Automatic band limited signal reconstruction

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Summary

We implement the popular Adaptive weights Conjugate gradients on Toeplitz (ACT) algorithm for signal reconstruction. This algorithm is fast and accurate, and we show its effectiveness in several typical trace regularization situations. This algorithm requires an estimate of the signal bandwidth as input, and overestimating the bandwidth can cause spurious high frequency noise in the reconstruction. As an improvement, we implement the Multi-level ACT to perform automatic bandwidth detection on its input by performing ACT iteratively to estimate the optimum reconstruction bandwidth. Results show that Multi-level ACT is effective when the signal bandwidth can not be accurately estimated. A toolbox has been assembled that can be requested from the authors.

ACT Method

Given an irregularly sampled signal $s_j = s(t_j)$, for $j = 1, 2, \ldots, N$, we can compute the simple DFT of the observed samples,

$$S_k = \sum_{j=1}^{N} s_j e^{-2\pi i t_j k/N}. \qquad (1)$$

These are not the true Fourier coefficients of the signal, which can be derived from the signal by the inverse DFT equation,

$$s_j = \frac{1}{N}\sum_{m=-M}^{M} \hat{S}_m e^{2\pi i t_j m/N}, \qquad (2)$$

when the signal is band limited by M. Combining these two equations gives a linear system of equations relating the true Fourier coordinates and the computed coordinates,

$$S_k = \frac{1}{N}\sum_{m=-M}^{M} \hat{S}_m \sum_{j=1}^{N} e^{2\pi i t_j (m-k)/N}. \qquad (3)$$

This matrix is Toeplitz, as its downward diagonals have constant value.

When the sampling pattern is very irregular, the reconstruction can be biased towards areas where the sampling is the most dense. To combat this we add a set of weights defined by the distance between a point's two nearest neighbors. This will cause the densely sampled points to have lower weight in the inversion. These weights are given by,

$$w_j(x) = \begin{cases} \frac{1+t_2}{2} & \text{if } j = 1 \\ \frac{N-t_{j-1}}{2} & \text{if } j = N \\ \frac{t_{j+1}-t_{j-1}}{2} & \text{otherwise} \end{cases}. \qquad (4)$$

Multiplying the samples by these weights gives the linear system on which the ACT method is based.

$$S_k = \frac{1}{N}\sum_{m=-M}^{M} \hat{S}_m \sum_{j=1}^{N} w_j e^{2\pi i t_j (m-k)/N}. \qquad (5)$$

This matrix is used to form the normal equations, which are symmetric, positive-definite and Toeplitz. This means that we can use a fast version of the conjugate gradient method, and we only have to store the first row of the matrix.

Uniformly decimated harmonic

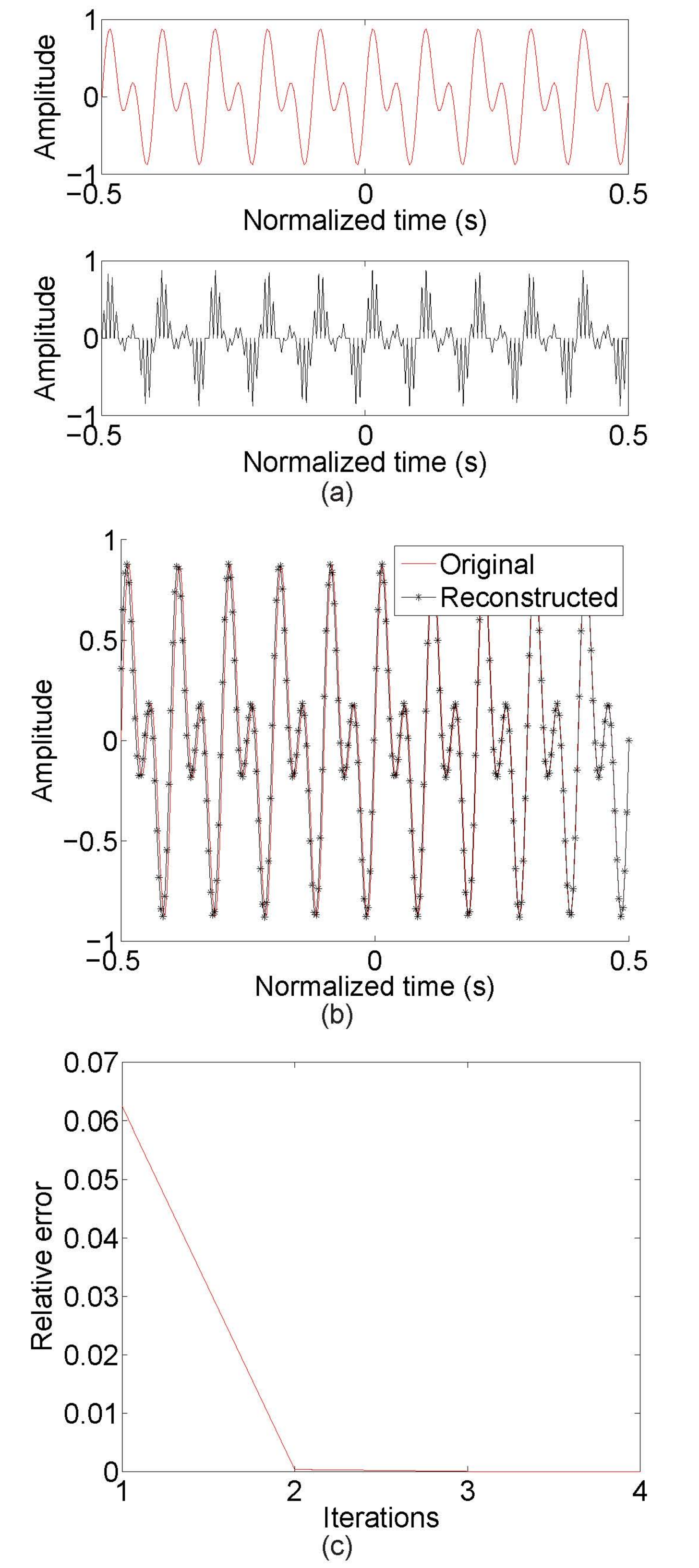

Figure 1: (a) A simple harmonic and a uniform 50% decimation of that harmonic. (b) The original signal and the ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.

Uniformly decimated trace

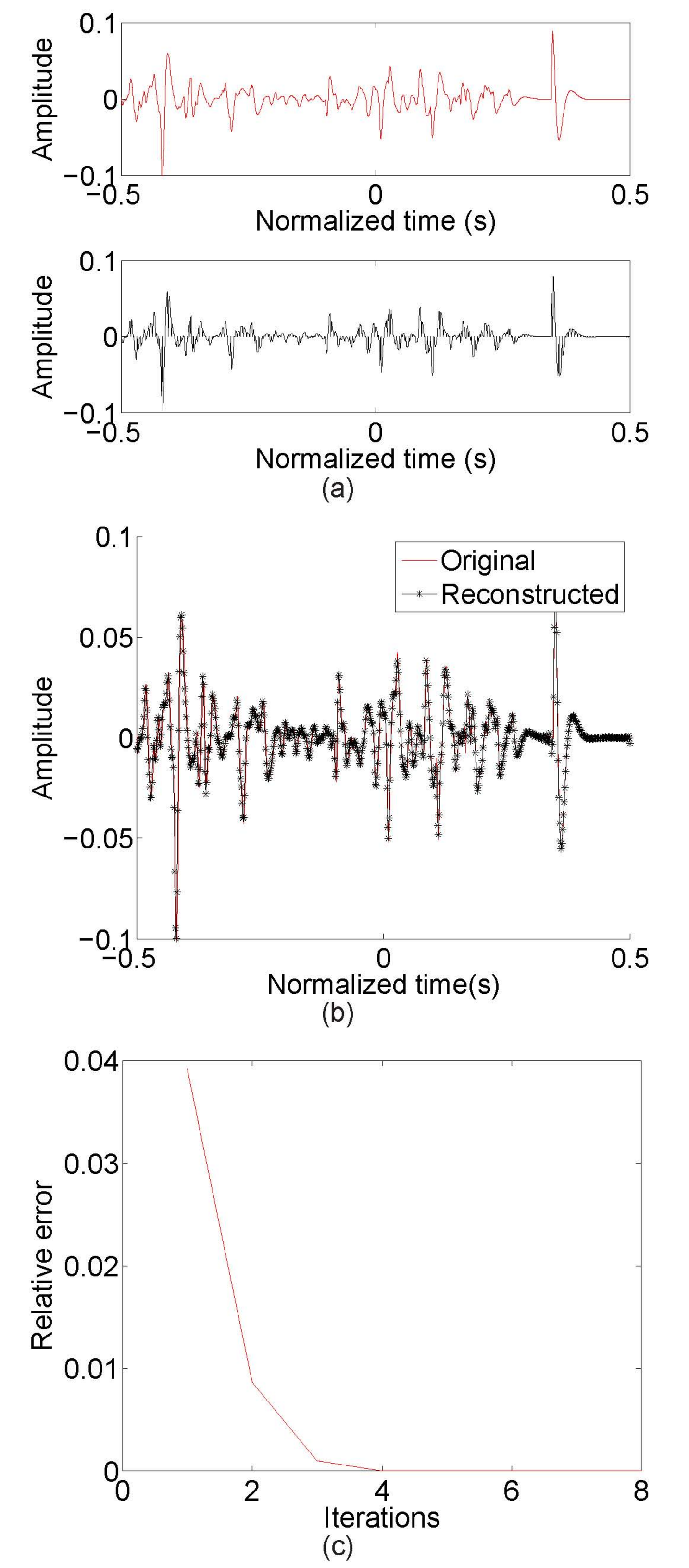

Figure 2: (a) A stationary seismic trace and a uniform 50% decimation of that trace. (b) The original signal and the ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.

Successful 70% reconstruction

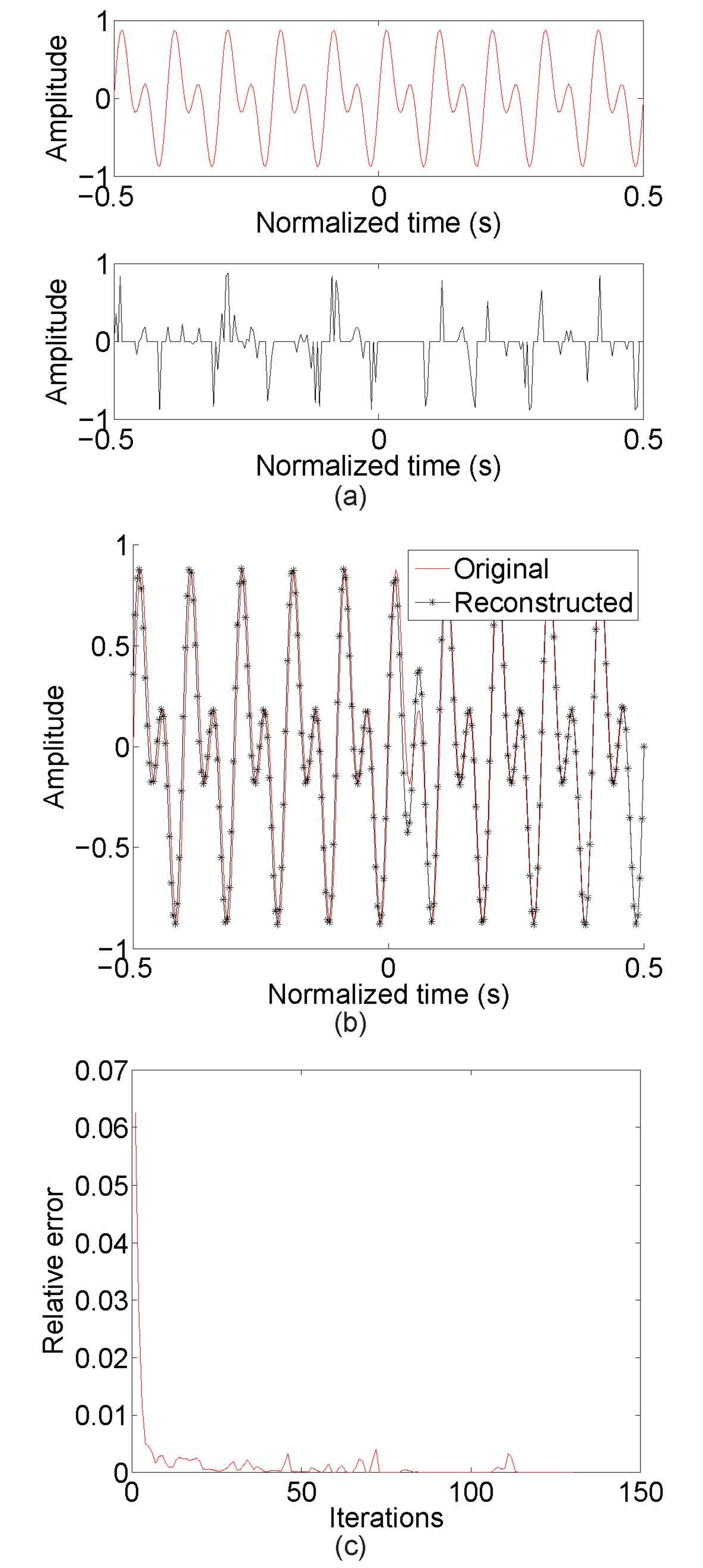

Figure 3: (a) A simple harmonic and a random 70% decimation of that harmonic. (b) The original signal and a successful ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.

Failed 70% reconstruction

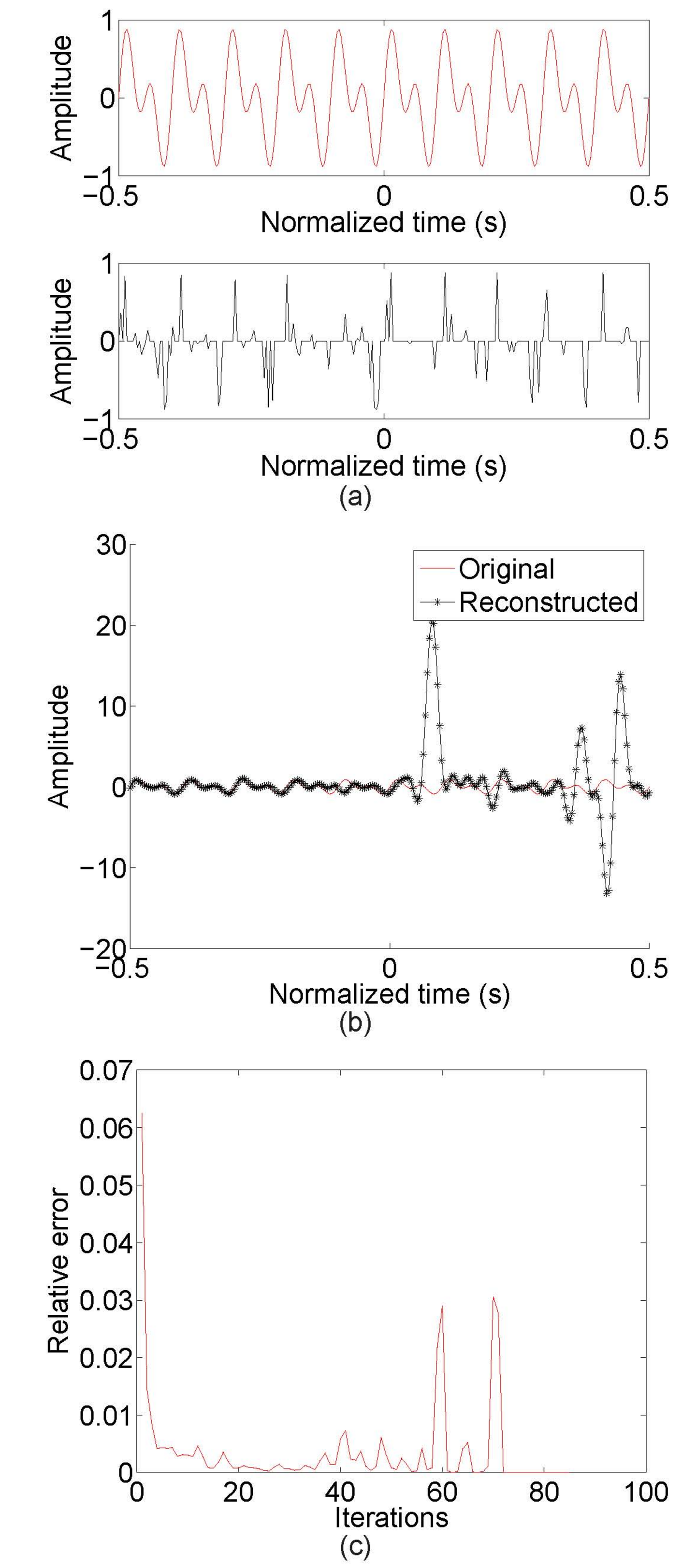

Figure 4: (a) A simple harmonic and a random 70% decimation of that harmonic. (b) The original signal and an unsuccessful ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.

MLACT

ACT requires an estimate of the bandwidth. Overestimating the bandwidth can cause wild fluctuations in the reconstruction which are not consistent with the signal. Underestimating the bandwidth will give a smoother result, but the inversion will fail to model the higher frequency components of the system. To overcome this problem we can use Multi-level ACT, which performs the reconstruction iteratively, increasing the bandwidth until the output agrees with the known samples to within a user-defined tolerance.

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Automatic bandwidth detection

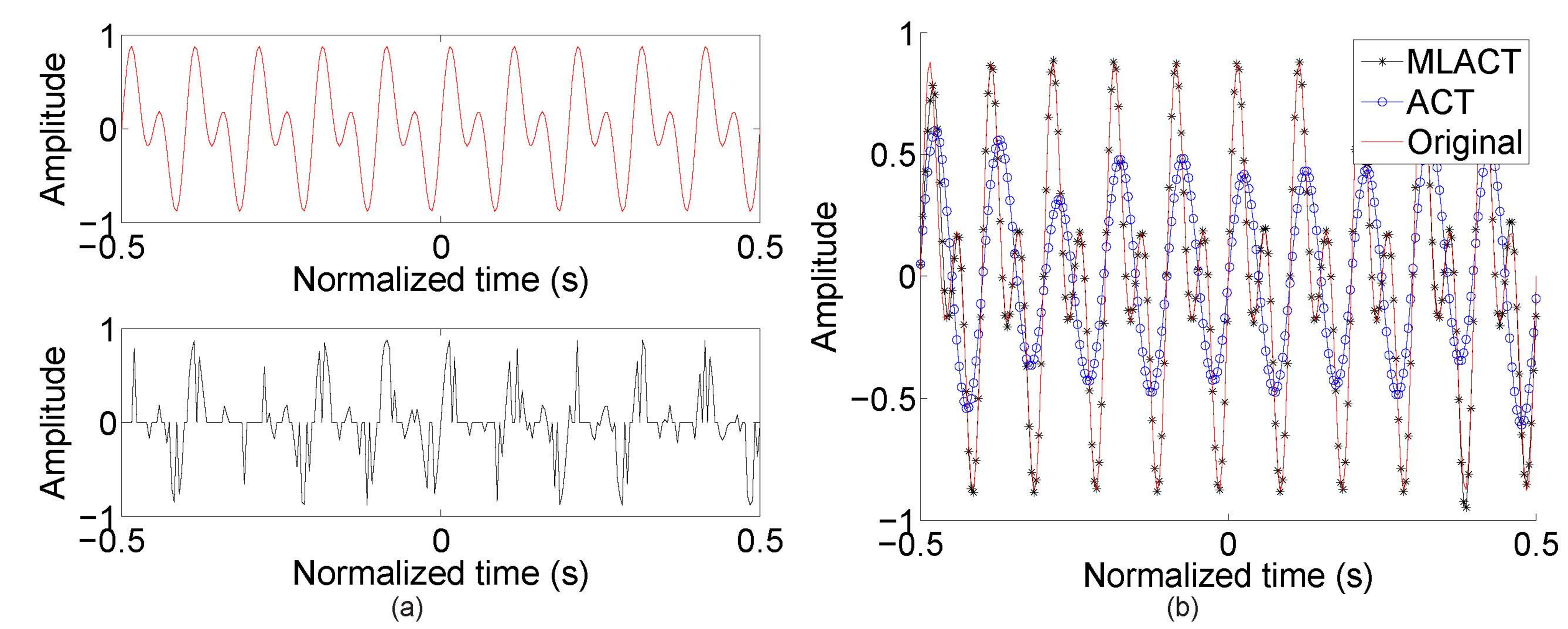

Figure 5: (a) A simple harmonic and a random 50% decimation of that harmonic. (b) The original signal, the ACT reconstruction with underestimated bandwidth, and the MLACT automatic bandwidth reconstruction.