CREWES

Phase Space Methods in Exploration Seismology

PIMS

Inverse Theory Summer School

Seattle, 2005

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Goals

- Examine some essential characteristics of seismic wavefields and their Fourier transforms
- Understand the concept of phase space for a wavefield
- Examine tools for manipulating a wavefield on its phase space:
  - raytracing, Fourier multipliers, pseudodifferential operators, Gabor multipliers
- Two extended examples
  - deconvolution: application of Gabor multipliers
  - wavefield extrapolation: application of pseudodifferential operators

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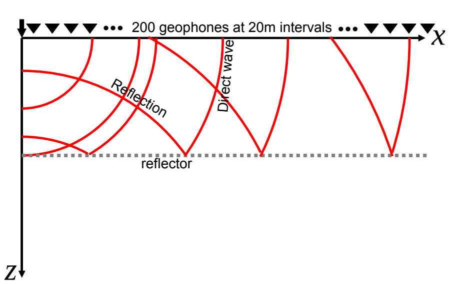
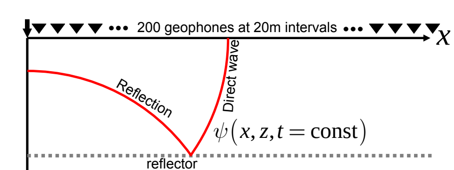
Part 1

Seismic Data

Seismic Shot Record



Seismic Shot Record



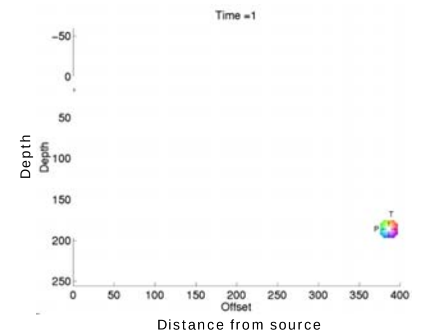
Geophones record $\psi(x,z=0,t)\in\mathbb{R}\times\mathbb{R}$

Want to know:

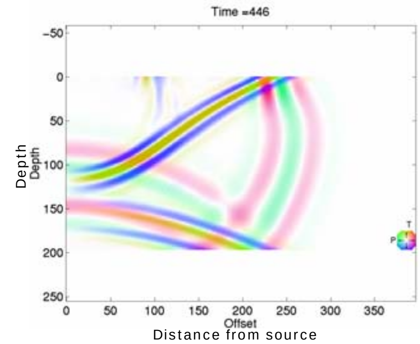
•Positions of reflectors

•Values of the reflection coefficients

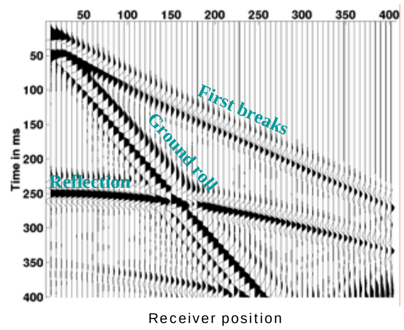
Purely elastic finite-difference simulation



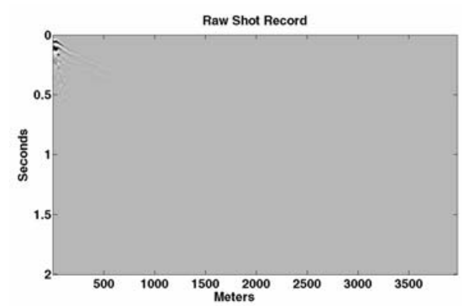
Purely elastic finite-difference simulation



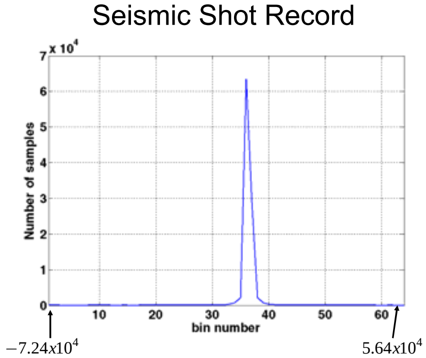
Surface recording of vertical motion



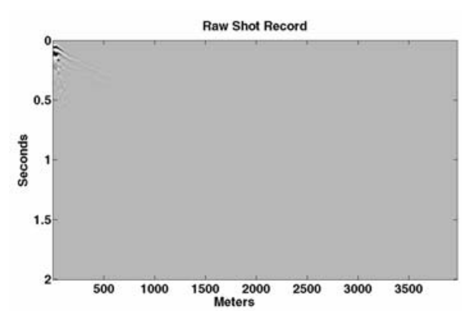
Seismic Shot Record



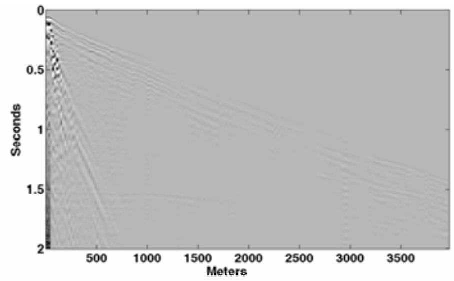
Seismic Shot Record



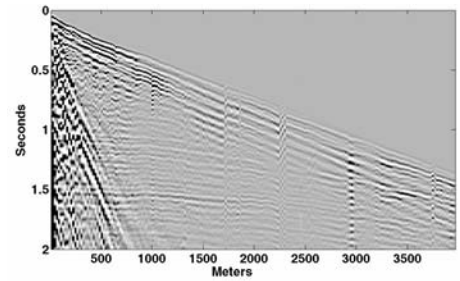
Seismic Shot Record



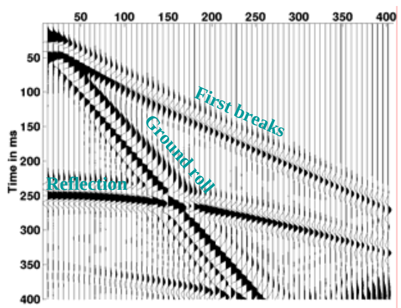
Seismic Shot Record



Seismic Shot Record



Surface recording of vertical motion



Part 2

Fourier Transforms

Fourier Transform (symplectic)

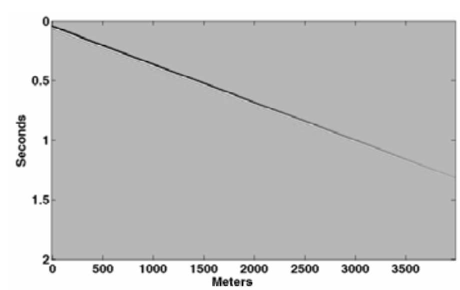
Forward transform over space and time

$$\hat{\hat{\psi}}(\xi_x, z=0, \omega) = \int_{\mathbb{R}^2} \psi(x, z=0, t) \mathrm{e}^{i(\xi_x x - \omega t)}\, dxdt$$

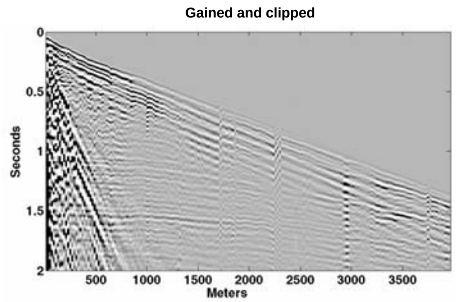
Inverse transform over wavenumber and frequency

$$\psi(x, z=0, t) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \hat{\hat{\psi}}(\xi_x, z=0, \omega) \mathrm{e}^{i(\omega t - \xi_x x)}\, d\xi_x d\omega$$

Synthetic First Break Event



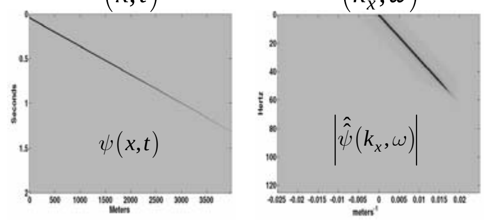
Seismic Shot Record



Fourier Transform

synthetic data

(x,t) (k_x,ω)



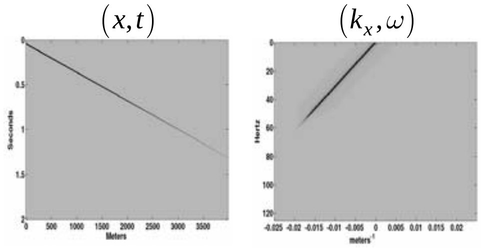
$$\hat{\hat{\psi}}(k_x,\omega)=\int_{\mathbb{R}^2}\psi(x,t)\mathrm{e}^{i(k_x x-\omega t)}\,dxdt$$

symplectic

Fourier Transform

synthetic data

(x,t) (k_x,ω)



$$\hat{\hat{\psi}}(k_x,\omega)=\int_{\mathbb{R}^2}\psi(x,t)\mathrm{e}^{-i(k_x x+\omega t)}\,dxdt$$

normal

Exercise

Model an ideal linear event, g(x,t), using the Delta distribution:

$$g(x,t)=\delta(px-t+c),\ p\in\mathbb{R}^+, c\in\mathbb{R}$$

where the Delta distribution has the property

$f(u_0)=\int_{\mathbb{R}}\delta(u-u_0)f(u)\,du$ for any f that we care about.

Show that the 2D (symplectic) Fourier transform of g(x,t) is

$$\hat{\hat{g}}(\xi_x,\omega)=2\pi\delta(\xi_x-p\omega)e^{-ic\xi_x}$$

use this to explain the preference stated in lecture for the symplectic Fourier transform. For $p\in[0,1]$ make a sketch showing where several typical events lie in both domains.

Exercise

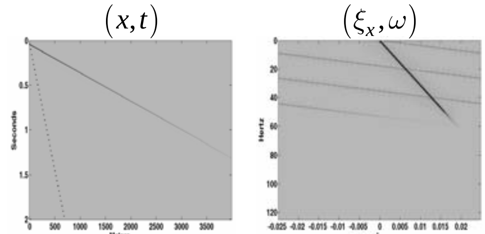
Conclusions from exercise:

- All events with the same slope (p-value) in (x,t) have the same amplitude spectrum in (ξ_x,ω).
- The slope of an event in (x,t) and the corresponding event in (ξ_x,ω) are inversely related.
- The value of p can be calculated directly from the ratio of ξ_x to ω in Fourier space.

Fourier Transform

synthetic data

(x,t) (ξ_x,ω)

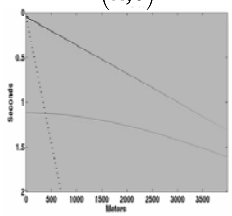


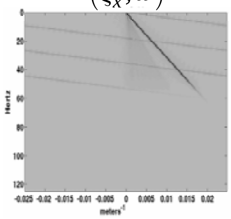
$$\hat{\hat{\psi}}(\xi_x,\omega)=\int_{\mathbb{R}^2}\psi(x,t)\mathrm{e}^{i(\xi_x x-\omega t)}\,dxdt$$

Fourier Transform

synthetic data

(x,t) (ξ_x,ω)



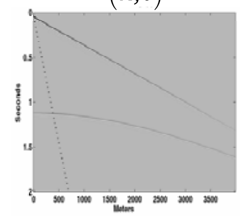


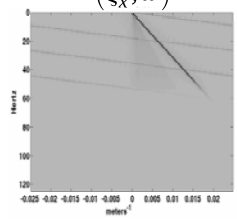
$$\hat{\hat{\psi}}(\xi_x,\omega)=\int_{\mathbb{R}^2}\psi(x,t)\mathrm{e}^{i(\xi_x x-\omega t)}\,dxdt$$

Fourier Transform

synthetic data

(x,t) (ξ_x,ω)



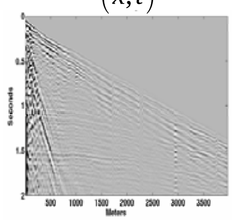


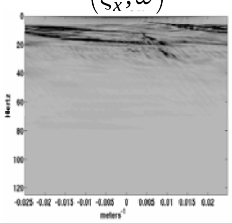
$$\hat{\hat{\psi}}(\xi_x,\omega)=\int_{\mathbb{R}^2}\psi(x,t)\mathrm{e}^{i(\xi_x x-\omega t)}\,dxdt$$

Fourier Transform

real data

(x,t) (ξ_x,ω)

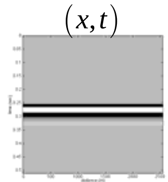


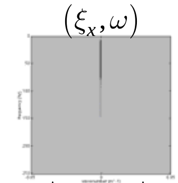


$$\hat{\hat{\psi}}(\xi_x,\omega)=\int_{\mathbb{R}^2}\psi(x,t)\mathrm{e}^{i(\xi_x x-\omega t)}\,dxdt$$

Fourier Transform Pairs

(x,t) (ξ_x,ω)

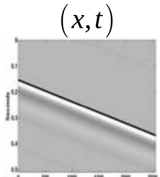


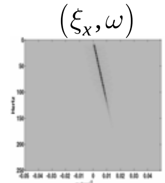


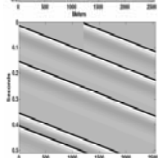
$\psi(x,t)$ $\left|\hat{\hat{\psi}}(\xi,\omega)\right|$

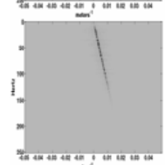
Fourier Transform Pairs

(x,t) (ξ_x,ω)



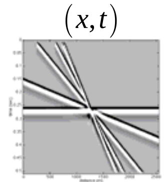


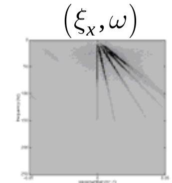


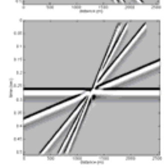


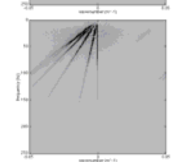
Fourier Transform Pairs

(x,t) (ξ_x,ω)



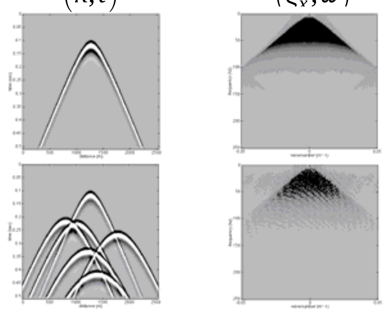






Fourier Transform Pairs

(x,t) (ξ_x,ω)

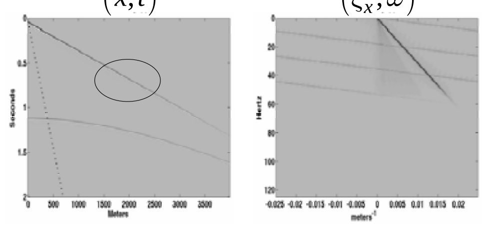


Part 3

Phase Space

Local Fourier Transforms

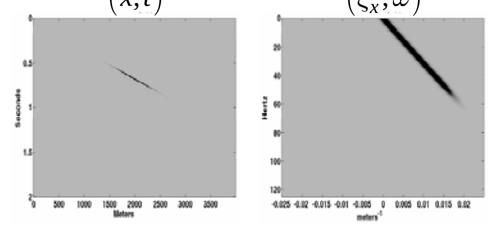
(x,t) (ξ_x,ω)



Apply a 2D Gaussian window in (x,t)

Local Fourier Transforms

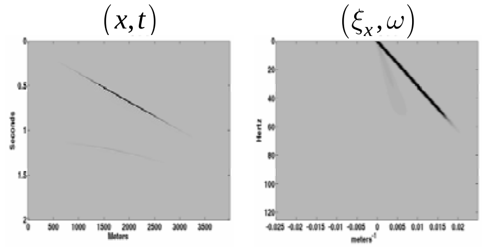
(x,t) (ξ_x,ω)



Localization in one domain causes blurring in the other

Local Fourier Transforms

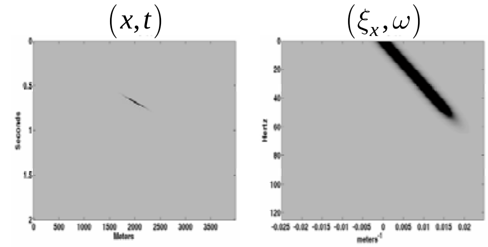
(x,t) (ξ_x,ω)



A larger window causes less blurring but is, of course, less local.

Local Fourier Transforms

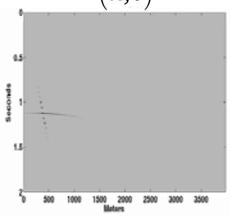
(x,t) (ξ_x,ω)

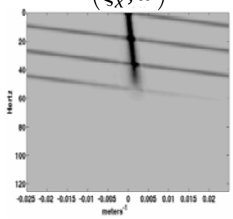


An even smaller window causes extreme blurring.

Local Fourier Transforms

$\left(x,t\right)$ $\left(\xi_x,\omega\right)$





Localizing somewhere else shows us a different spectrum.

Uncertainly Principle

Localization in (x,t) causes loss of detail in (ξ_x,ω). That is, we cannot precisely define the (ξ_x,ω) values at a precise (x,t) position. As Heisenberg showed in the context of quantum mechanics, this implies:

(uncertainty in (x,t))(uncertainty in (ξ_x,ω)) ≥ a constant

This is often stated as the time-width band-width theorem.

Question: Just what is meant by "uncertainty" in such a statement?

Time-width Band-width Theorem

Given any convenient measure of width, the time-width and bandwidth of a signal are inversely proportional.

$$E=\int_{\mathbb{R}}\left|s(x)\right|^2 dx$$

$$x_0=\left[\int_{\mathbb{R}} x\left|s(x)\right|^2 dx\right]E^{-1} \qquad \xi_0=\left[\int_{\mathbb{R}} \xi\left|\hat{s}(\xi)\right|^2 d\xi\right]E^{-1}$$

$$(\Delta x)^2=\left[\int_{\mathbb{R}} (x-x_0)^2\left|s(x)\right|^2 dx\right]E^{-1} \qquad (\Delta\xi)^2=\left[\int_{\mathbb{R}} (\xi-\xi_0)^2\left|\hat{s}(\xi)\right|^2 d\xi\right]E^{-1}$$

$$\Delta x\Delta\xi \geq \left(4\pi\right)^{-1}$$

The equality holds only for a Gaussian signal.

Time-limited Band-limited Theorem

If a signal, not identically zero, is compactly supported then its Fourier transform cannot be and vice-versa.

It follows that any finite length signal cannot be bandlimited.

Correspondence

- Associated with a neighborhood of a point in (x,t), there is a local Fourier spectrum. (Strictly speaking this depends upon the details of the localizing window.)
- Resolution in the local spectrum is directly proportional to the size (radius) of the neighborhood.

Phase Space

The phase space of a wavefield is the 8D manifold:

$$M:\left(x,y,z,t\right)\times\left(\xi_x,\xi_y,\xi_z,\omega\right)$$

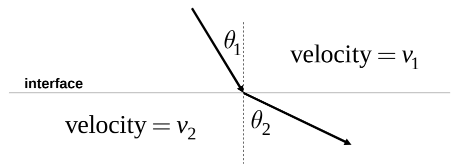
Methods that have been devised to directly manipulate a field on its phase space include:

- **Ray tracing**
- **Pseudodifferential operators**
- **Gabor Multipliers**
- **Nonstationary filters**

Part 3

Raytracing

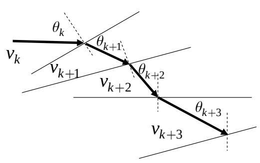
Ray Parameter



Snell's Law $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2} \equiv p$

Ray Parameter

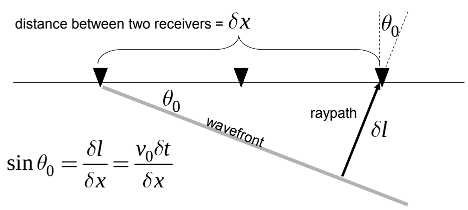
More generally: $\theta =$ angle between interface normal and ray



$$\frac{\sin\theta_k}{v_k} = \text{constant}$$

The ray parameter is conserved during propagation.

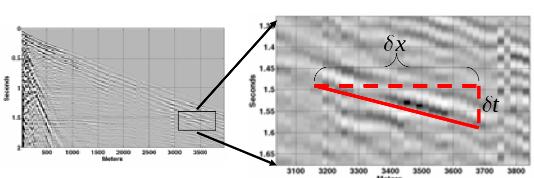
Measurement of Ray Parameter



$$\sin\theta_0 = \frac{\delta l}{\delta x} = \frac{v_0 \delta t}{\delta x}$$

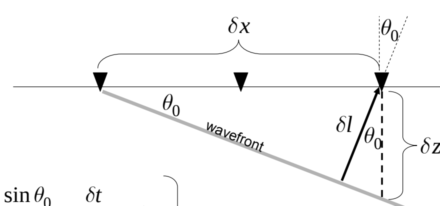
$$\Rightarrow \frac{\sin\theta_0}{v_0} = \frac{\delta t}{\delta x} = p$$

Measurement of Ray Parameter



$$\frac{\delta t}{\delta x}(3500m, 1.55s) \approx \frac{1.59s - 1.49s}{3680m - 3160m} \approx 1.92\times10^{-4}\,\frac{s}{m}$$

Recall



$$\left.\begin{aligned}\frac{\sin\theta_0}{v_0} &= \frac{\delta t}{\delta x} = p\\ \frac{\cos\theta_0}{v_0} &= \frac{\delta t}{\delta z} = q\end{aligned}\right\} \quad \left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2 = \frac{1}{v_0^2}$$

The eikonal equation: the foundation of ray tracing

Ray Theory
- A high frequency approximation -

Start with the wave equation

$$\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

Assume "plane wave" solution

$$\psi = A(\vec{x})e^{i\omega(t - T(\vec{x}))}$$

Ray Theory
- A high frequency approximation -

$$(\nabla T)^2 = \frac{1}{v^2}$$

The eikonal equation gives traveltimes.

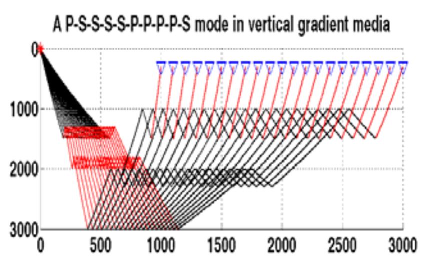
$$\frac{A}{2}\nabla^2 T = \nabla A \bullet \nabla T$$

The transport equation gives amplitudes

This approximate theory gets better with higher frequency. In highly heterogeneous media, the theory is notoriously "touchy".

Ray Theory
- Snell's Law -

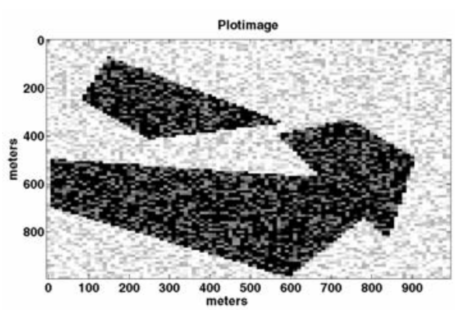
The eikonal equation is equivalent to Snell's law.



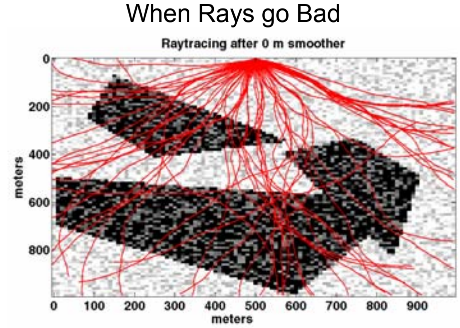
Ray Theory
- Problems and Limitations -

- Traveltimes are usually better determined than amplitudes.
- This approximate theory gets better with higher frequency. In highly heterogeneous media, the theory is notoriously "touchy".
  - Diffractions are not usually simulated.
  - Random media must often be smoothed to simulate wave behaviour.

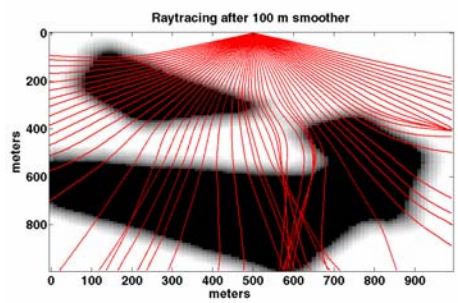
Velocity model



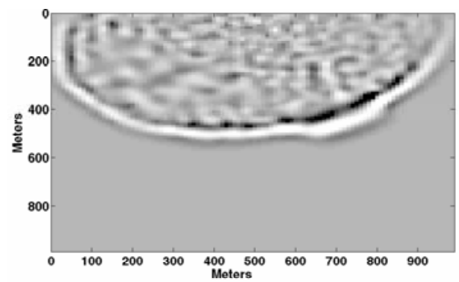
Raytracing in unsmoothed model
When Rays go Bad



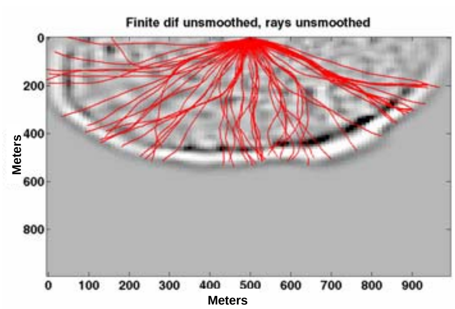
Raytracing in smoothed model



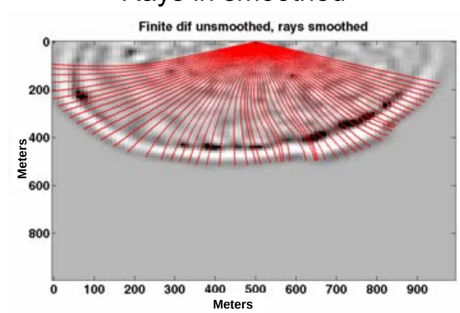
Waves in unsmoothed media



Waves and rays in unsmoothed



Waves in unsmoothed
Rays in smoothed



Part 4

Stationary Filters and Fourier Methods

Stationary Filters

We define “signals” as 1D functions in Schwartz space:

$$s(t), r(t), w(t) \in S$$

A 1D stationary filter operation is, for example,

$$s(t) = \int_{\mathbb{R}} w(t-\tau)\, r(\tau)\, d\tau \equiv (C_w r)(t)$$

which is a convolution integral.

Stationary Filters

Stationarity, or translation invariance, means that the "impulse response" of the system is "temporally invariant", eg:

$$r(t) = \sum_{j\in\mathbb{Z}} r_j \delta\left(t - t_j\right)$$

$$s(t) = \int_{\mathbb{R}} w(t-\tau) \sum_{j\in\mathbb{Z}} r_j \delta\left(\tau - t_j\right) d\tau$$

$$= \sum_{j\in\mathbb{Z}} r_j \int_{\mathbb{R}} w(t-\tau)\delta\left(\tau - t_j\right) d\tau = \sum_{j\in\mathbb{Z}} r_j w\left(t - t_j\right)$$

Stationary Filters

This concept of stationary filters can be generalized in many ways including:

- Extension to signals in *S'*, the space of tempered distributions.
- Extension to discrete sequences (digital signal theory).
- Inverse filter theory, Wiener filters.
- Fourier multipliers.

We consider the last of these explicitly.

Fourier Multipliers

Every stationary convolution operator has a corresponding Fourier multiplier:

$$s(t) = (C_w r)(t) = \left(F^{-1} M_{\hat{w}} F r\right)(t)$$

or more simply

$$s = C_w r = F^{-1} M_{\hat{w}} F r$$

where:

$$M_a b \equiv ab$$
$$\hat{w} \equiv Fw$$
$$F = \text{the Fourier transform}$$

Fourier Multipliers
Inverse Operators

A Fourier multiplier has a simple inverse, if

$$s = F^{-1} M_{\hat{w}} F r$$

then

$$r = F^{-1} M_{\hat{w}^{-1}} F s$$

provided that $\hat{w} \neq 0$

Fourier Multipliers
Inverse Operators

If $\hat{w} = 0$ somewhere in its domain, or is very small, then a common practice is to seek an approximate inverse such as

then

$$r \approx F^{-1} M_{\hat{w}_I} F s$$

where

$$\hat{w}_I = \frac{1}{\hat{w} + \mu \sup(\hat{w})}, \mu \in (0,1)$$

Fourier Multipliers
Solution of PDE's

$$\frac{\partial^2 \psi}{\partial z^2} = \left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\psi$$

$$\frac{\partial^2 \psi}{\partial z^2} = \frac{1}{4\pi^2}\left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\int_{\mathbb{R}^2} \hat{\hat{\psi}}(\xi_x, z, \omega) e^{i(\omega t - \xi_x x)} d\xi_x d\omega$$

$$\frac{\partial^2 \psi}{\partial z^2} = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \alpha_2(\xi_x, \omega) \hat{\hat{\psi}}(\xi_x, z, \omega) e^{i(\omega t - \xi_x x)} d\xi_x d\omega$$

$$\alpha_2(\xi_x, \omega) = \xi_x^2 - \frac{\omega^2}{v^2}$$ Fourier multiplier or symbol for the second z derivative.

Fourier Multipliers
Solution of PDE's

Now, we deduce two alternative expressions for the first z derivative:

$$\left(\frac{\partial\psi}{\partial z}\right)^{\pm} = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \alpha_1^{\pm}(\xi_x,\omega)\hat{\hat{\psi}}(\xi_x,z,\omega)e^{i(\omega t-\xi_x x)}\,d\xi_x d\omega$$

$$\alpha_1^{\pm}(\xi_x,\omega) = \pm\sqrt{\alpha_2(\xi_x,\omega)} = \pm i\xi_z \qquad \xi_z = \begin{cases}\sqrt{\frac{\omega^2}{v^2}-\xi_x^2}, \frac{\omega^2}{v^2}\geq \xi_x^2\\ i\sqrt{\xi_x^2-\frac{\omega^2}{v^2}}, \xi_x^2 > \frac{\omega^2}{v^2}\end{cases}$$

These are examples of one-way wave equations. They are exact for v=constant and $\omega^2 v^{-2} \geq \xi_x^2$

However, this approach fails if v is not constant.

Fourier Multipliers
Solution of PDE's

Solutions to either of these one-way wave equations are also solutions to the two-way wave equation

Let ψ^+ satisfy $\dfrac{\partial\psi^+}{\partial z} = F_2^{-1}M_{\alpha_1^+}F_2\psi^+$

$$\frac{\partial}{\partial z}\left(\frac{\partial\psi^+}{\partial z}\right) = \psi = \left[F_2^{-1}M_{\alpha_1^+}F_2\right]\frac{\partial}{\partial z}\psi$$

$$= F_2^{-1}M_{\alpha_1^+}F_2F_2^{-1}M_{\alpha_1^+}F\psi = F_2^{-1}M_{\alpha_2}F_2\psi = \frac{\partial^2\psi}{\partial z^2}$$

Exercise

Show that $\psi^{\pm} = \dfrac{1}{4\pi^2}\displaystyle\int_{\mathbb{R}^2} A(\xi_x,\omega)e^{i(\omega t-\xi_x x\pm\xi_z z)}\,d\xi_x d\omega$

(where *A* is arbitrary) solves the one-way wave equations on the previous slides. Then show that the + sign corresponds to waves traveling in the -z direction and the - sign gives waves traveling in the +z direction.

What happens with this approach when v depends on x?

Problem

- We need wavefield analysis and filtering methods that adapt rapidly to spatial and temporal variations in the wavefield but still retain high fidelity.
- Raytracing offers rapid adaptation but poor fildelity.
- Fourier methods give high fidelity but poor spatial adaptivity.

Part 5

Pseudodifferential Operators

Pseudodifferential Operators

Consider the previous example when v=v(x):

$$\frac{\partial^2\psi}{\partial z^2} = \frac{1}{4\pi^2}\left[\frac{1}{v(x)^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\int_{\mathbb{R}^2}\hat{\hat{\psi}}(\xi_x,z,\omega)e^{i(\omega t-\xi_x x)}\,d\xi_x d\omega$$

$$\frac{\partial^2\psi}{\partial z^2} = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\alpha_2(\xi_x,x,\omega)\hat{\hat{\psi}}(\xi_x,z,\omega)e^{i(\omega t-\xi_x x)}\,d\xi_x d\omega$$

$$\alpha_2(\xi_x,x,\omega) = \xi_x^2 - \frac{\omega^2}{v(x)^2}$$

This integral is no longer an inverse Fourier transform but is instead an example of pseudodifferential operator, specifically of the Kohn-Nirenberg (standard) calculus.

Pseudodifferential Operators

Kohn-Nirenberg standard form:

$$g_s(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \alpha(x,\xi)\hat{h}(\xi)e^{ix\xi}d\xi \equiv \left(F_\alpha^I \hat{h}\right)(x)$$

Kohn-Nirenberg anti-standard form:

$$\hat{g}_a(\xi) = \int_{\mathbb{R}} \alpha(x,\xi)h(x)e^{-ix\xi}dx \equiv (F_\alpha h)(\xi)$$

In general $g_a \neq g_s$, although you should be able to find an obvious case when they are equal.

Pseudodifferential Operators

These operators extend the idea of Fourier multipliers to the "nonstationary" setting.

Definition: The x dependence of the symbol will be called its nonstationary dependence.

Definition: A "stationary limit" of a pseudodifferential operator is any limiting form of the operator in which the nonstationary dependence of the symbol becomes constant.

Pseudodifferential Operators

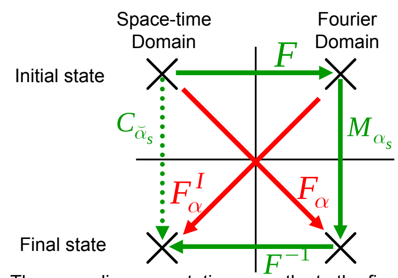
We have:

$$\lim_{stat} F_\alpha^I = F^{-1}M_{\alpha_s}$$

$$\lim_{stat} F_\alpha = M_{\alpha_s}F$$

where $$\lim_{stat} \alpha = \alpha_s$$

Pseudodifferential Operators



The green lines are stationary paths to the final state while the red lines are nonstationary. In general, the red paths give a different result if the same symbol is used.

Spaces and Symbol Classes

Usually pseudodifferential operators can be extended to mappings:

$$T_\alpha : S' \to S'$$

Symbols are classified by the order of their polynomial growth at infinity:

We say $\alpha \in S_m$

if $$\frac{\partial^\rho \alpha}{\partial \xi^\rho} = O\left(\left[1+|\xi|^2\right]^{m/2}\right), \rho \in \mathbb{N}, m \in \mathbb{Z}$$

Symbols are also classified by their growth in x.

Pseudodifferential Operators

Back to the wave equation, in the variable velocity case, we might still hope that

$$\left(\frac{\partial \psi}{\partial z}\right)^{\pm} = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \alpha_1^{\pm}(\xi_x,x,\omega)\hat{\hat{\psi}}(\xi_x,z,\omega)e^{i(\omega t-\xi_x x)}d\xi_x d\omega$$

$$\alpha_1^{\pm}(\xi_x,x,\omega) = \pm\sqrt{\alpha_2(\xi_x,x,\omega)}$$

It turns out that this is still a useful approximate one-way wave equation but its solutions are not solutions to the two-way equation.

Pseudodifferential Operators

To see this, write

$$\left(\frac{\partial \psi}{\partial z}\right)^{+} = T_{\alpha_1^+}\psi$$

and ask whether

$$\frac{\partial^2 \psi}{\partial z^2} = T_{\alpha_2}\psi \underset{?}{=} T_{\alpha_1^+} \circ T_{\alpha_1^+}\psi$$

That is, does the composition of $T_{\alpha_1^+}$ with itself give T_{α_2} ?

Pseudodifferential Operators Composition Theorem

Let T_α T_β be two pseudodifferential operators with suitably nice symbols. Then

$$T_\beta \circ T_\alpha = T_\gamma \qquad \alpha \in S_m, \beta \in S_n \Rightarrow \gamma \in S_{m+n}$$

where γ has the asymptotic expansion

$$\gamma \sim \alpha\beta - i\frac{\partial \beta}{\partial \xi}\frac{\partial \alpha}{\partial x} + \cdots$$

This expansion is written for 1D but generalizes to any number of dimensions.

Pseudodifferential Operators

So,

$$T_{\alpha_1^+} \circ T_{\alpha_1^+}\psi = T_\gamma \psi$$

where

$$\gamma \sim \left(\alpha_1^+\right)^2 - i\frac{\partial \alpha_1^+}{\partial \xi}\frac{\partial \alpha_1^+}{\partial x} + \cdots$$

Thus, if we take $\alpha_1^+ = +\sqrt{\alpha_2}$ then we do not get an exact factorization (i.e. $\gamma \neq \alpha_2$).

However, it is still possible to find an exact factorization in certain cases (e.g. Fishman …).

Pseudodifferential Operators

A problem with attempting this factorization using pseudodifferential operator theory is that the theory assumes the relevant symbols are *elliptic*.

Definition: A pseudodifferential symbol is said to be elliptic if there exists a constant C such that:

$$\left|\alpha(x,\xi)\right| > C\left|\xi\right|, \forall x \in \mathbb{R}$$

Symbol $\alpha_2(\xi_x, x, \omega) = \xi_x^2 - \dfrac{\omega^2}{v(x)^2}$ is not elliptic.

Part 6

Separable Symbols

and

The Gabor Transform

KN Formalism

Recall

$$s(x) = (T_\alpha r)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \alpha(x,\xi)\hat{r}(\xi)e^{ix\xi}d\xi$$

If

$$\lim_{stat} \alpha(x,\xi) = \alpha_0(\xi)$$

Then

$$\lim_{stat} s = \underbrace{F^{-1}M_{\alpha_0}F}_{\text{a Fourier multiplier}} r$$

Approximate symbols

Consider an arbitrary symbol $\alpha(x,\xi)$
One can always find a partition of $\mathbb{R}, \{x_k\}, k \in \mathbb{Z}$
and corresponding functions $\{\alpha_k\}$ such that

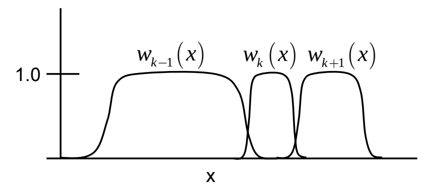
$$\left\| \alpha(x,\xi) - \sum_{k\in\mathbb{Z}} \chi_k(x)\alpha_k(\xi) \right\|_{L2} < \varepsilon$$

$$\chi_k(x) = \begin{cases} 1, x \in [x_k, x_{k+1}) \\ 0, \text{otherwise} \end{cases}$$

Piecewise Stationary Symbols

Suppose the symbol is separable such that

$$\alpha(x,\xi) = \sum_{k\in\mathbb{Z}} w_k(x)\alpha_k(\xi) \qquad w_k(x) \in C_0^\infty$$



Piecewise Stationary Symbols
Standard Calculus

Then we have

$$\alpha(x,\xi) = \sum_{k\in\mathbb{Z}} w_k(x)\alpha_k(\xi)$$

$$(T_\alpha r)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} w_k(x)\alpha_k(\xi)\right]\hat{r}(\xi)e^{ix\xi}d\xi$$

$$(T_\alpha r)(x) = \sum_{k\in\mathbb{Z}} w_k(x)\frac{1}{2\pi}\int_{\mathbb{R}} \alpha_k(\xi)\hat{r}(\xi)e^{ix\xi}d\xi$$

$$T_\alpha r = \sum_{k\in\mathbb{Z}} w_k F^{-1} M_{\alpha_k} F r$$

superposition of windowed Fourier multipliers

Piecewise Stationary Symbols
Anti-Standard Calculus

Alternatively:

$$\alpha(x,\xi) = \sum_{k\in\mathbb{Z}} w_k(x)\alpha_k(\xi)$$

$$\widehat{\left(\tilde{T}_\alpha r\right)}(\xi) = \int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} w_k(x)\alpha_k(\xi)\right] r(x)e^{-ix\xi}dx$$

$$\widehat{\left(\tilde{T}_\alpha r\right)}(x) = \sum_{k\in\mathbb{Z}} \alpha_k(\xi)\int_{\mathbb{R}} w_k(x)r(x)e^{-ix\xi}dx$$

$$\tilde{T}_\alpha r = \sum_{k\in\mathbb{Z}} F^{-1} M_{\alpha_k} F w_k r$$

superposition of Fourier multipliers of a windowed function

Equivalent forms

For a separable symbol

$$\alpha(x,\xi) = \sum_{k\in\mathbb{Z}} w_k(x)\alpha_k(\xi)$$

The following are equivalent

$$T_\alpha r = \sum_{k\in\mathbb{Z}} w_k F^{-1}\alpha_k \hat{r}$$

$$T_\alpha r = \sum_{k\in\mathbb{Z}} w_k C_{\check{\alpha}_k} r$$

$$T_\alpha r = F^{-1}\left(\sum_{k\in\mathbb{Z}} C_{\hat{w}_k}\alpha_k \hat{r}\right)$$

Exercise: Derive the corresponding formulae for $\tilde{T}_\alpha$

Piecewise Stationary Symbols
Windowing Analogs

$$\alpha(x,\xi) = \sum_{k\in\mathbb{Z}} w_k(x)\alpha_k(\xi)$$

$$T_\alpha r = \sum_{k\in\mathbb{Z}} w_k F^{-1} M_{\alpha_k} F r$$

Windowed inverse Fourier transform

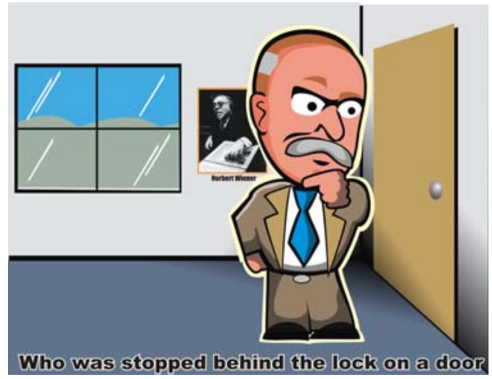
$$\tilde{T}_\alpha r = \sum_{k\in\mathbb{Z}} F^{-1} M_{\alpha_k} F w_k r$$

Windowed forward Fourier transform

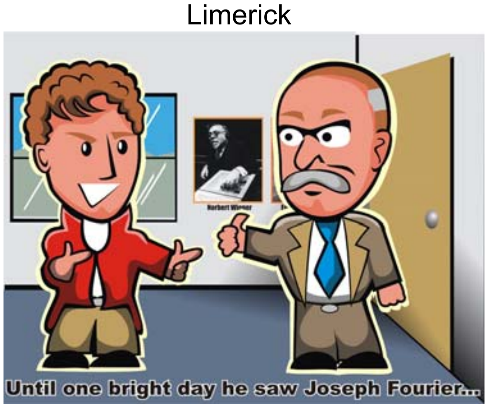
This suggests the Gabor Transform!

Limerick

Limerick

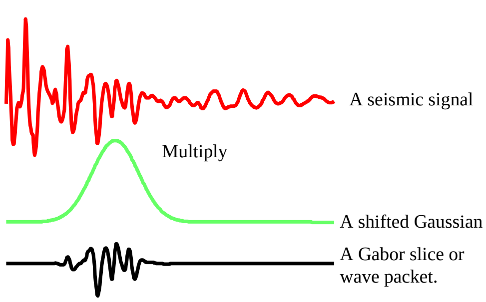


Limerick



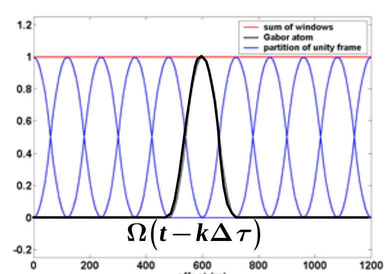
Limerick

The Gabor Idea

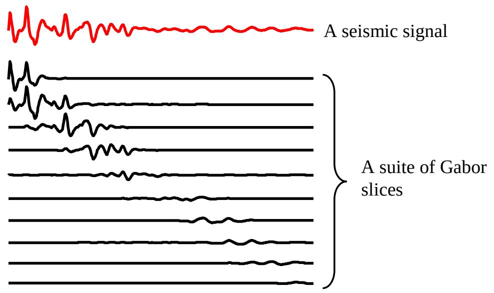


A Partition of Unity

The key to a fast, robust Gabor transform



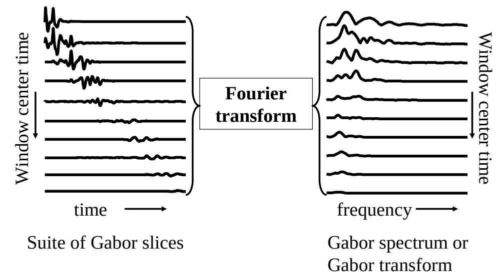
The Gabor Idea



The suite of Gabor slices will sum to recreate the original signal with high fidelity because of the partition of unity.

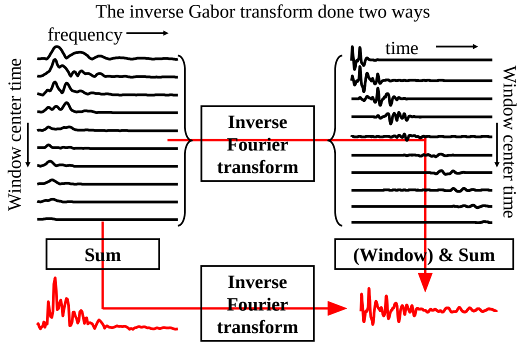
The Gabor Idea

The Gabor transform



The Gabor Idea

The inverse Gabor transform done two ways



Gabor Transform

Partition of unity

$$\sum_{k\in\mathbb{Z}} \Omega_k(x) = 1, \quad \Omega_k(x) \in C_0^\infty$$

Let $\underbrace{g_k(x) = \Omega_k^{\,p}(x)}_{\text{analysis window}}$ and $\underbrace{\gamma_k(x) = \Omega_k^{\,1-p}(x)}_{\text{synthesis window}},\ p \in [0,1]$

Then, the Gabor transform is defined by

$$V_g s(k,\xi) = F(g_k s)(\xi) : L^2(\mathbb{R}) \to L^2(\mathbb{Z}\times\mathbb{R})$$

This particular Gabor transform is partially discrete by design. Fully discrete and fully analytic algorithms are easily derived.

Gabor Transform Inverse

Given $\quad V_g s(k,\xi) = F(g_k s)(\xi) \in \mathbb{Z}\times\mathbb{R}$

The signal is recovered with a windowed inverse Fourier transform and a summation over windows.

$$V_\gamma^{-1}(V_g s) = \sum_{k\in\mathbb{Z}} \gamma_k F^{-1} F g_k s = \sum_{k\in\mathbb{Z}} \gamma_k g_k s = s$$

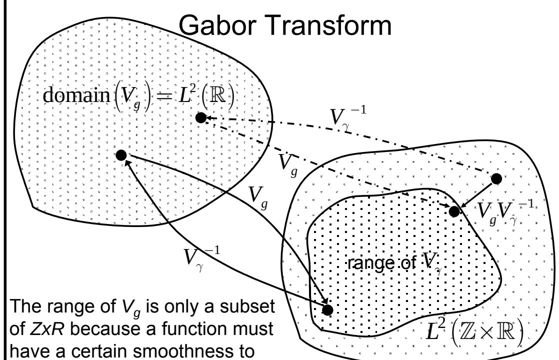
Note that:

$$V_\gamma^{-1} V_g = 1 \in L^2(\mathbb{R})$$

$$V_g V_\gamma^{-1} = P \neq 1 \in L^2(\mathbb{Z}\times\mathbb{R})$$

where P is a projection operator onto the range of the forward Gabor transform.

Gabor Transform



The range of V_g is only a subset of *ZxR* because a function must have a certain smoothness to be a Gabor transform.

Gabor Multipliers

Given $V_g s(k,\xi) = F(g_k s)(\xi) \in \mathbb{Z}\times\mathbb{R}$

$\alpha_k(\xi) \in \mathbb{Z}\times\mathbb{R}$

We define a Gabor multiplier through the operation

$$r = V_\gamma^{-1} M_{\alpha_k} V_g s$$

Exercise

Consider the Gabor multiplier

$$r = V_\gamma^{-1} M_{\alpha_k} V_g s$$

If

$$\lim_{stat} M_{\alpha_k} = M_\alpha$$

Show that

$$\lim_{stat} V_\gamma^{-1} M_{\alpha_k} V_g s = F^{-1} M_\alpha F s$$

That is, the stationary limit of the Gabor multiplier is the Fourier multiplier.

Exercise

Consider the standard form K-N operator

$$s(x) = (T_\alpha r)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \alpha(x,\xi)\hat{r}(\xi) e^{ix\xi} d\xi$$

suppose that the symbol can be written as

$$\alpha(x,\xi) = \sum_k \gamma_k(x)\beta(k,\xi) \quad \text{with} \quad \sum_k \gamma_k(x) = 1$$

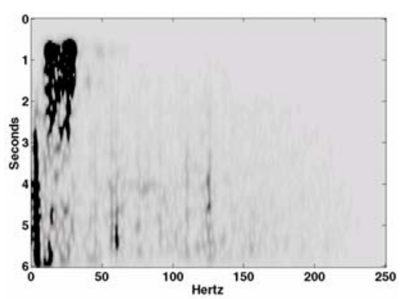
Show that

$$T_\alpha r = V_\gamma^{-1} M_\beta V_g r \quad \text{with} \quad g(x) = 1$$

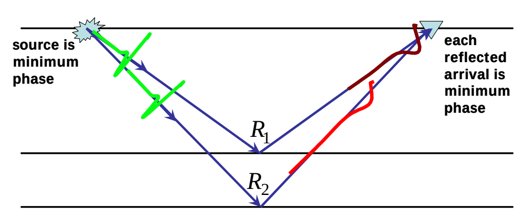
Part 7

Gabor Deconvolution

Gabor magnitude spectrum of a raw seismic trace



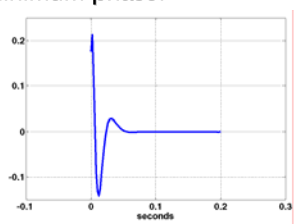
Attenuation and nonstationarity



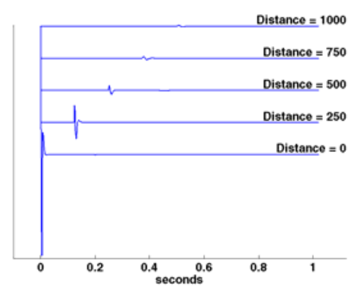
Attenuation depends on path length. Therefore the seismic recording is inherently nonstationary, being a linear superposition of many different minimum-phase arrivals with differing degrees of attenuation.

Attenuation and minimum phase

Futterman (1962) showed that wave attenuation in a causal, linear theory is always minimum phase.

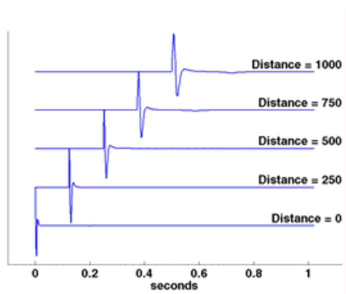


Attenuation Simulation



True Amplitude

Attenuation Simulation



Normalized

Attenuation model

$$\hat{s}(\omega) = \hat{w}(\omega) \int_0^\infty \alpha(\tau,\omega) r(\tau) e^{-i\omega\tau} d\tau$$

Physical arguments show that seismic attenuation can be modelled as a pseudodifferential operator where

$\hat{s}(\omega)$ Fourier transform of the seismic trace

$\hat{w}(\omega)$ Fourier transform of the source signature

$r(\tau)$ Reflectivity

Attenuation model

$$\alpha(\tau,\omega)$$

is the pseudodifferential symbol whose form is

$$|\alpha(\tau,\omega)| \sim \exp\left(\frac{-|\omega||\tau|}{2Q}\right)$$

$$phase[\alpha(\tau,\omega)] \sim H\left(\ln|\alpha(\tau,\omega)|\right)$$

The phase calculation is known as the "minimum phase assumption" and is a true for a causal, invertible time series.

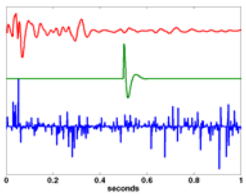
Attenuation model

So we have a pseudodifferential symbol that is:

complex-valued
highly smoothing
exponentially decaying in τ and ω
Q(τ) is unknown

Nonstationary seismic trace model

$$\hat{s}(\omega) = \hat{w}(\omega) \int_0^\infty \alpha(\tau,\omega) r(\tau) e^{-i\omega\tau} d\tau$$



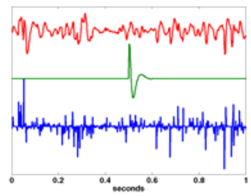
seismogram with attenuation

source signature

reflectivity

Stationary seismic trace model

$$s(t) = \int_0^\infty w(t-\tau) r(\tau) d\tau$$

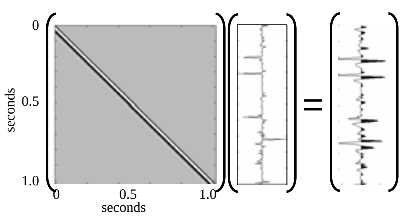


seismogram without attenuation

source signature

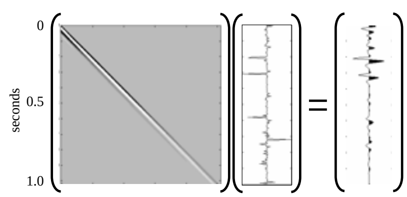
reflectivity

Stationary seismic trace model



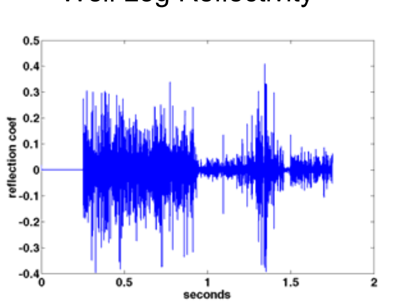
Stationary superposition process.

Nonstationary seismic trace model

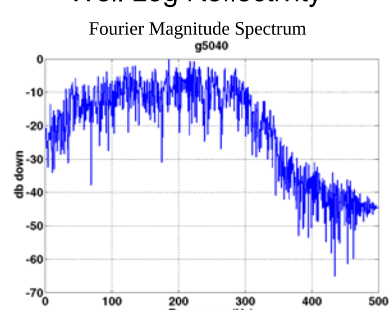


Nonstationary superposition process.

Well Log Reflectivity

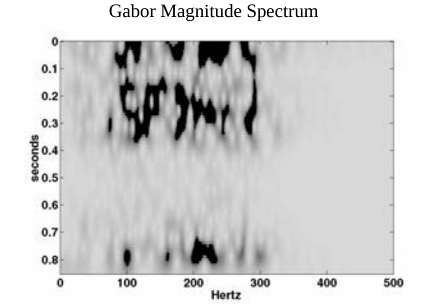


Well Log Reflectivity



Well Log Reflectivity

Gabor Magnitude Spectrum



Problem statement

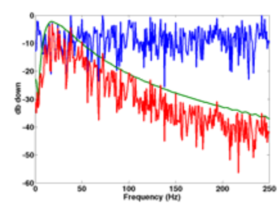
Given only the seismic trace, the physical model just presented, and the assumption of a random reflectivity, then estimate that reflectivity.

What did Wiener do?

Stationary seismic trace model

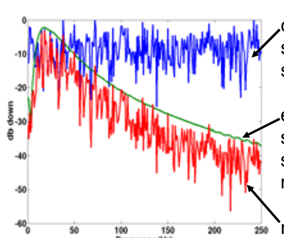
in the Fourier domain

$$\hat{s}(\omega) = \hat{w}(\omega)\hat{r}(\omega)$$



Wiener's deconvolution

actually due to Enders Robinson



Observation

Wiener's algorithm is enabled because the Fourier transform factorizes the convolution integral.

The Gabor transform induces an approximate factorization (diagonalization) of the pseudodifferential operator model. This suggests a parallel to Wiener's method using the Gabor transform.

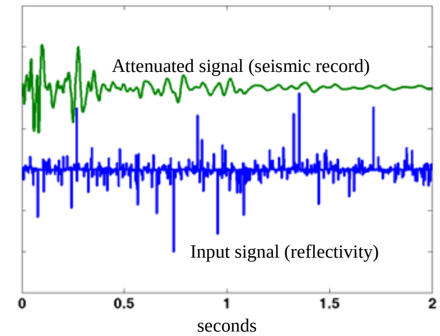
Nonstationary seismic trace model

Thus we expect that

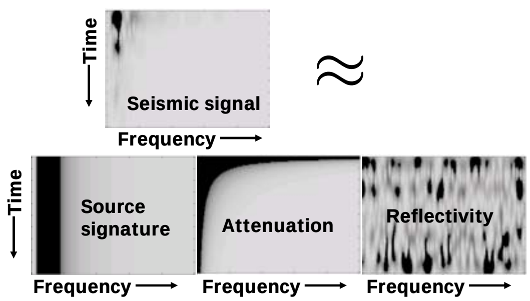
$$\underbrace{V_g s(\tau,f)}_{\text{Gabor transform of seismic signal}} \approx \underbrace{V_g V_\gamma^{-1}}_{\text{Projection}} \underbrace{\left[\hat{w}(f)\alpha(\tau,f)\right]}_{\text{Stuff we want to get rid of}} \underbrace{V_g r(\tau,f)}_{\text{Gabor transform of reflectivity}}$$

Strategy: Estimate the “stuff we want to get rid of” from the Gabor spectrum of the seismic trace. Then develop a Gabor multiplier that is the algebraic inverse of the estimated stuff.

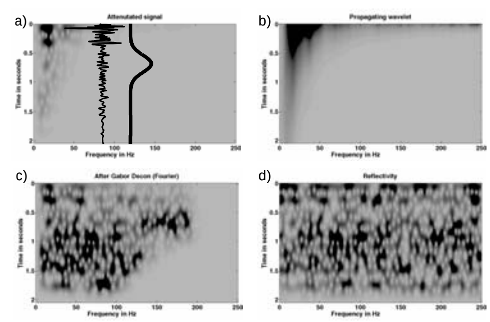
Synthetic Example



Gabor Factorizes Nonstationary Trace Model

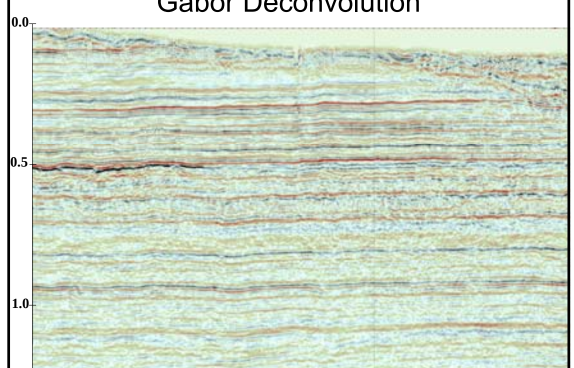


Gabor Deconvolution

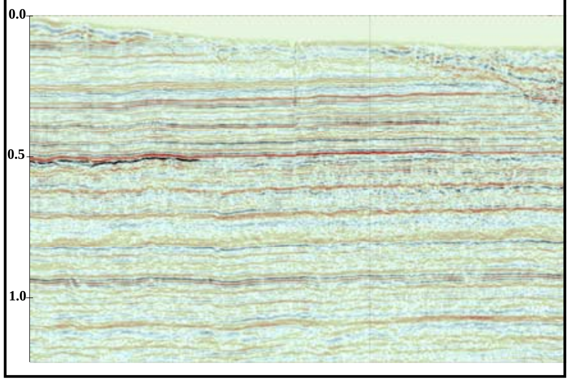


Comparison on Synthetic

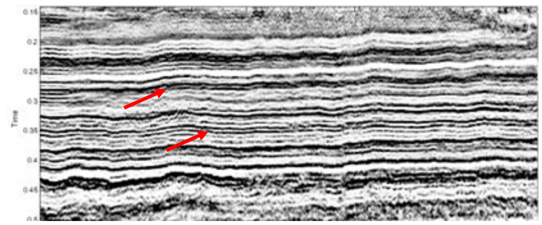
Gabor Deconvolution



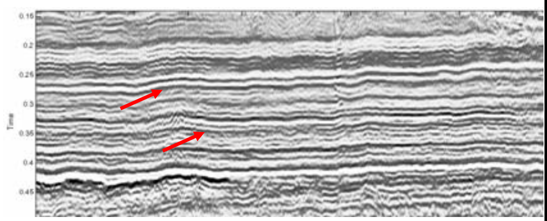
State-of-the-art alternative



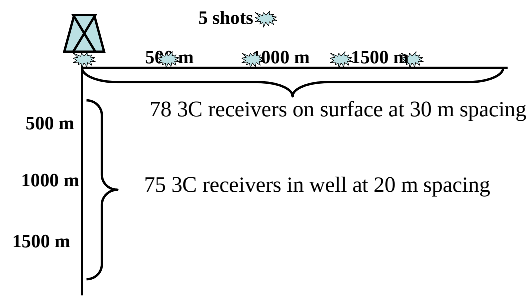
Gabor Deconvolution



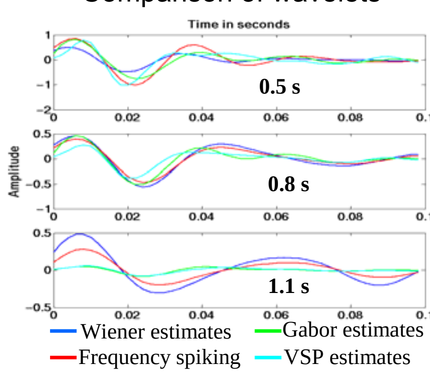
State-of-the-art alternative



Validation of Gabor wavelets with a VSP



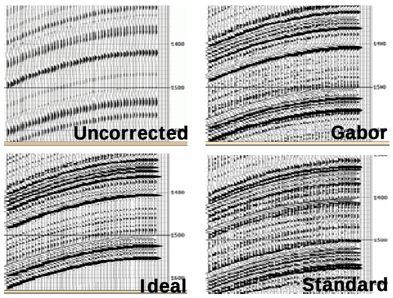
Comparison of wavelets



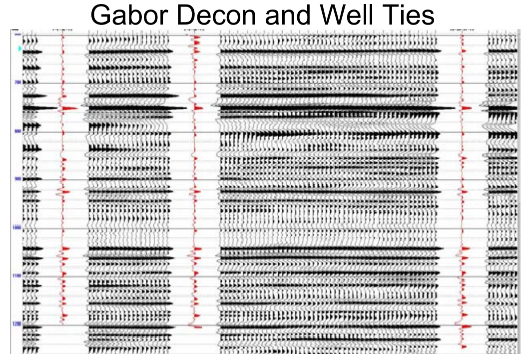
Ongoing Industry Testing

Husky Energy Ltd. and Geo-X are conducting an extensive and detailed test of Gabor deconvolution versus their standard practice. This is not an "easy win" for Gabor because standard practice has long included a number of ad-hoc correction algorithms, most notably TVSW, designed to address first-order nonstationary effects.

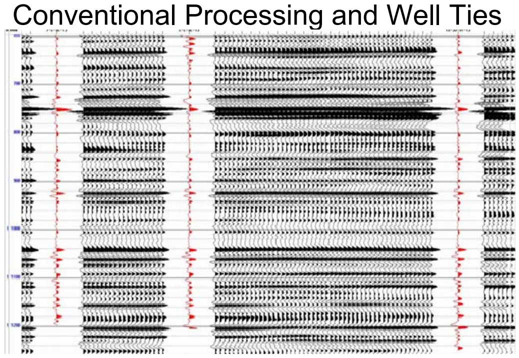
Ongoing Test
Synthetic Data



Ongoing Test
Gabor Decon and Well Ties



Ongoing Test
Conventional Processing and Well Ties



Ongoing Test
Observations

- Gabor is much better than standard practice on synthetic data.
- On real data, the methods are comparable with no clear cut winner (yet).
- Both Gabor and standard practice apparently have residual phase errors.
- Ideas are emerging to improve the Gabor process.

Summary

We have demonstrated that a complex-valued Gabor multiplier can be derived from seismic data to correct for attenuation effects.

On synthetic tests, this amounts to in inversion of a pseudodifferential operator by a Gabor multiplier.

Our method generalizes that of Wiener to nonstationary seismic records.

Part 8

Wavefield extrapolation

Wavefield Extrapolators

Recall the one-way wave equation

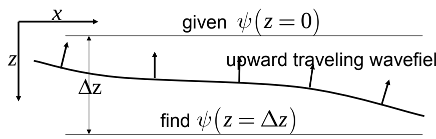
$$\left(\frac{\partial\psi}{\partial z}\right)^{\pm} = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\alpha_1^{\pm}\left(\xi_x,x,\omega\right)\hat{\hat{\psi}}\left(\xi_x,z,\omega\right)\mathrm{e}^{i\left(\omega t-\xi_x x\right)}\,d\xi_x d\omega$$

$$\alpha_1^{\pm}\left(\xi_x,x,\omega\right)\approx\pm\sqrt{\alpha_2\left(\xi_x,x,\omega\right)}\equiv\pm i\xi_z\left(\xi_x,x,\omega\right)$$

$$\xi_z=\begin{cases}\sqrt{\frac{\omega^2}{v^2}-\xi_x^2}, & \frac{\omega^2}{v^2}\geq\xi_x^2\\ i\sqrt{\xi_x^2-\frac{\omega^2}{v^2}}, & \xi_x^2>\frac{\omega^2}{v^2}\end{cases}$$

Wavefield Extrapolators

We wish to solve the wavefield extrapolation problem:



$$\frac{\partial\hat{\psi}}{\partial z}=\frac{i}{2\pi}\int_{\mathbb{R}^2}\xi_z\left(\xi_x,x,\omega\right)\hat{\hat{\psi}}\left(\xi_x,z,\omega\right)\mathrm{e}^{-i\xi_x x}\,d\xi_x$$

one-way equation for upward traveling waves.

or $$\frac{\partial\hat{\psi}}{\partial z}=T_{\alpha_1}\hat{\psi}\left(\xi_x,z,\omega\right)$$

Wavefield Extrapolators

Formal Taylor series

$$\hat{\psi}\left(\Delta z\right)=\hat{\psi}\left(0\right)+\Delta z\left.\frac{\partial\hat{\psi}}{\partial z}\right|_{z=0}+\frac{\left(\Delta z\right)^2}{2}\left.\frac{\partial^2\hat{\psi}}{\partial z^2}\right|_{z=0}+\cdots+\frac{\left(\Delta z\right)^k}{k!}\left.\frac{\partial^k\hat{\psi}}{\partial z^k}\right|_{z=0}+\cdots$$

which can be rewritten as

$$\hat{\psi}\left(\Delta z\right)=\hat{\psi}_0+\Delta z T_{\alpha_1}\hat{\psi}_0+\frac{\left(\Delta z\right)^2}{2}T_{\alpha_1}\circ T_{\alpha_1}\hat{\psi}_0+\ldots$$

Wavefield Extrapolators

According to the composition theorem

$$\underbrace{\left(T_{\alpha_1}\circ T_{\alpha_1}\cdots\right)}_{n\text{ times}}\hat{\psi}_0\equiv T_{\alpha_n}\hat{\psi}_0$$

is a pseudodifferential operator whose symbol has a first order approximation:

$$\alpha_n\approx\alpha_1^n$$

So, with an unknown error, we approximate the Taylor series as

$$\hat{\psi}\left(\Delta z\right)=\frac{1}{2\pi}\int_{\mathbb{R}}\left(1+\Delta z\alpha_1+\frac{\left(\Delta z\alpha_1\right)^2}{2}+\ldots\right)\hat{\hat{\psi}}_0 e^{-i\xi_x x}d\xi_x$$

Wavefield Extrapolators

Summing the series gives

$$\hat{\psi}\left(\Delta z\right)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{\Delta z\alpha_1}\hat{\hat{\psi}}_0 e^{-i\xi_x x}d\xi_x$$

or

$$\hat{\psi}\left(x,\Delta z,\omega\right)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{i\Delta z\xi_z\left(x,\xi_x,\omega\right)}\hat{\hat{\psi}}\left(\xi_x,0,\omega\right)e^{-i\xi_x x}d\xi_x$$

or

$$\hat{\psi}\left(x,\Delta z,\omega\right)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{W}\left(k\left(x\right),\xi_x,\Delta z\right)\hat{\hat{\psi}}\left(\xi_x,0,\omega\right)e^{-i\xi_x x}d\xi_x$$

$$k\left(x\right)=\frac{\omega}{v\left(x\right)}$$

This is known as the GPSPI (generalized phase shift plus interpolation) wavefield extrapolator.

Wavefield Extrapolators

The GPSPI extrapolator

$$\hat{\psi}\left(x,\Delta z,\omega\right)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{W}\left(k\left(x\right),\xi_x,\Delta z\right)\hat{\hat{\psi}}\left(\xi_x,0,\omega\right)e^{-i\xi_x x}d\xi_x$$

Summary of approximations:

(1) $\alpha_1^{\pm}\left(\xi_x,x,\omega\right)\approx\pm\sqrt{\alpha_2\left(\xi_x,x,\omega\right)}$ True only for homogeneous medium.

(2) $\alpha_n\approx\alpha_1^n$ Only asymptotically valid even if the first derivative symbol is exact.

(3) $\alpha_2\left(\xi_x,x,\omega\right)$ is elliptic Elliptic means bounded away from zero and this is false.

(4) The Taylor series converges It does in some specific cases but we don't know in general.

Wavefield Extrapolators

The GPSPI extrapolator

$$\hat{\psi}(x,\Delta z,\omega)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{W}\left(k(x),\xi_x,\Delta z\right)\hat{\hat{\psi}}(\xi_x,0,\omega)e^{-i\xi_x x}d\xi_x$$

Things we know (or think we do):

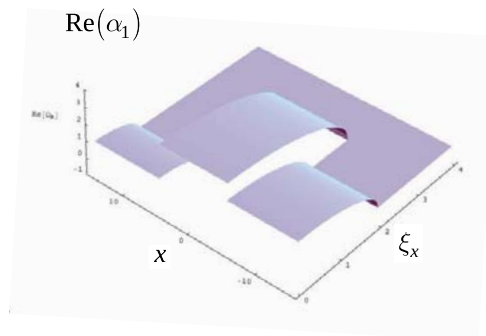
(1) Any explicit finite difference method is an approximation to GPSPI.

(2) "Screen" methods are approximations to GPSPI.

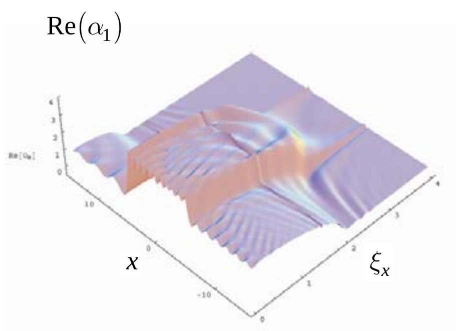
(3) GPSPI produces very high quality seismic images but it is computationally expensive.

(4) More accurate methods can be formulated simply as operators with different symbols.

Fishman: Locally Homogeneous Approximation 3-Layer Profile



Fishman: Exact Operator Symbol 3-Layer Profile



Exercise

Schwartz Kernel of a Pseudodifferential Operator

given: $$s(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\alpha(x,\xi)\hat{r}(\xi)e^{-i\xi x}d\xi$$

show by formal manipulation (don't worry about conversion etc) that this is equivalent to

$$s(x)=\int_{\mathbb{R}}A(x,y)r(y)dy \qquad A(x,y)=\frac{1}{2\pi}\int_{\mathbb{R}}\alpha(x,\xi)e^{-i\xi(x-y)}d\xi$$

The quantity A(x,y) is called the Schwartz kernel of the pseudodifferential operator and the integral applying A is called a singular integral operator.

Singular Integral form of a ΨDO

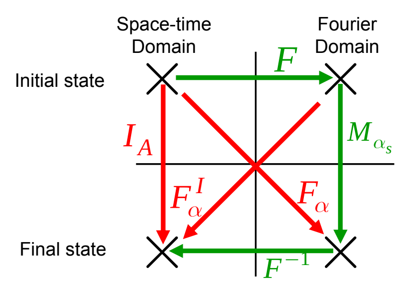
Given: $$s=T_\alpha r$$

then, with suitable circumstances, it follows that

$$s(x)=(I_A r)(s)\equiv\int_{\mathbb{R}}A(x,y)r(y)dy$$

where $$A(x,y)=\frac{1}{2\pi}\int_{\mathbb{R}}\alpha(x,\xi)e^{-i\xi(x-y)}d\xi$$

Pseudodifferential Operators



Exercise

Schwartz Kernel of a Fourier Multiplier

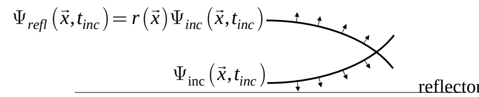
Given:

$$s = F^{-1} M_{\alpha} F r \qquad \alpha(\xi): \mathbb{R} \to \mathbb{R}$$

show that the Schwartz kernal depends only on x-y (translation invariance) and that the resulting singular integral operator is just a convolution.

Seismic Imaging Paradigm

A common seismic imaging methodology is derivable from first-order inverse Born scattering



$$\frac{\Psi_{refl}(\vec{x}, t_{inc})}{\Psi_{inc}(\vec{x}, t_{inc})} = r(\vec{x})$$ A reflectivity estimate.

Seismic Imaging Paradigm

Seismic imaging typically is done in the frequency domain and uses depth steps not time steps, so a more common imaging condition is:

$$r(x, y, \Delta z) = \sum_k \frac{\psi_{refl}(x, y, z = \Delta z, k\Delta\omega)}{\psi_{inc}(x, y, z = \Delta z, k\Delta\omega)}$$

Seismic Imaging Paradigm

So for each depth, we must calculate two fields:

$\psi_{refl}(x, y, n\Delta z, \omega)$ The reflected field comes from mathematically marching the recorded data down into the earth.

$\psi_{inc}(x, y, n\Delta z, \omega)$ The incident field comes from a mathematical model of the source wavefield that is also marched down.

In both cases, the wavefield marching is done through a "background" velocity filed that is presumed known.

Wavefield Extrapolators

The Schwartz kernel of the GPSPI extrapolator is

$$W(k(x), x - x', \Delta z) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{W}(k(x), \xi_x, \Delta z) e^{-i\xi_x(x - x')} d\xi_x$$

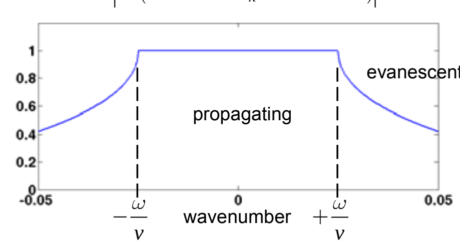
So wavefield extrapolation is also accomplished with

$$\hat{\psi}(x, \Delta z, \omega) = \int \hat{\psi}(x', z = 0, \omega) W(k(x), x - x', \Delta z) dx'$$

It turns out that W is not compactly supported. If a complctly supported, accurate approximation can be found then we will have an efficient implementation of GPSPI.

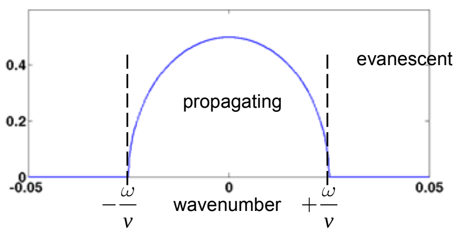
Wavefield Extrapolators

$$\left|\hat{W}(k = \text{const}, \xi_x, \Delta z = \text{const})\right|$$



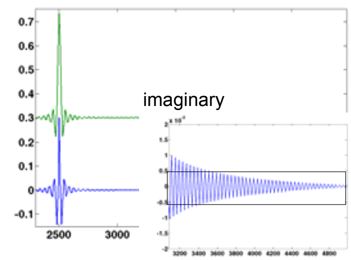
Wavefield Extrapolators

$$phase\left[\hat{W}\left(k=\text{const},\xi_x,\Delta z=\text{const}\right)\right]$$



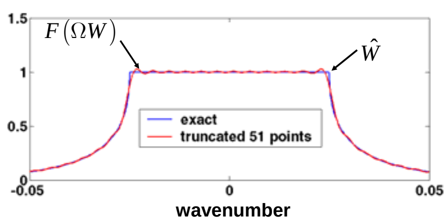
Wavefield Extrapolators

In the space-frequency domain



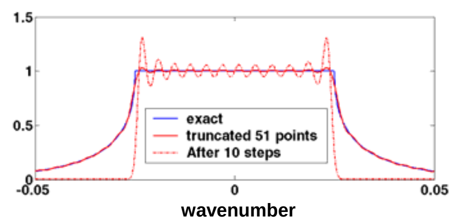
Wavefield Extrapolators

Back to the wavenumber domain



Wavefield Extrapolators

Back to the wavenumber domain



Stabilization by Wiener Filter

Two useful properties

$$\hat{W}\left(k,\xi_x,\Delta z\right)=\hat{W}\left(k,\xi_x,\frac{\Delta z}{2}\right)\hat{W}\left(k,\xi_x,\frac{\Delta z}{2}\right)$$

Product of two half-steps make a whole step.

$$\hat{W}^{-1}\left(k,\xi_x,\Delta z\right)=\hat{W}^{*}\left(k,\xi_x,\Delta z\right),\quad \frac{\omega^{2}}{v^{2}}>\xi_x^2$$

The inverse is equal to the conjugate in the wavelike region.

Stabilization by Wiener Filter

A windowed forward operator for a half-step

$$\tilde{W}\left(\Delta z/2\right)=\Omega W\left(\Delta z/2\right)$$

Solve by least squares for *WI*

$$\tilde{W}\left(\Delta z/2\right)\bullet WI=F^{-1}\left[\left|\hat{W}\left(\Delta z/2\right)\right|^{\eta}\right]$$

$$0\le\eta\le 2$$

Stabilization by Wiener Filter

WI is a band-limited inverse for $\tilde{W}\left(\Delta z/2\right)$
Both have compact support

Form the FOCI™ approximate operator by

$$W_F\left(\Delta z\right) = WI^* \bullet \tilde{W}\left(\Delta z/2\right) \approx W\left(\Delta z\right)$$

FOCI™ is an acronym for

Forward Operator with Conjugate Inverse.

Properties of FOCI operator

Let

$$n_{inv} = length\left(WI_n\right) \quad n_{for} = length\left(\tilde{W}_n\left(z/2\right)\right)$$

Then

$$length\left(W_{nF}\left(z\right)\right) = n_{op} = n_{for} + n_{inv} - 1$$

Properties of FOCI operator

n_{for} determines phase accuracy.

n_{inv} determines stability.

Empirical observation:

$$n_{inv} \approx 1.5 n_{for}$$

Properties of FOCI operator

Amount of evanescent filtering is inversely related to stability

$$\eta = \begin{cases} 0 \cdots \text{no evanescent filtering } (\sim 1000 \text{ steps}) \\ 1 \cdots \text{half evanescent filtering } (\sim 100 \text{ steps}) \\ 2 \cdots \text{full evanescent filtering } (\sim 50 \text{ steps}) \end{cases}$$

Operator tables

Since the operator is purely numerical, migration proceeds by construction of operator tables.

$k_{\min}$	$W_{nF}\left(k_{\min}\right)$
$k_{\min} + \Delta k$	$W_{nF}\left(k_{\min} + \Delta k\right)$
$k_{\min} + 2\Delta k$	$W_{nF}\left(k_{\min} + 2\Delta k\right)$
$\cdots$	$\cdots$
$k_{\max}$	$W_{nF}\left(k_{\max}\right)$

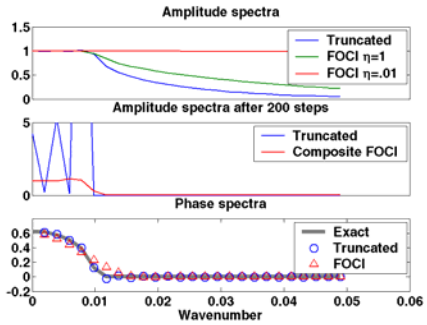
$$k_{\min} = \frac{\omega_{\min}}{v_{\max}} \quad \Delta k = \frac{\Delta\omega}{mean(v)} \quad k_{\max} = \frac{\omega_{\max}}{v_{\min}}$$

Operator tables

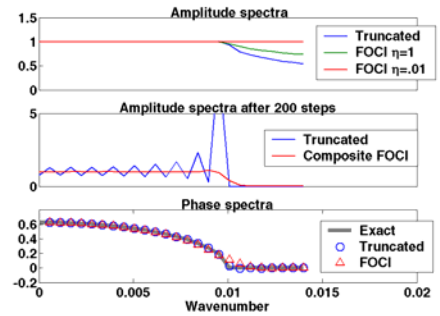
We construct two operator tables for small and large η. The small η table is used most of the time, with the large η being invoked only every n^th^ step.

$$\eta = \begin{cases} 0 \cdots \text{no evanescent filtering } (\sim 1000 \text{ steps}) \\ 1 \cdots \text{half evanescent filtering } (\sim 100 \text{ steps}) \\ 2 \cdots \text{full evanescent filtering } (\sim 50 \text{ steps}) \end{cases}$$

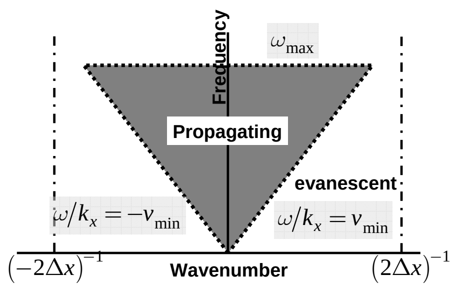
Operator Design Example



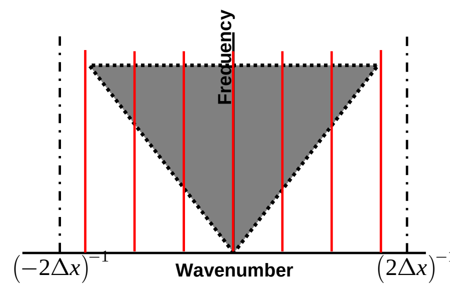
Improved Operator Design



Spatial Resampling

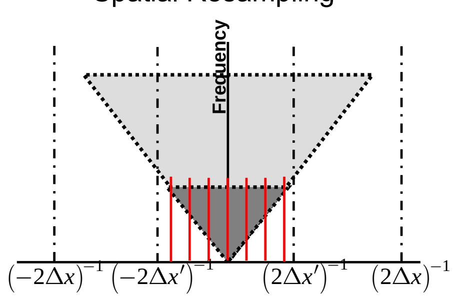


Spatial Resampling



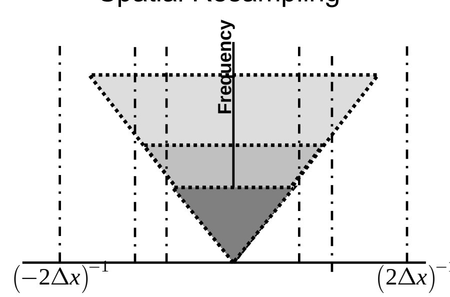
In red are the wavenumbers of a 7 point filter

Spatial Resampling



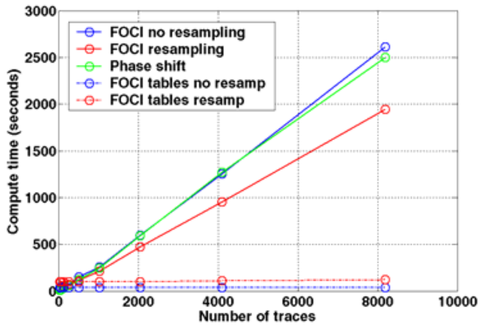
Downsampling for the lower frequencies uses the filter more effectively

Spatial Resampling

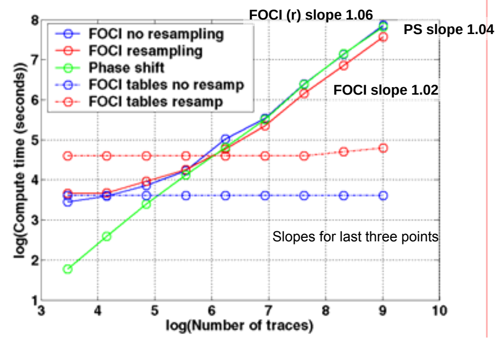


Spatial resampling is done in frequency “chunks”.

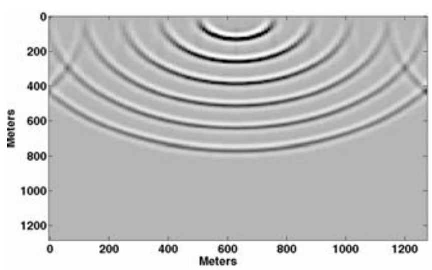
Run Time Experiment



Run Times log-log Scale

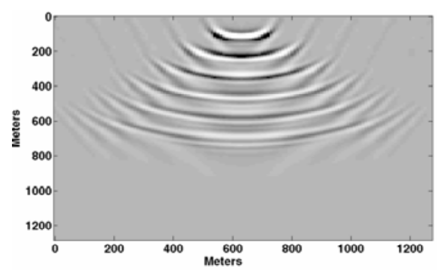


Phase Shift Impulse Response



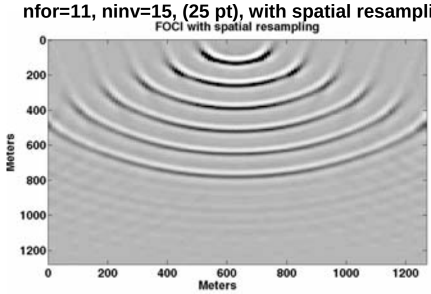
FOCI Impulse Response

nfor=11, ninv=15, (25 pt), no spatial resampling



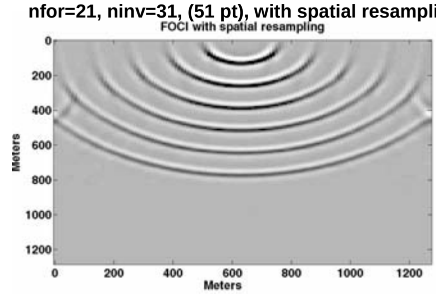
FOCI Impulse Response

nfor=11, ninv=15, (25 pt), with spatial resampling

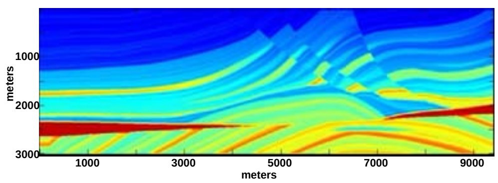


FOCI Impulse Response

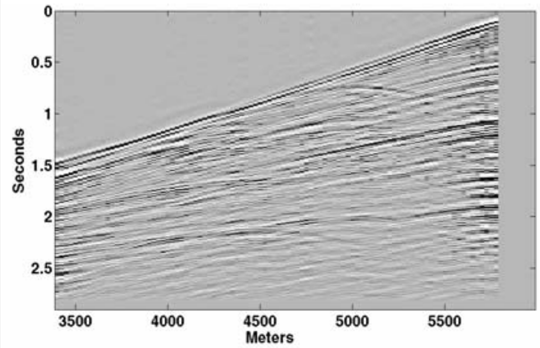
nfor=21, ninv=31, (51 pt), with spatial resampling



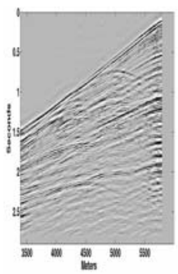
Marmousi Movie
Real Story



Marmousi Data



Marmousi Data

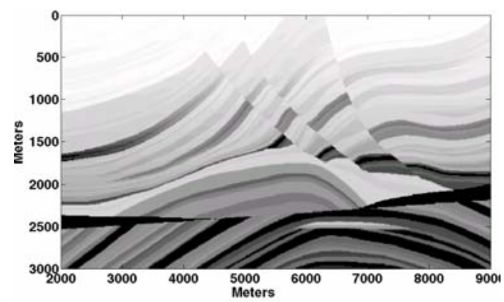


240 shots
96 receivers/shot
726 samples/receiver
8 bytes/samples

Dataset size= 240*96*726*8 ~ 134 Mbytes

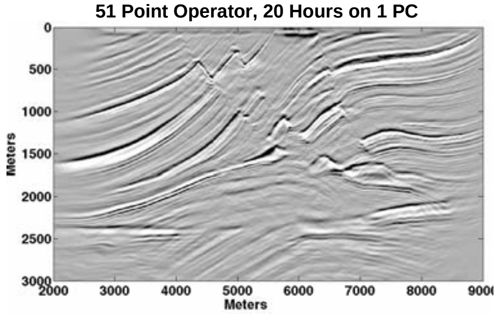
Real datasets have 1000’s of shots, 1000’s of receivers/shot, and 1000’s of samples/receiver.

Marmousi Velocity Model



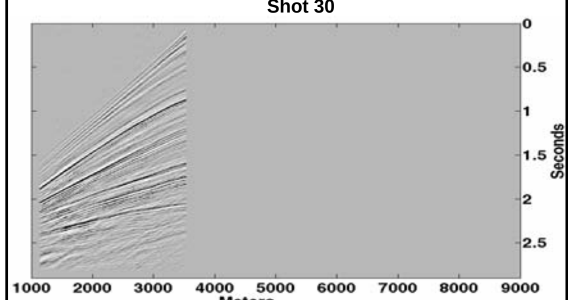
FOCI Pre-Stack Migration

51 Point Operator, 20 Hours on 1 PC

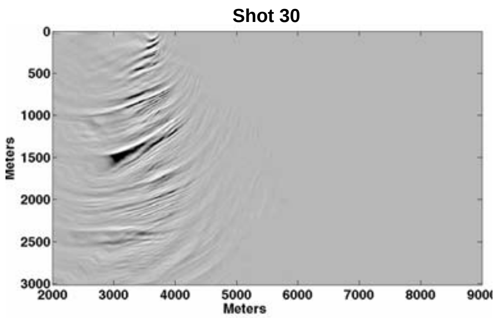


FOCI Pre-Stack Migration

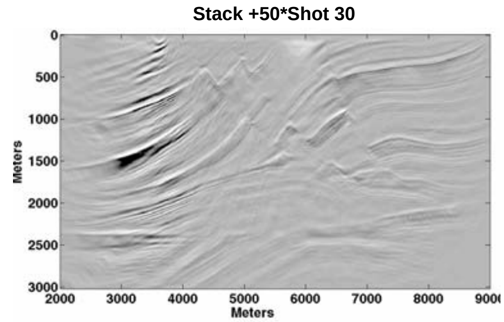
Shot 30



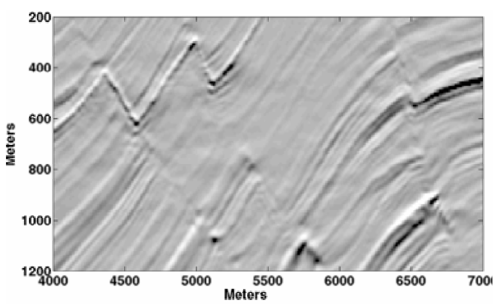
FOCI Pre-Stack Migration



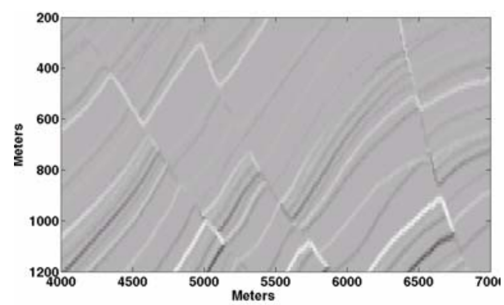
FOCI Pre-Stack Migration



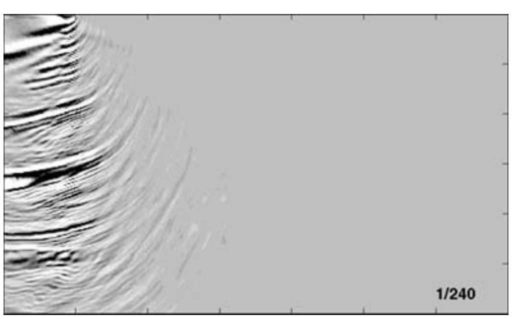
Detail of Pre-Stack Migration



Marmousi Reflectivity Detail



Depth Migration Movie



Marmousi run times

Full prestack depth migrations of Marmousi on a single 2.5GHz PC using Matlab code.

20 hours for the best result

1 hour for a usable result

Conclusions

Explicit wavefield extrapolators can be made local and stable using Wiener filter theory.

The FOCI method designs an unstable forward operator that captures the phase accuracy and stabilizes this with a band-limited inverse operator.

Reducing evanescent filtering increases stability.

Spatial resampling increases stability, improves operator accuracy, and reduces runtime.

The final method appears to be ~O(NlogN).

Very good images of Marmousi have been obtained.

Overall Conclusions

The manipulation of a wavefield on its phase space offers new possibilities for improving seismic imaging.

Pseudodifferential operators and Gabor multipliers are powerful new signal processing tools.

There are lots of new things waiting to be done!

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