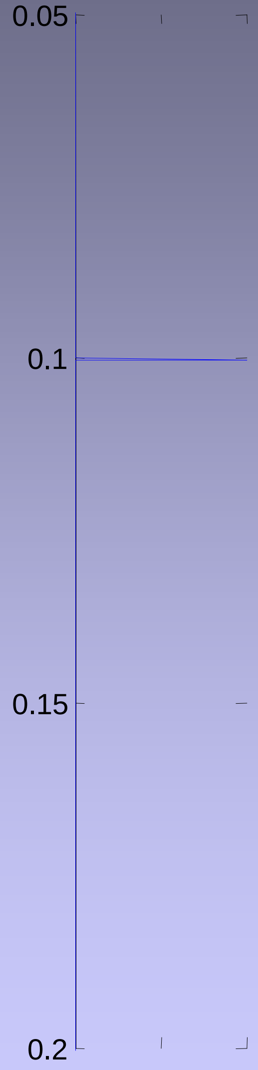
Coarse grid Followed by GS



Spectral Performance



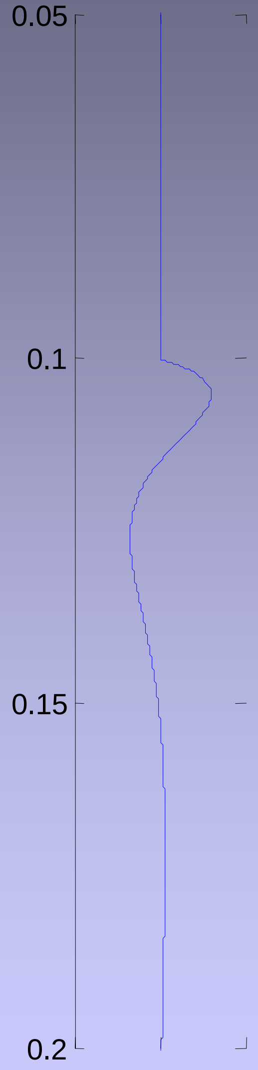
Resolution Experiment



Single spike at

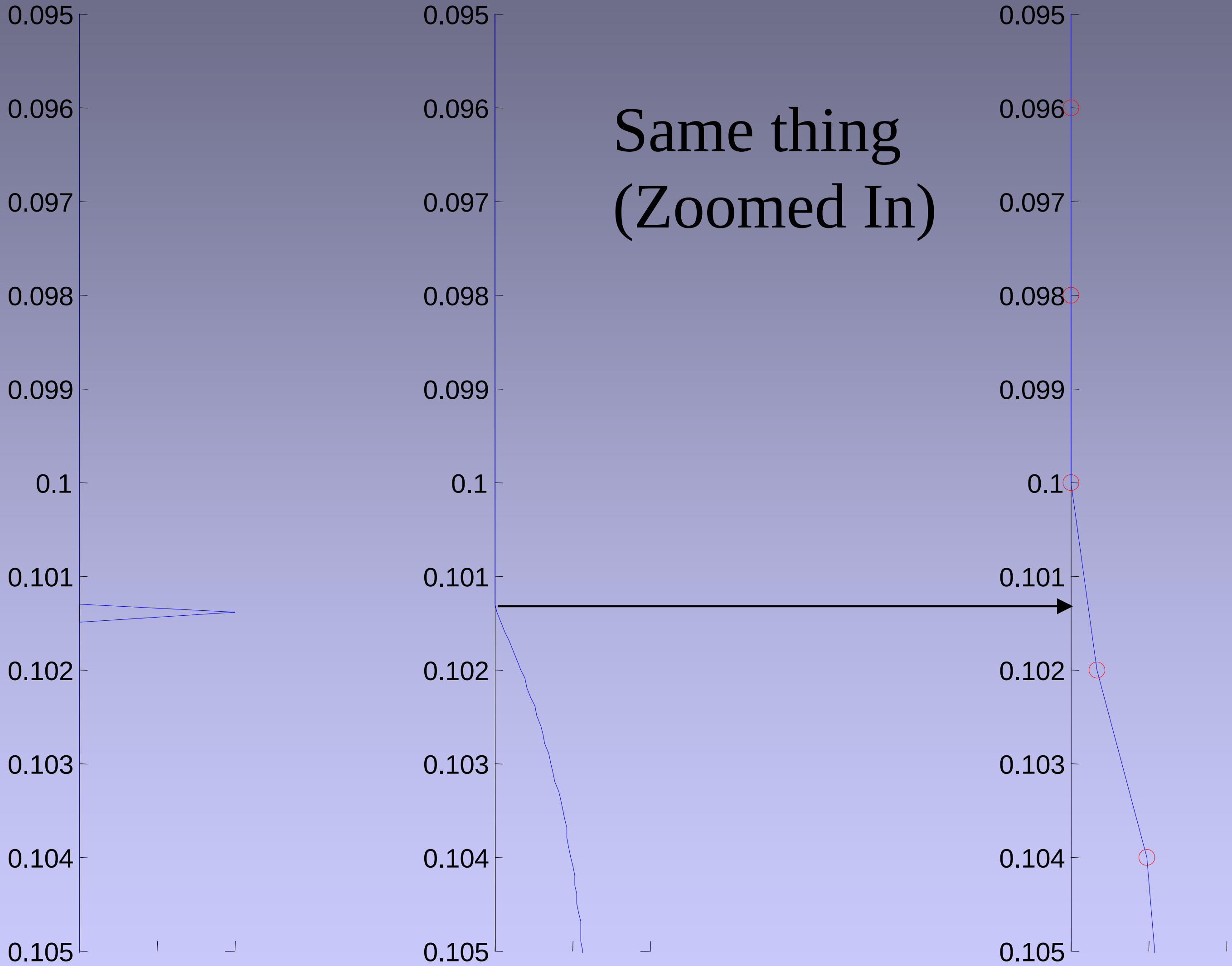
$$t = 0.1014s$$

$$\Delta t = 0.0001s$$



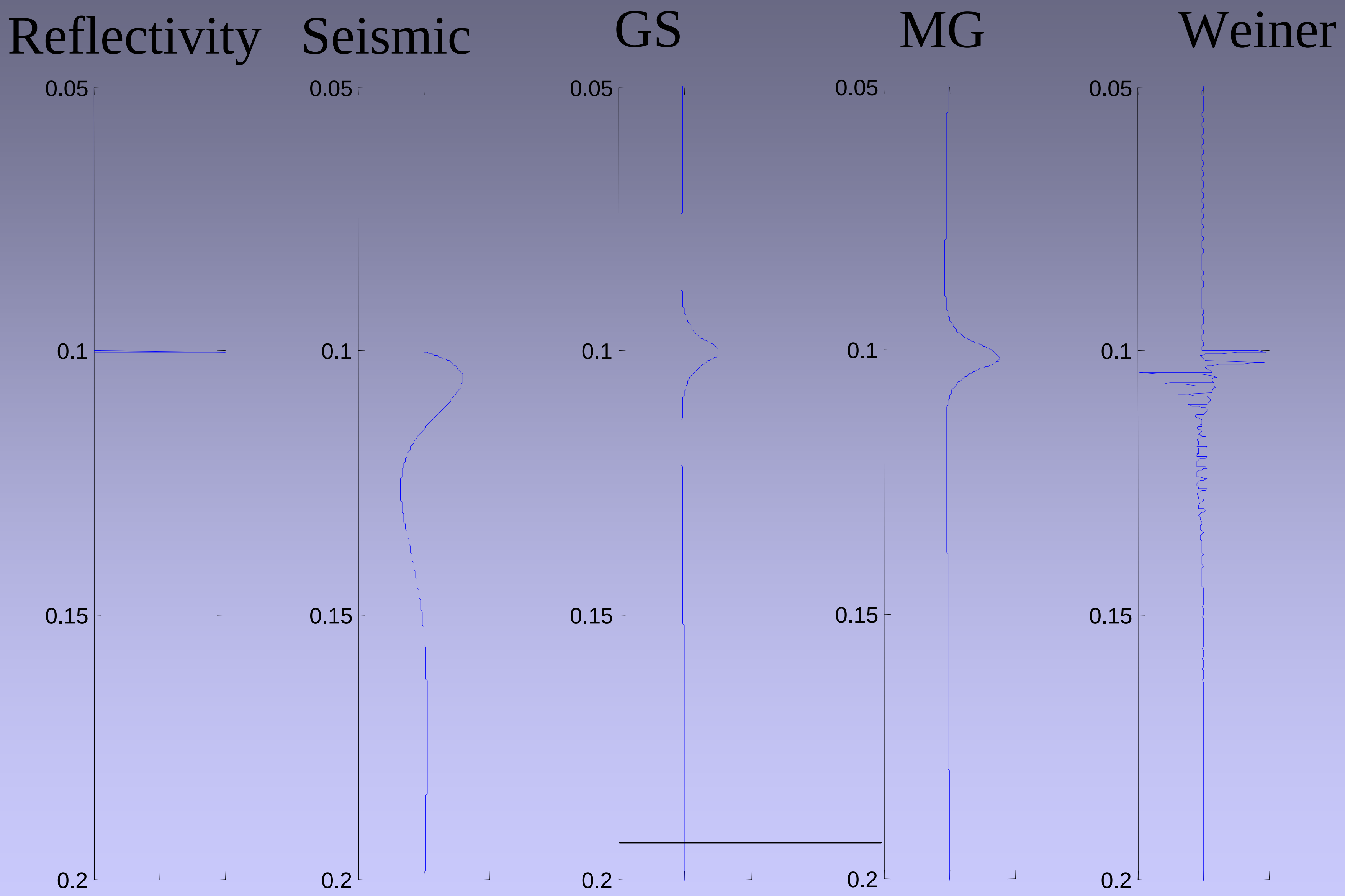
Convolved with Minimum phase wavelet (all frequencies up to Nyquist)

Resolution Experiment

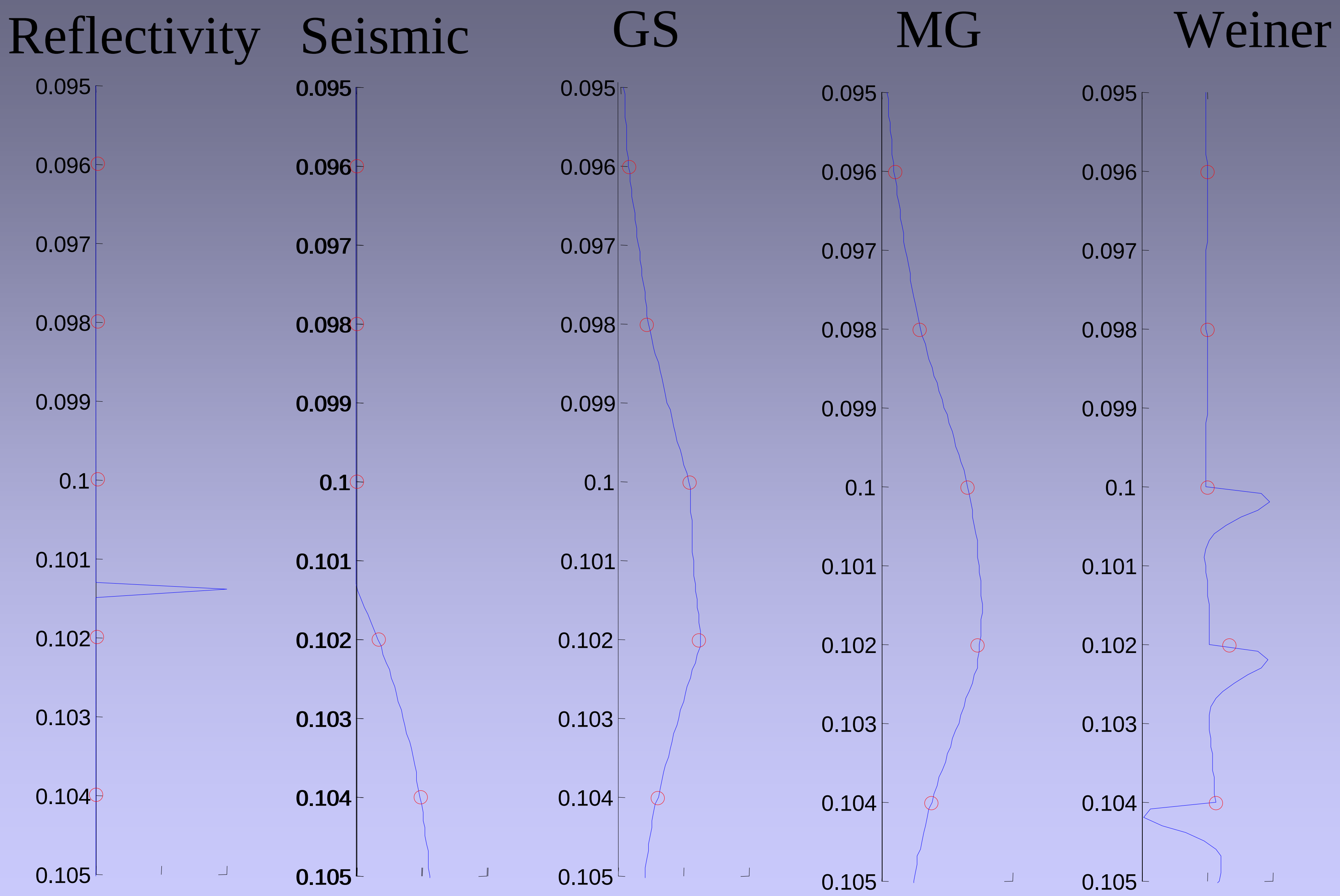


Re-sampled to 2 ms data, such that reflectivity spike sits in between samples

Resolution Results



Resolution Results Zoom



The Surface Consistent Equations

$$
\begin{bmatrix}
s_1 & & & r_1 & & & & & & & \\
s_1 & & & & r_2 & & & & & & \\
s_1 & & & & & r_3 & & & & & \\
s_1 & & & & & & r_4 & & & & \\
 & s_2 & & + & & r_3 & & & & & \\
 & s_2 & & & & & r_4 & & & & \\
 & s_2 & & & & & & r_5 & & & \\
 & s_2 & & & & & & & r_6 & & \\
 & & s_3 & & & & & r_5 & & & \\
 & & s_3 & & & & & & r_6 & & \\
 & & s_3 & & & & & & & r_7 & \\
 & & s_3 & & & & & & & & r_8
\end{bmatrix}
=
\begin{bmatrix}
t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12}
\end{bmatrix}
$$

The Surface Consistent Equations

$$
\begin{bmatrix}
1 & & & 1 & & & & & & & \\
1 & & & & 1 & & & & & & \\
1 & & & & & 1 & & & & & \\
1 & & & & & & 1 & & & & \\
 & 1 & & & & 1 & & & & & \\
 & 1 & & & & & 1 & & & & \\
 & 1 & & & & & & 1 & & & \\
 & 1 & & & & & & & 1 & & \\
 & & 1 & & & & & 1 & & & \\
 & & 1 & & & & & & 1 & & \\
 & & 1 & & & & & & & 1 & \\
 & & 1 & & & & & & & & 1
\end{bmatrix}
\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{bmatrix}
=
\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{bmatrix}
$$

$$\mathbf{As} = \mathbf{t}$$

The Surface Consistent Equations

Use least mean squares equations …

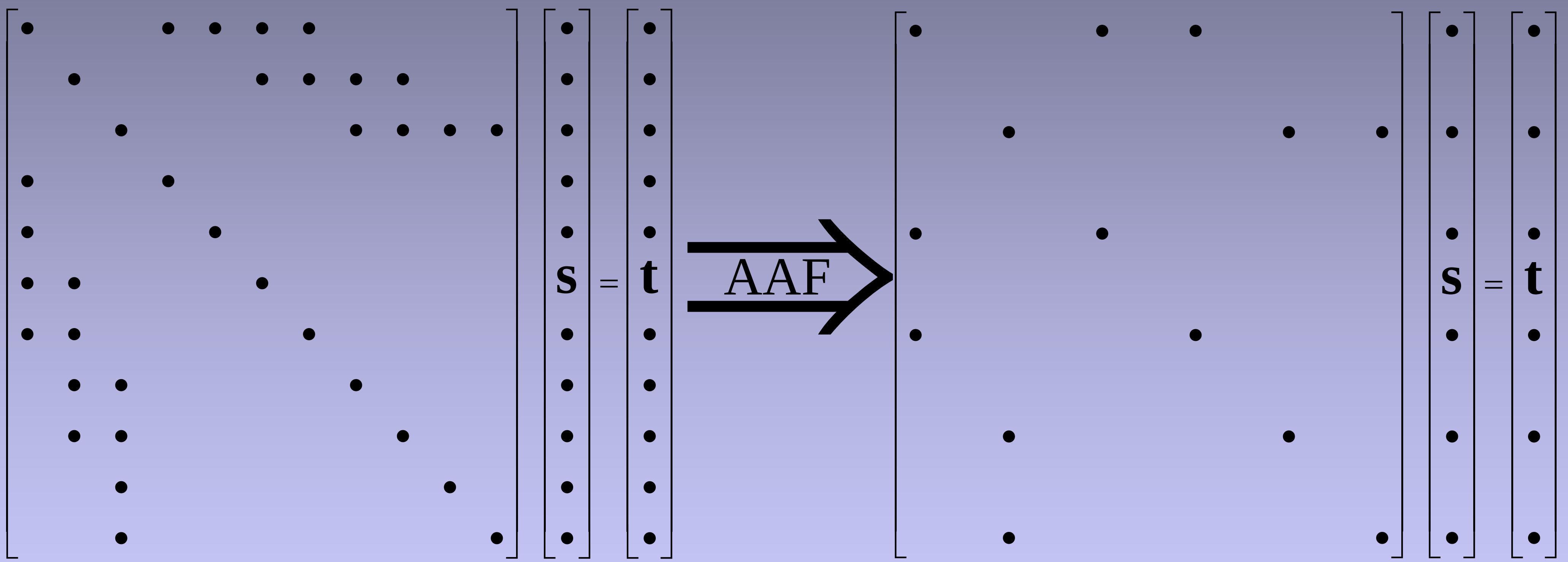
$$\mathbf{A}^T\mathbf{A}\mathbf{s} = \mathbf{A}^T\mathbf{t}$$

$$\mathbf{s} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{t}$$

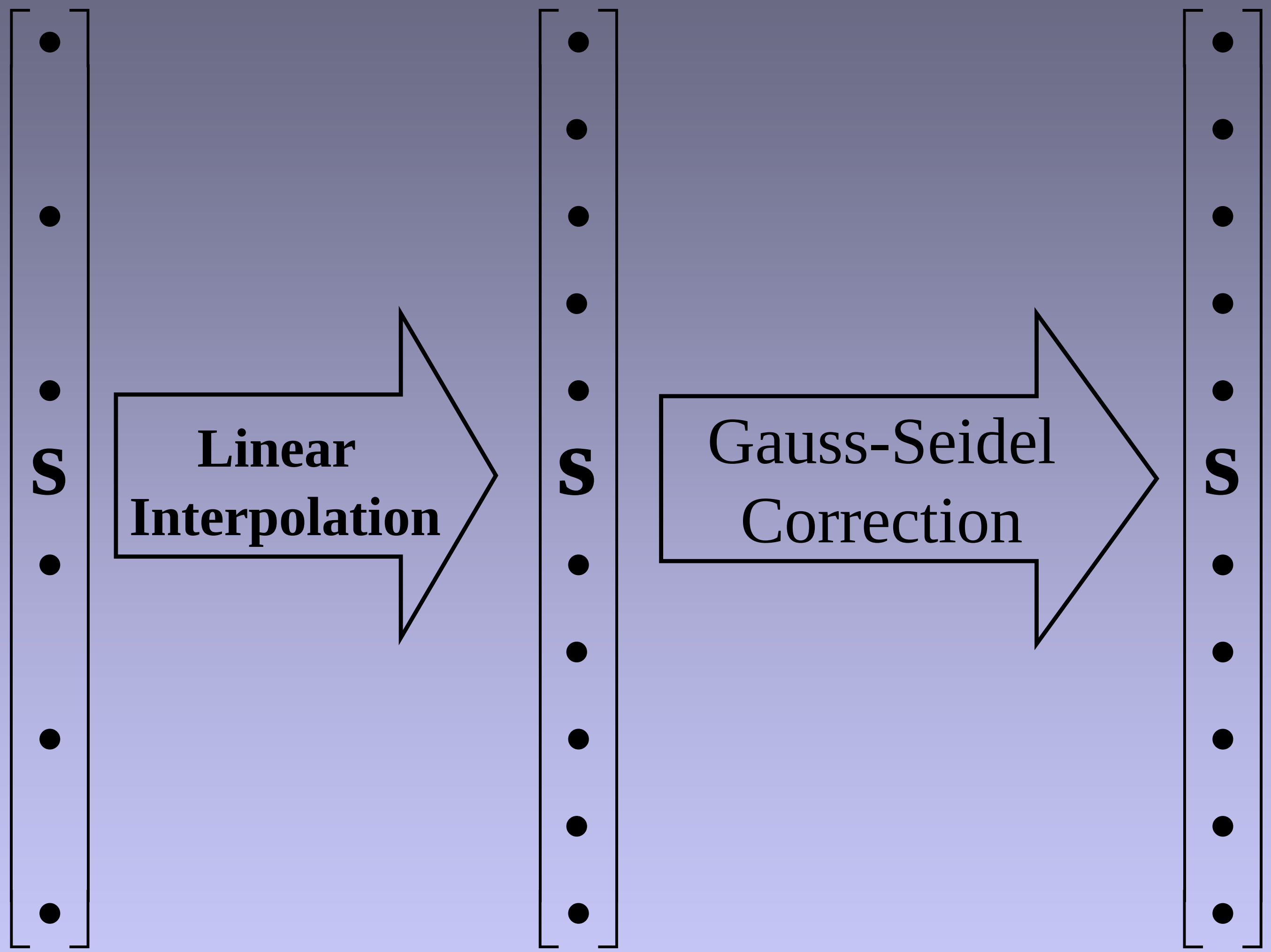
$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix}
4 & & & 1 & 1 & 1 & 1 & & & & \\
 & 4 & & & & 1 & 1 & 1 & 1 & & \\
 & & 4 & & & & & 1 & 1 & 1 & 1 \\
1 & & & 1 & & & & & & & \\
1 & & & & 1 & & & & & & \\
1 & 1 & & & & 2 & & & & & \\
1 & 1 & & & & & 2 & & & & \\
 & 1 & 1 & & & & & 2 & & & \\
 & 1 & 1 & & & & & & 2 & & \\
 & & 1 & & & & & & & 1 & \\
 & & 1 & & & & & & & & 1
\end{bmatrix}$$

Restrict the matrix

Anti-Alias Filter & Down Sampling



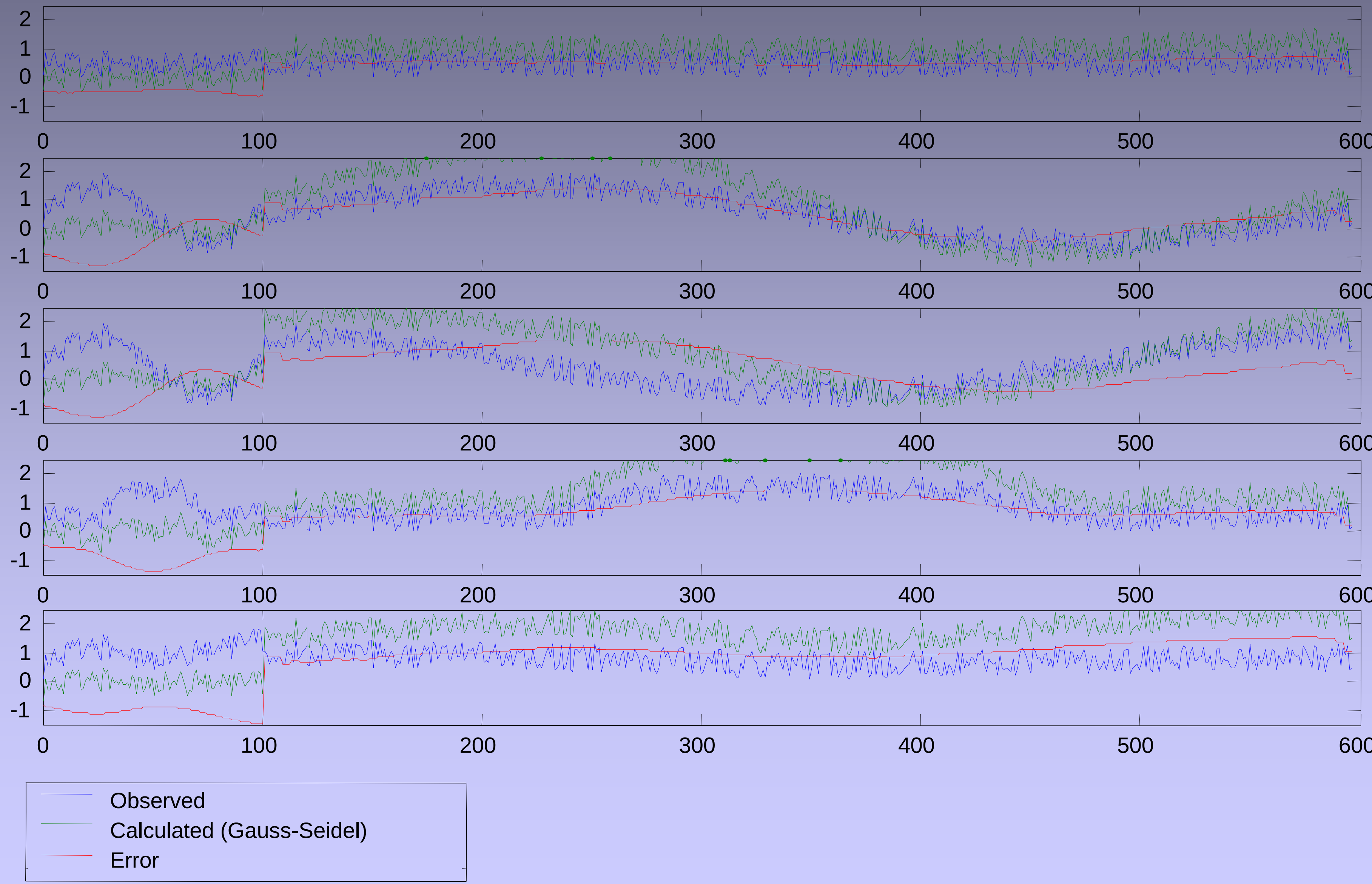
Interpolate the solution



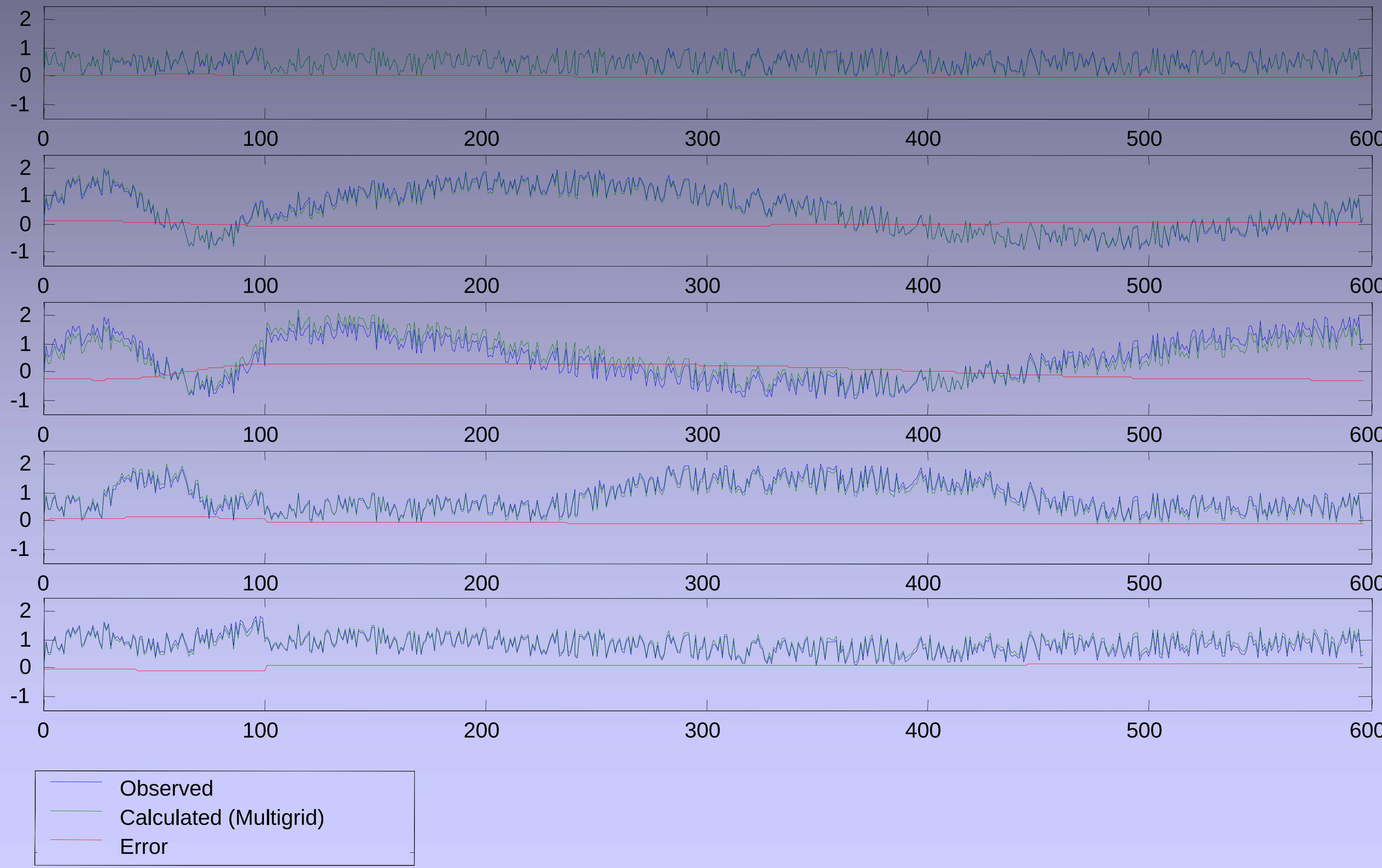
Interpolation introduces errors with approximately the same wavelength as the grid.

Gauss-Seidel Statics

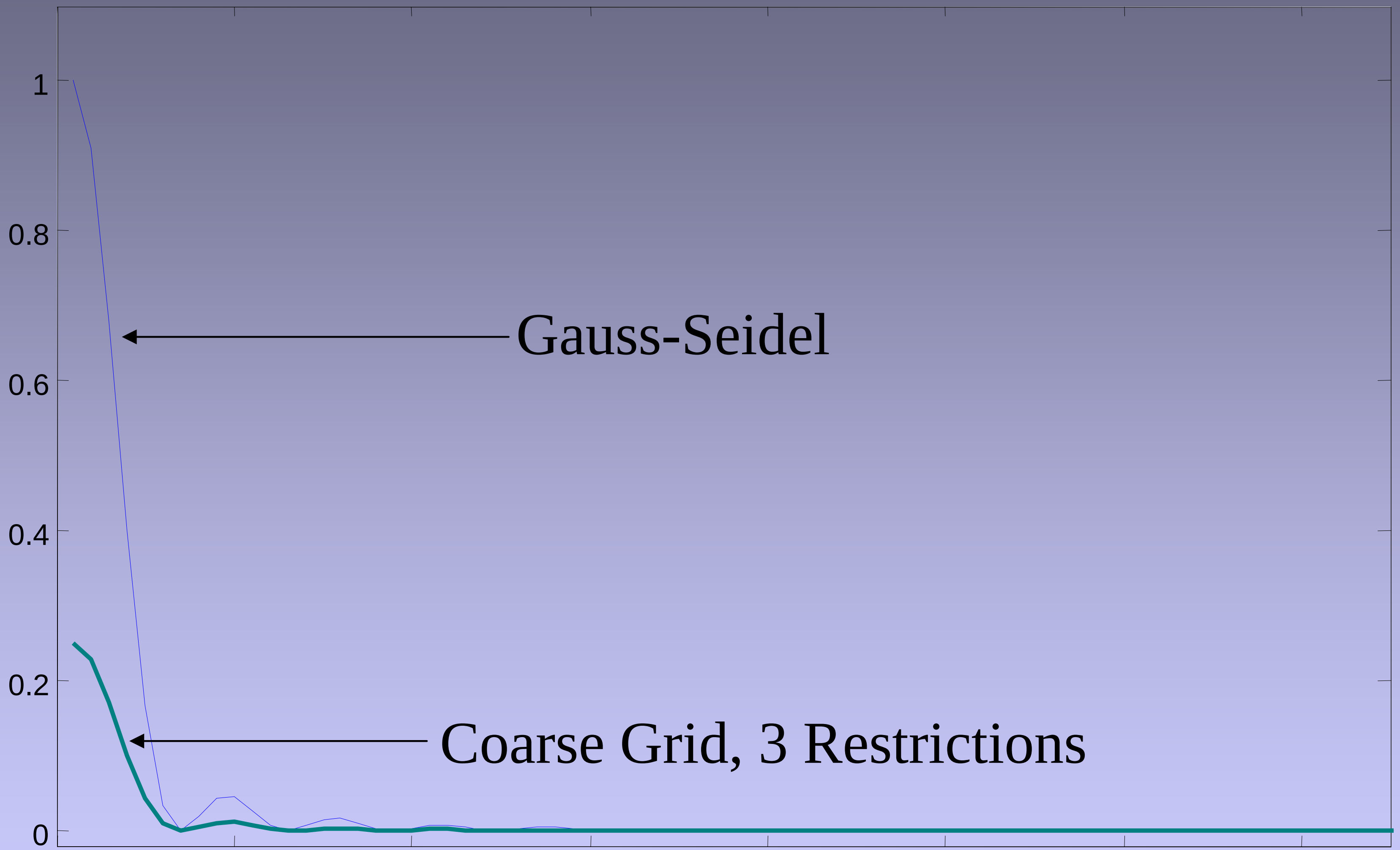
100 shots, 100 live receivers per shot, shot spacing of 4



Multigrid Statics



Spectral Performance



Decon Conclusions

- Multigrid deconvolution converges with the same limits of resolution as other methods, ie. the quality of the decon rests on the ability to estimate the wavelet
- For standard uses, may not be fast enough to be practical
- Extension to non-stationary wavelets straightforward
- May be of benefit where Fourier methods fail eg. Aliased data
- Provide a direct method to place reflectors in between samples, although presented results are ideal
- Insight into behavior of interpolation and restriction of seismic data

Statics Conclusions

- Will effectively estimate long wavelength component of surface consistent problems
- Faster compute time than Gauss-Seidel

In General …

- Multigrid methods will soon be applied to larger, non-linear problems where the calculation time becomes more of a factor.

Thanks

CREWES Project Sponsors and Members

Students and Instructors at the U of C

The Audience

References

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