A multiple model and Padé approximation

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1. Introduction

The numerical experiments described in this work were conceived as a test of how the technique of least squares Padé approximation could be used in a typical seismic data processing application. We considers rational Padé approximation of the Z-transform function of a time dependent minimum phase signal. We review the derivation of reflection multiples for a double interface, and observe the multiple signal can be modelled by an infinite impulse response (IIR) filter of a simple form, with coefficients determined by the reflection and transmission parameters. The Padé method features a numerical approach that works directly on the data - there is no need to transform to the Fourier (or other) domain. We set up the Padé approximation problem using the seismic data directly, with some choice on the rational function form to reduce the dimension of the solution space. The rational ([p, q]-Padé) approximation of the Z-transform function is formulated as a constrained least squares minimization problem with regularization constraints provided by the minimum phase signal. Results of some numerical experiments in building the Padé approximating filter, and its use as an inverse filter to remove the multiples demonstrate the effectiveness of the presented approach.

2. Multiple Reflections and Transmissions

Single Interface With an incoming wave $e^{i(\omega t+k_1 x)}$ on the right of the interface at $x=0$, the transmitted and reflected waves are generated as $Te^{i(\omega t+k_2 x)}$ and $Re^{i(\omega t-k_1 x)}$, respectively. The continuity of the total waveforms and the normal derivatives at the point $x=0$ gives

$$T-R=1, \quad rT+R=1, \tag{1}$$

where $r=k_2/k_1$ is the relative index of refraction with the wave numbers k_1 (right medium) and k_2 (left medium). The reflection and transmission coefficients are given by

$$R_{\leftarrow}=\frac{1-r}{1+r}, \qquad T_{\leftarrow}=\frac{2}{1+r}. \tag{2}$$

If we reverse directions, then we have

$$R_{\rightarrow}=-R_{\leftarrow}, \qquad T_{\rightarrow}=rT_{\leftarrow}. \tag{3}$$

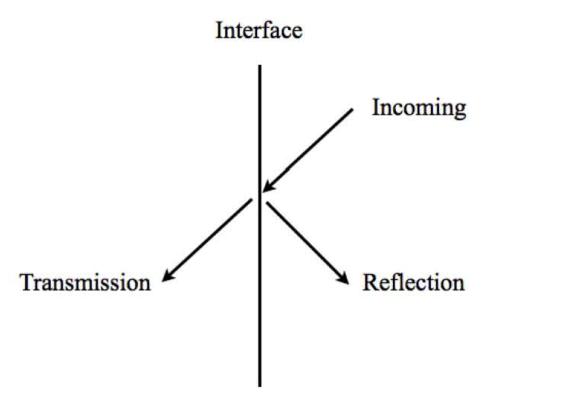

Figure 1: *A reflection and transmission event across a single interface at $x=0$.*

Two Interface There will be multiple internal reflections. The total reflectivity is given by

$$R_{total}=R^1_{\leftarrow}+\sum_{n=0}^{\infty}T^1_{\rightarrow}DR^2_{\leftarrow}D[R^1_{\rightarrow}DR^2_{\leftarrow}D]^nT^1_{\leftarrow}=R^1_{\leftarrow}-\frac{T^1_{\rightarrow}}{R^1_{\rightarrow}}T^1_{\leftarrow}+\frac{T^1_{\rightarrow}}{R^1_{\rightarrow}}\left(1-R^1_{\rightarrow}DR^2_{\leftarrow}D\right)^{-1}T^1_{\leftarrow}$$

where

$$R_1=R^1_{\leftarrow}, R_2=T^1_{\rightarrow}DR^2_{\leftarrow}DT^1_{\leftarrow}, R_3=T^1_{\rightarrow}DR^2_{\leftarrow}D[R^1_{\rightarrow}DR^2_{\leftarrow}D]T^1_{\leftarrow}. \tag{4}$$

The coefficients T's and R's are constants, R_{total} can be written as

$$R_{total}=a+b\left(1-cD^2\right)^{-1}, \quad \text{for some constants } b',\ c. \tag{5}$$

The transmission terms can be derived analytically as

$$\begin{aligned} T_1 &= T^2_{\leftarrow}DT^1_{\leftarrow} \\ T_2 &= T^2_{\leftarrow}D[R^1_{\rightarrow}DR^2_{\leftarrow}D]T^1_{\leftarrow} \\ T_3 &= T^2_{\leftarrow}D[R^1_{\rightarrow}DR^2_{\leftarrow}D]^2T^1_{\leftarrow} \end{aligned}$$

so that the effective transmission T_{total} is given as

$$\begin{aligned} T_{total} &= \sum_{n=1}^{\infty}T^2_{\leftarrow}D[R^1_{\rightarrow}DR^2_{\leftarrow}D]^nT^1_{\leftarrow} \\ &= T^2_{\leftarrow}D\left(1-R^1_{\rightarrow}DR^2_{\leftarrow}D\right)^{-1}T^1_{\leftarrow}. \end{aligned} \tag{6}$$

Again noting the constants in (6), T_{total} can also be written as

$$T_{total}=b'D\left(1-cD^2\right)^{-1}, \quad \text{for constants } b',\ c. \tag{7}$$

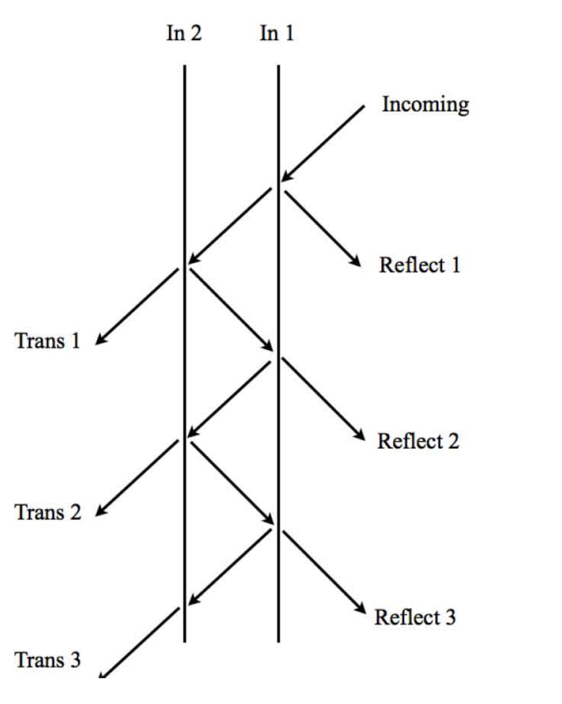

Figure 2: *Reflection and transmission events across two interfaces.*

3. Modelling with IIR filters

The two responses R_{total} and T_{total} can be modelled with Infinite Impulse Response (IIR) filters of very similar form. The related IIR filters of R_{total} and T_{total} are given by the following rational functions

$$G(z)=a+\frac{b}{1-cz^{2d}}=\frac{\alpha+\beta z^{2d}}{1+\eta z^{2d}}, \qquad H(z)=\frac{b'z^d}{1-cz^{2d}}, \tag{8}$$

respectively, where $\alpha=a+b$, $\beta=-ac$, and $\eta=-c$ are related to the multiple reflectivities, and d is an integer that models the delay of the signal through the gap. The filters $G(z)$ and $H(z)$ are stable since $c=R^1_{\rightarrow}R^2_{\leftarrow}<1$. The integer d can be computed from the width of the gap, the velocity of sound in the gap, and the sample rate of the sampled signal, which is given by

$$d=\frac{\text{length}}{\text{velocity}}*\text{ sample rate}. \tag{9}$$

4. Padé Approximation for Inversion

The developed numerical inversion algorithm for constructing Padé approximation of the function G(z) is given as.

$$G(z)\simeq G_{[p,q]}(z)=\frac{a(z)}{b(z)}=\frac{a_0+a_1z+a_2z^2+\cdots+a_pz^p}{b_0+b_1z+b_2z^2+\cdots+b_qz^q} \qquad (b_0=1, \quad p\le q) \tag{10}$$

where a_l $(l=0,1,\ldots,p)$ and b_j $(j=0,1,\ldots q)$ are real coefficients of two polynomials $a(z)$ and $b(z)$ of orders p and q, respectively. The approximation $G_{[p,q]}(z)$ of the IIR filter $G(z)$ implies that the input wavelet $\{w_l\}_{l=0}^{\infty}$ and the output signal $\{s_k\}_{k=0}^{\infty}$ must satisfy the recursion formula

$$s_k=\sum_{l=0}^{p}a_lw_{k-l}-\sum_{j=1}^{q}b_js_{k-j}, \ k=0,1,2,.... \tag{11}$$

A finite number $N\gg p+q+1$ must be chosen in order to reconstruct the full Padé coefficients a_l's and b_j's given the data $\{w_l\}_{l=0}^{\infty}$ and $\{s_k\}_{k=0}^{\infty}$. Therefore, the system (11) for the unknown coefficients a_l's and b_j's becomes:

$$a_0w_k+a_1w_{k-1}+...+a_pw_{k-p}-b_1s_{k-1}-b_2s_{k-2}-...-b_qs_{k-q}=s_k, \tag{12}$$

for $k=0,1,2,...,N$ and $p\le q\ll N$. Some coefficients a_l's and b_j's in the middle terms of $a(z)$ and $b(z)$ can be assumed to be zero. For a fixed integer number m (e.g., $m=6$), we suppose that

$$\begin{aligned} a_m=a_{m+1}=...=a_{p-(m-2)}=a_{p-(m-1)}=0, \\ b_{m+1}=b_{m+2}=...=b_{q-(m-2)}=b_{q-(m-1)}=0. \end{aligned}$$

Therefore, the system (12) can be rewritten as

$$\mathbf{Ac}:=[\mathbf{A_1}\ \mathbf{A_2}]\begin{bmatrix}\mathbf{c_1}\\ \mathbf{c_2}\end{bmatrix}:=s, \qquad \text{where } \mathbf{s}=(s_0,s_1,s_2,...,s_N)^{\top} \tag{13}$$

$$\mathbf{c_1}=(a_0,a_1,...,a_{(m-1)},a_{p-m},...,a_p)^{\top}, \ \ \mathbf{c_2}=(b_1,b_2,...,b_m,b_{q-m},...,b_q)^{\top}, \tag{14}$$

and the matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ with entries in terms of data $\{w_l\}_{l=0}^{N}$ and $\{s_k\}_{k=0}^{N}$, respectively. The reconstruction problem (13) is ill-posed and requires regularization to develop a stable numerical algorithm. To construct a real solution vector **c** of the reduced Padé coefficients for the inverse problem (13), we introduce a penalization term in the Tikhonov regularization functional $\mathcal{T}^{\lambda}(\mathbf{c},\mathbf{s})$, so that the problem (13) can be formulated as the following constrained least squares minimization problem with the regularization parameter $\lambda>0$ chosen properly

$$\min_{\mathbf{c}}\mathcal{T}^{\lambda}(\mathbf{c},\mathbf{s})=\min_{\mathbf{c}}\{||\mathbf{Ac}-\mathbf{s}||^2+\lambda^2||\mathbf{c}||^2\}$$

$$\text{subject to } |u_k-r_0|<\delta_0, |v_j-r_1|<\delta_1, k=1,2,\ldots,p, j=1,2,\ldots,q. \tag{15}$$

The parameters u_k and v_j in the constraints (15) are zeros and poles of the reconstructed $[p,q]$-Padé approximation $G_{[p,q]}(z)$ of $G(z)$, $r_0=(-\alpha/\beta)^{1/2d}$, and $r_1=(-1/\eta)^{1/2d}$. The minimizer (solution) of the problem (15) is given by

$$\mathbf{c}=\{\mathbf{A}^{\top}\mathbf{A}+\lambda\mathbf{I}_{4m+2}\}^{-1}\{\mathbf{A}^{\top}\mathbf{s}\} \tag{16}$$

where $\mathbf{I}_{4m+2}$ denotes the $(4m+2)\times(4m+2)$ identity matrix. After reconstruction of the real coefficient vector **c**, we can extend it to a full coefficients of the function $G_{[p,q]}(z)$ by inserting zero coefficients, this gives $[p,q]$-Padé approximation of $G(z)$. The reconstructed function $G_{[p,q]}(z)$ can be used to estimate the reflectivity parameters and to identify the impulse wavelet using inverse filtering.

5. Results

To simulate the synthetic data - impulse response signal $\{s_j\}_{j=0}^{\infty}$, a Ricker wavelet with the dominant frequency 25Hz was generated for the input signal, and the parameters a, b and c were chosen as $a=\frac{4}{45}$, $b=\frac{11}{18}$, $c=\frac{9}{10}$, $d=50$ leading to the polynomial coefficients a_0, a_{100}, b_{100} represented by $\alpha=0.7$, $\beta=-0.08$, $\eta=-0.9$, respectively. The sample rate of the Ricker wavelet is 0.003 seconds, the wavelet length is 3.0 seconds, so that the total number of data is $N=1001$. The left Fig. 3 shows the reconstruction of the 26 reduced Padé coefficients for $m=6$ compared with the true coefficients of G(z) for the order of $p=q=104$ chosen in the inversion algorithm. The valid recovered poles of $G_{[p,q]}(z)$ using the constraints in (15) are illustrated in the right Fig. 3. Here $\delta_1=\delta_2=0.06$, all poles and zeros of $G(z)$ lie on the unit circle with radius $r_1=1.0011$ and $r_2=1.022$ in the complex z-plane, respectively. The true and computed output signals $\{s_j\}_{j=0}^{\infty}$ (the IIR filtered Ricker wavelet) using the recovered $[p,q]$-Padé coefficients fit fairly well for data with 8% noise (left Fig. 4). The recovered function $G_{[p,q]}(z)$ as an inverse filter was used to reconstruct the original input Ricker wavelet (right Fig. 4). The reconstruction of the input Ricker wavelet is almost identical, with no difference between the theoretical and reconstructed functions when there is no noise in the data. Even for input data with adding 8% noise, the original input Ricker wavelet in the time interval $[0,1.5]$ seconds was recovered very well using the reconstructed function $G_{[p,q]}(z)$ as an inverse filter. However, there are some oscillation events that occurred with amplitudes changing rapidly after $t=0.15$ seconds, which need to be further studied.

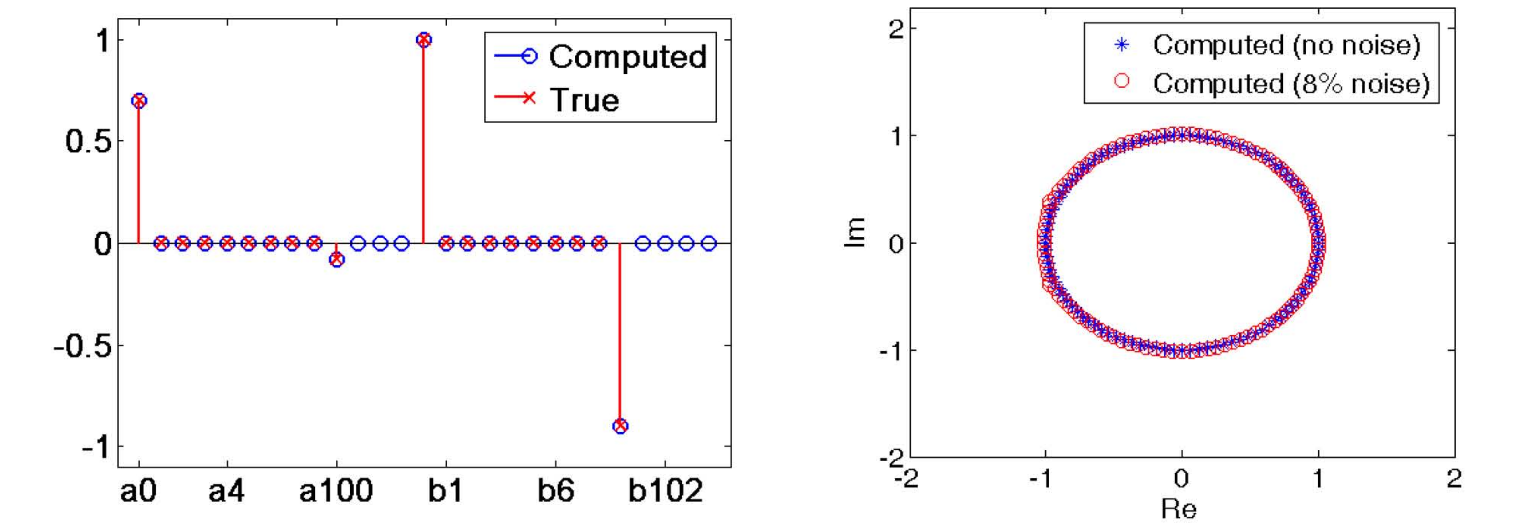

Figure 3: *Recovery of Padé coefficients for data with no noise (left) and calculation of poles for function G(z) (right).*

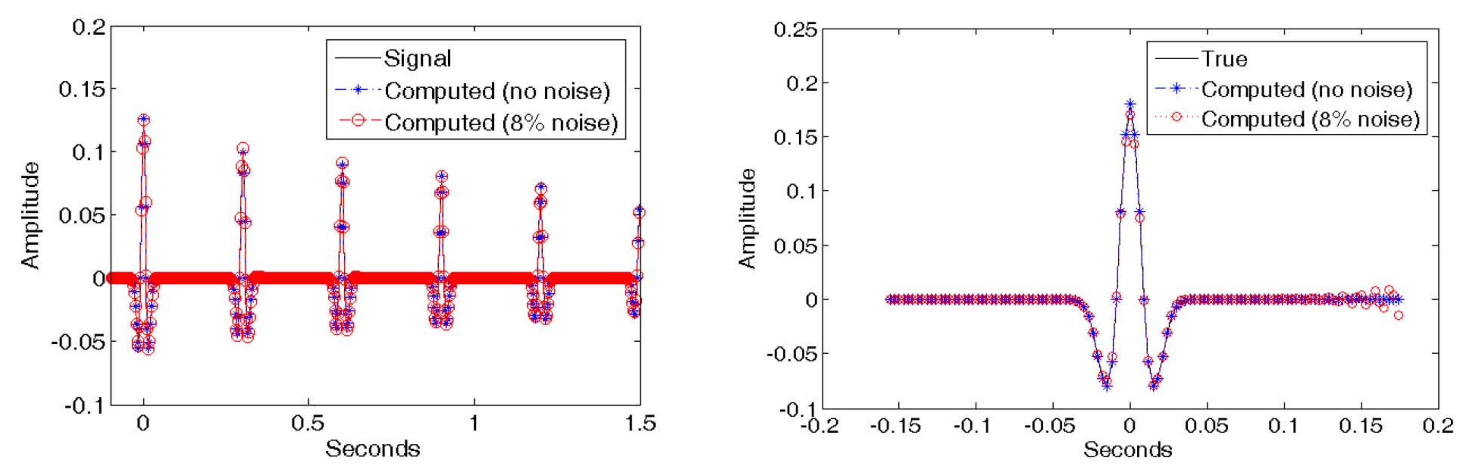

Figure 4: *Reconstruction of the IIR filtered Ricker wavelet (left) and the Ricker wavelet with a 25 Hz dominant frequency (right).*

6. Conclusions

We developed a new numerical inversion method for reconstruction of the Z-transform function of a time-dependent minimum phase signal using Padé approximation. The approach is based on rational ($[p,q]$-Padé) approximation of the Z-transform function in the complex plane. The problem is formulated as a constrained least squares minimization problem with regularization constraints provided by the minimum phase signal (all poles and zeros of Z-transform function lie outside the unit circle). The method was tested using a Ricker wavelet to generate a minimum phase signal (IIR filtered wavelet) as synthetic input data. The performed numerical experiments for reconstruction of the Z-transform function and its use as an inverse filter show the effectiveness of the presented approach.

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