Hidden nonlinearities in the Aki-Richards approximation

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Introduction

The Aki-Richards approximation comes in two forms, one involving the incidence angle and the other involving the average of the incidence and transmission angles. The first of these may be straightforwardly derived by expanding a matrix form of the Knott-Zoeppritz equation in series and truncating. The second is formally a linearization but is more reasonably interpreted as being nonlinear, and this can be quantified by expanding the average angle in series about the P-wave velocity perturbation. The Aki-Richards approximation is often discussed in terms of P-wave, S-wave, and density reflectivities. The average angle too may be expressed in terms of the incidence angle and the P-wave reflectivity, with the latter perturbing the former.

Two versions of the Aki-Richards approximation

There are two Aki-Richards approximations. They both look the same, but each has a slightly different definition of *angle* on their respective right-hand sides. One, which we will call version A, uses the incidence angle. The other, which we will call version B, uses an average of the incidence angle and the transmission angle. In Figures 1a and c version A is illustrated in red, for a particular set of elastic parameters against the exact R_{PP} for comparison. In Figures 1b and d version B is illustrated in blue. Let us discuss both in turn.

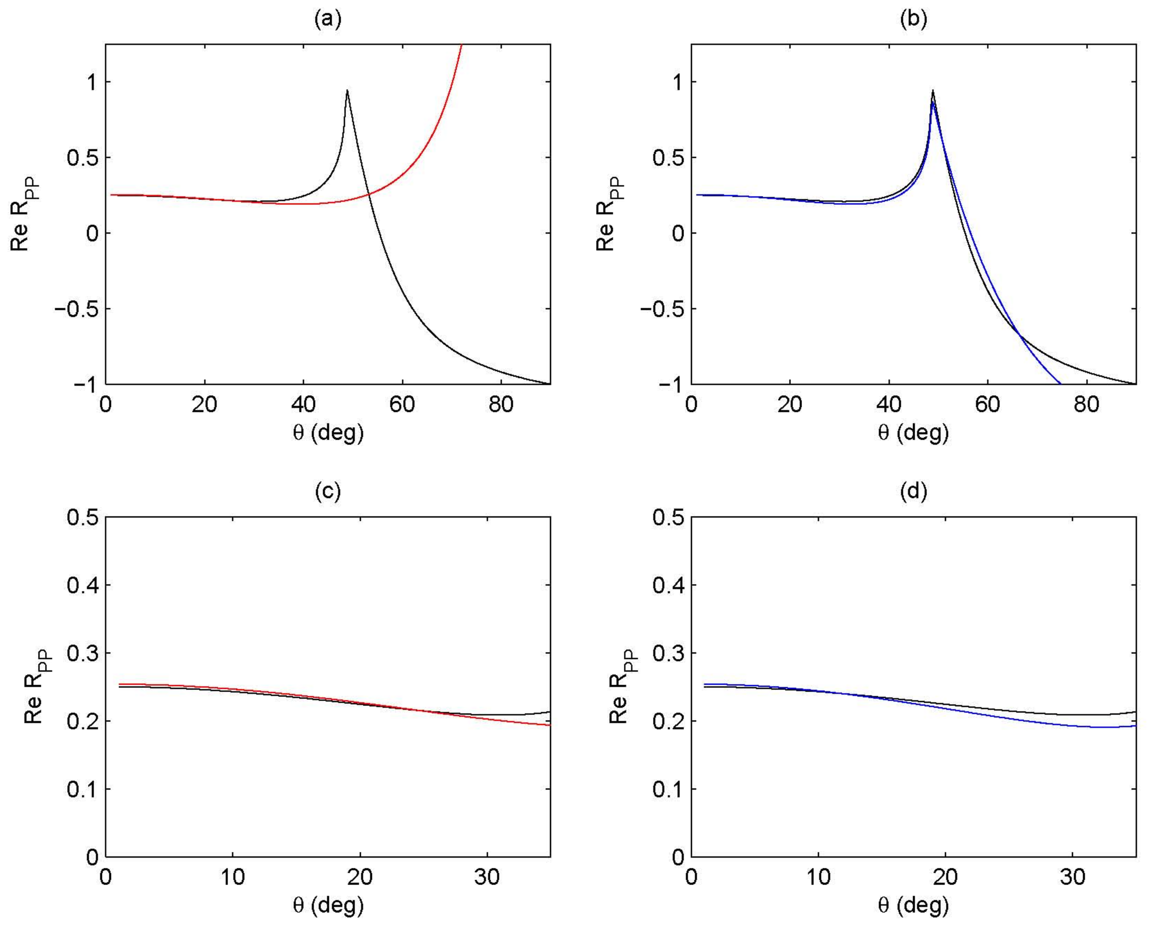

Figure: 1. Numerical behaviour of the two versions of the Aki-Richards approximation.

Version A.

The version actually presented by Aki and Richards is given by

$$R_{PP} \approx \frac{1}{2}\left(1+\tan^2\theta\right)\frac{\Delta V_P}{V_P} - 4\frac{V_S^2}{V_P^2}\sin^2\theta\frac{\Delta V_S}{V_S} + \frac{1}{2}\left(1+4\frac{V_S^2}{V_P^2}\sin^2\theta\right)\frac{\Delta\rho}{\rho},$$

where θ is the angle of incidence.

Version B.

The other involves a different angle on the right hand side:

$$R_{PP} \approx \frac{1}{2}\left(1+\tan^2\theta'\right)\frac{\Delta V_P}{V_P} - 4\frac{V_S^2}{V_P^2}\sin^2\theta'\frac{\Delta V_S}{V_S} + \frac{1}{2}\left(1+4\frac{V_S^2}{V_P^2}\sin^2\theta'\right)\frac{\Delta\rho}{\rho},$$

where θ' is the average of the incidence and transmission angles:

$$\theta' = \frac{1}{2}\left[\theta + \sin^{-1}\left(\frac{V_{P_1}}{V_{P_0}}\sin\theta\right)\right].$$

Comparing versions A (red) and B (blue) in Figure 1 it is evident that a significant up-tick in accuracy is achieved at large angles by using B.

Nonlinearity of version B

How can two linearizations of the same equations exhibit such strong differences in accuracy? There are two equally legitimate answers:

1. They are both linear, but in different things.
2. They cannot. Version II. A. is linear, and version II. B. is nonlinear.

Consider interpretation (1). Inspection reveals, trivially, that version B is linear, if θ' and the perturbations $\Delta V_P/V_P$, $\Delta V_S/V_S$ and $\Delta\rho/\rho$ vary independently. Likewise version A, as long as θ and the perturbations vary independently. So, both are linear, but assuming independence of different variables. Add a requirement: that the approximation be linear *in target medium properties*. After all, for a geophysicist, a practical R_{PP} approximation answers the question "what happens to R_{PP} when my target medium properties change?" For version II.A, this requirement changes nothing. θ does not depend on target properties, so version A is linear in target properties. Not so version II.B. θ' depends on target properties, because the transmission angle depends on target properties. For practical purposes, then, interpretation (2) holds. Version B is nonlinear.

Quantifying the nonlinearity

In the corresponding CREWES report, we analyze the Aki-Richards approximation by expanding the Zoeppritz equations in terms of dimensionless perturbations. For instance, if V_P varies from V_{P_0} to V_{P_1} across the boundary, the corresponding perturbation is

$$a_{VP} = 1 - \frac{V_{P_0}^2}{V_{P_1}^2}.$$

The nonlinearity of version B can be discussed formally by expanding θ' in terms of a_{VP}. The average angle is

$$\theta' = \frac{1}{2}\left[\theta + \sin^{-1}\left(\frac{V_{P_1}}{V_{P_0}}\sin\theta\right)\right]. \tag{1}$$

The second term above may be replaced, using a_{VP}, with

$$\sin^{-1}\left(\frac{V_{P_1}}{V_{P_0}}\sin\theta\right) = \left[\left(1+\frac{a_{VP}}{2}\right)\sin\theta\right] + \frac{1}{6}\left[\left(1+\frac{a_{VP}}{2}\right)\sin\theta\right]^3 + \ldots \tag{2}$$

Introducing equation (2) to version B of the Aki-Richards approximation, we now have on the right an expression in incidence angle only, but with series in powers of a_{VP}, or, equivalently, $\Delta V_P/V_P$. This exposes and quantifies the nonlinearity.

Perturbing θ with the P-wave reflectivity

Shuey and others (see the corresponding CREWES report and the bibliography) discuss the Aki-Richards approximation in terms of *reflectivities*. We may treat the average angle part of version B in terms of incidence angles and reflectivities also.To see this, we return to the series expansion of part of θ'. Re-arranging, we find a subseries, in θ only, that corresponds to the expansion of a simpler $\sin^{-1}$, and a subseries with a_{VP} as a common factor also. Up to first order in a_{VP} we obtain

$$\sin^{-1}\left(\frac{V_{P_1}}{V_{P_0}}\sin\theta\right) = \left[\sin\theta + \frac{1}{6}\sin^3\theta + \ldots\right] + \frac{a_{VP}}{2}\sin\theta\left[1 + \frac{1}{2}\sin^2\theta + \ldots\right]$$
$$\approx \theta + \frac{a_{VP}}{2}\tan\theta.$$

Hence for reasonably small contrasts and reasonably small angles, the average angle θ' can be replaced with the incidence angle θ and a correction term in a_{VP}, or alternatively in terms of the P-wave reflectivity $\Delta V_P/V_P$:

Perturbing θ with the P-wave reflectivity continued

$$\theta' \approx \frac{1}{2}\left[\theta + \theta + \frac{a_{VP}}{2}\tan\theta\right] \approx \theta + \frac{1}{2}\frac{\Delta V_P}{V_P}\tan\theta. \tag{3}$$

In Figure 2a–b, three curves are plotted: in black, exact R_{PP}, in blue, version II.A of the Aki-Richards approximation, and in (a) the θ-reflectivity form of version II.B in red, and in (b) the original version II.B in red. In Figure 2c the two instances of version II.B are compared, and in Figure 2d their difference is plotted. The error grows with angle but is quite small.

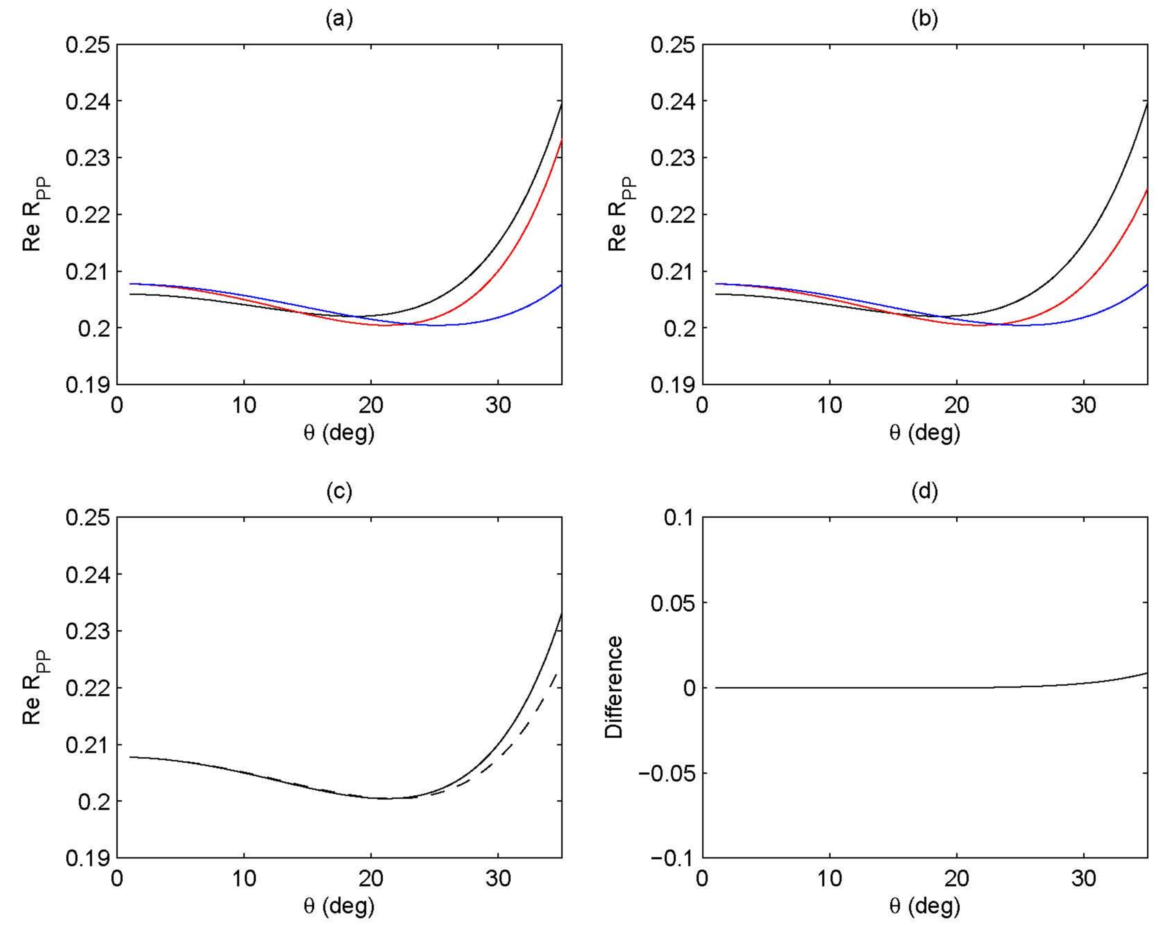

Figure: Aki-Richards version B parametrized in two different ways. (a) Three curves are plotted: in black, exact R_{PP}, in blue, version A of the Aki-Richards approximation, and the θ-reflectivity form of version B in red; (b) as in (a) but with the original version B in red. (c) The two instances of version B are compared; (d) their difference is plotted.

Conclusions

The first of two Aki-Richards approximation forms may be straightforwardly derived by expanding a matrix form of the Knott-Zoeppritz equation in series and truncating. The second is nonlinear, which is quantified by expanding the average angle in series about the P-wave velocity perturbation.

The Aki-Richards approximation is often discussed in terms of P-wave, S-wave, and density reflectivities. We have shown that the average angle too may be expressed in terms of the incidence angle and the P-wave reflectivity.

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