Continuous wavelet transforms and Lipschitz exponents as a means for analysing seismic data

Hormoz Izadi*, Kris Innanen and Michael Lamoureux
hizadi@ucalgary.ca

Introduction

In seismic signal analysis, regions of abrupt change classifiable as "edges", contain considerable amount of a signal's information, thus making edge detection a potentially appropriate and efficient tool for obtaining information from seismic data (Innanen, 2003). Edge detection requires analysis of local properties of corresponding edges.

The Wavelet transform characterises the local regularity of a signal by decomposing signals into fundamental building blocks localised in space and frequency. Applying advanced mathematical techniques namely continuous wavelet transform enables us to obtain the modulus maxima from seismic data and estimate the Lipschitz exponents which in turn allows us to measure the local regularity of functions and differentiate the intensity profile of different edges (Mallat and Zhong, 1992; Mallat and Hwang, 1992).

A robust estimation of Lipschitz exponents from seismic data, alongside prior geological information, could potentially lead to processing and inversion algorithms able to discern and characterise such targets.

Theoretical Background

I. Wavelet transform

Although a powerful tool for analysing periodic functions, the Fourier transform fails to provide time-frequency localisation (Daubeschies, 1992; Kaiser, 1994; Qian, 2002). One possible solution is to cut *f* into blocks and subsequently perform the Fourier transform on a block by block basis which will provide information in regards to the signal's frequency content during the time frame covered by the corresponding window (Qian, 2002). This method referred to as short-time Fourier transform can be described by the following mathematical relation (Daubeschies, 1992),

$$(T^{win}\tilde{f})(\omega,t)=\int_{-\infty}^{+\infty} f(s)g(s-t)e^{-i\omega s}ds \quad (1)$$

where *f(s)* is an arbitrary signal and *g(s)* is the windowed function designed to localise signals in time. An underlying problem with this method relates to the flexibility or the fixed size of the window.

The wavelet transform provides a solution to some of the shortcomings associated with the Fourier and the short-time Fourier transform by utilising a scalable modulated window and providing a time-scale representation of the signal by calculating the spectrum at every position and shifting the scalable window along the signal. Mathematically the wavelet transform for a given function *f(t)* at some scale *s* is given by

$$Wf(s,\tau)=\int_{-\infty}^{+\infty} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-\tau}{s}\right)dt \quad (2)$$

where $\psi(t)$ is the *"mother wavelet"* dilated and time-shifted.

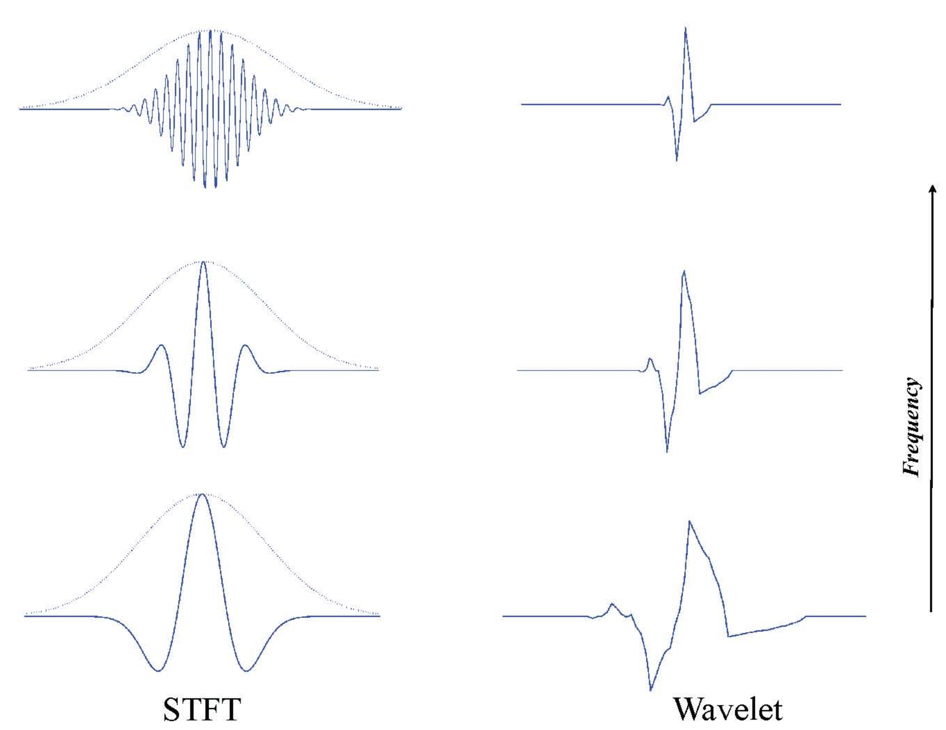

FIG. 1. The wavelet (right side) varies the width while keeping the number of oscillations constant. The short-time Fourier transform (left side) has a fixed window size independent of oscillations.

II. Lipschitz regularity

The local regularity of a function *f* is often measured by the corresponding Lipschitz exponent (Mallat and Zhong, 1992; Hong et al., 2002). Based on wavelet transform, a function *f(x)* is said to be uniformly Lipschitz α over $[a,b]$ if and only if there exists a constant $A > 0$ such that the wavelet transform satisfies the following (Mallat and Zhong, 1992; Innanen, 2003),

$$|W_s f(x)| \le As^{\alpha} \quad (3)$$

where $|W_s f(x)|$ is the modulus maxima of the function *f(x)* at various scales $s=2^j$. Based on the following properties a distinction could be made between singular and differentiable functions (Mallat and Zhong, 1992):

- If *f(x)* is Lipschitz α, then its integral *g(x)* has an associated Lipschitz exponent equal to $\alpha + 1$.
- A function *f (x)* is singular if the associated Lipschitz exponent, α, is less than 1.
- The Lipschitz exponent α, associated with a continuously differentiable function *f (x)* is equal to or greater than 1.
- The Lipschitz regularity of a delta function is equal to 1, since its associated modulus maxima decreases with scale.

III. Estimating the Lipschitz Exponent

To estimate α, one could find linearise equation (3) and obtain the slope

$$\log_2|W_s f(x)| \le \log_2 A + \alpha\log_2(s) \quad (4)$$

Additionally one could estimate α and *A* by forming the following objective function

$$\phi(A,\alpha)=\sum_{i,j=1}^{n}(\log_2(a_i)-(\log_2 A+\alpha\log_2(s)))^2 \quad (5)$$

and minimise to get

$$\begin{pmatrix} A \\ \alpha \end{pmatrix} = \begin{pmatrix} n & \sum_{j=1}^{n}\log_2(s_j) \\ \sum_{j=1}^{n}\log_2(s_j) & \sum_{j=1}^{n}(\log_2(s_j))^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{n}\log_2(a_i) \\ \sum_{j=1}^{n}\log_2(s_j).\sum_{i=1}^{n}\log_2(a_i) \end{pmatrix} \quad (6)$$

However, the use of the linear model given in (5) would be limited to single events. In seismic signal analysis due to absorption and loss of energy with progression of time, a single event resembling a delta function would gradually obtain spectral characteristics of a Gaussian with increasing variance. Such a function could be modelled as a delta function smoothed by a Gaussian with variance σ^2. Thus, the new model is non-linear and requires the minimisation of the following objective function,

$$\phi(A,\alpha,\sigma)=\sum_{i,j=1}^{n}[\log_2|a_i|-\log_2(A)-j+\frac{\alpha-1}{2}(\log_2(\sigma^2+2^{2j})]^2 \quad (7)$$

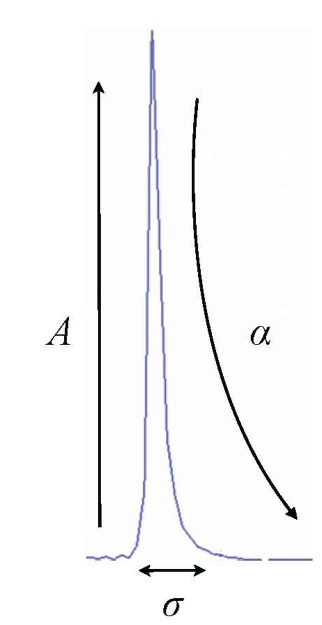

FIG. 2. Three parameters α, A and σ, measuring a pulse' decay, amplitude and width respectively.

Conclusion

The continuous wavelet transform and the associated Lipschitz regularity provide a potentially efficient and powerful tool for analysing singularities in a signal. For a single event, a linear model enables us to estimate the Lipschitz exponent and characterise the singularity with relative ease.

However, for practical applications, a seismic event would have to be modelled as delta function smoothed by a Gaussian, thus leading to a non-linear model. In order to estimate the Lipschitz exponent, one would have to form the objective function and minimise, using a relatively time consuming and computationally expensive method such as the steepest descent or conjugate gradient.

Our primary concern is to analyse closely spaced events, develop a model of attenuation and dispersion in order to estimate *Q* values.

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