Pseudospectral-element modelling of elastic waves

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Strong form

- The i^{th} component of the strong (differential) form of the full elastic wave equation, in an isotropic medium $\Omega \in \mathbb{R}^d$, is

$$\rho_i \ddot{u}_i = \partial_j \sigma_{ij} + f_i, \ \mathbf{x} \in \Omega, \ t > 0. \quad (1)$$

- $\mathbf{u} = (u_1, ..., u_d)$ is the displacement vector
- $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$
- ρ is the density
- *dot* denotes time differentiation
- σ_{ij} are the stresses
- $\partial_j := \partial / \partial x_j$
- f_i is the i^{th} component of the applied force
- sum over repeated indices
- $\sigma_{ij} = \lambda(\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu\varepsilon_{ij}$
- $\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$
- λ and μ are the Lamé parameters

Weak form

- Multiply both sides of equation (1) by an arbitrary function $v(\mathbf{x})$ and integrate by parts to obtain the weak (integral) form,

$$\int_\Omega \rho \ddot{u}_i v d\Omega + \int_\Omega \sigma_{ij} \partial_j v d\Omega = \int_\Omega f_i v d\Omega + \oint_{\partial\Omega} \sigma_{ij} v \cdot \mathbf{n} dS \quad (2)$$

- Pseudospectral methods choose a set of points $\{\mathbf{x}_0, ..., \mathbf{x}_N\}$ in Ω and a set of functions $\{\phi_0, ..., \phi_N\}$ in $L^2(\Omega)$ with the property

$$\phi_m(\mathbf{x}_n) = \delta_{mn}$$

- Write the displacements as linear combinations of the basis vectors

$$u_i(\mathbf{x}, t)) = \sum_{n=0}^{N} u_i(\mathbf{x}_n, t)\phi_n(\mathbf{x})$$

- Equation (2) is enforced for $v = \phi_m(\mathbf{x})$, for all $m = 0, ..., N$

Boundary conditions in 2D

- Split the surface integral over the $\alpha = N, S, E$ and W boundaries

$$\oint_{\partial\Omega} \sigma_{ij} \cdot \mathbf{n} v dS = \sum_\alpha \oint_{\Gamma_\alpha} \sigma_{ij} \cdot \mathbf{n} v dS$$

- The free surface condition $\sigma_{ij} \cdot \mathbf{n} = 0$ implies

$$\oint_{\Gamma_N} \sigma_{ij} v \cdot \mathbf{n} dS = 0.$$

- Second order absorbing boundary conditions along a vertical boundary at $x = x_{max}$ can be enforced by substituting into the stresses

$$\partial_1 u = -\frac{1}{Vp}\dot{u} - \frac{Vp - Vs}{Vp}\partial_2 w, \quad \partial_1 w = -\frac{1}{Vs}\dot{w} - \frac{Vp - Vs}{Vs}\partial_2 u.$$

- Similarly, at $z = z_{max}$ the substitution is

$$\partial_2 u = -\frac{1}{Vs}\dot{u} - \frac{Vp - Vs}{Vs}\partial_1 w, \quad \partial_2 w = -\frac{1}{Vp}\dot{w} - \frac{Vp - Vs}{Vp}\partial_1 u.$$

- At $x = 0$ the signs are switched.

Time-integration

- Substituting the boundary conditions into equation (2) produces a system of ordinary differential equations

$$M\ddot{\mathbf{U}}(t) + A\dot{\mathbf{U}}(t) + K\mathbf{U}(t) = \mathbf{F}(t).$$

which can be time-stepped numerically.

Domain decomposition

- In domain decomposition the model parameters are split up into smaller constant regions. This can be done by averaging the parameters at the 4 corners, or fitting a polynomial to the original model and evaluating at the cell centers.

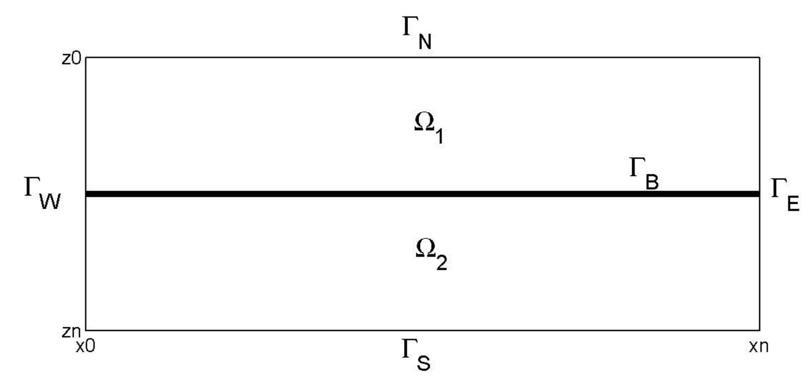

Figure: A two element decomposition sharing an interior boundary.

- At the interface between elements we enforce the conditions
  - Continuity of displacement: $u_i|_{\Omega_1} = u_i|_{\Omega_2}$
  - Continuity of traction: $\sigma_{ij} \cdot \mathbf{n}|_{\Omega_1} = \sigma_{ij} \cdot \mathbf{n}|_{\Omega_2}$
- The first is done by making the functions $\phi_m(\mathbf{x})$ piecewise continuous at the interfaces.

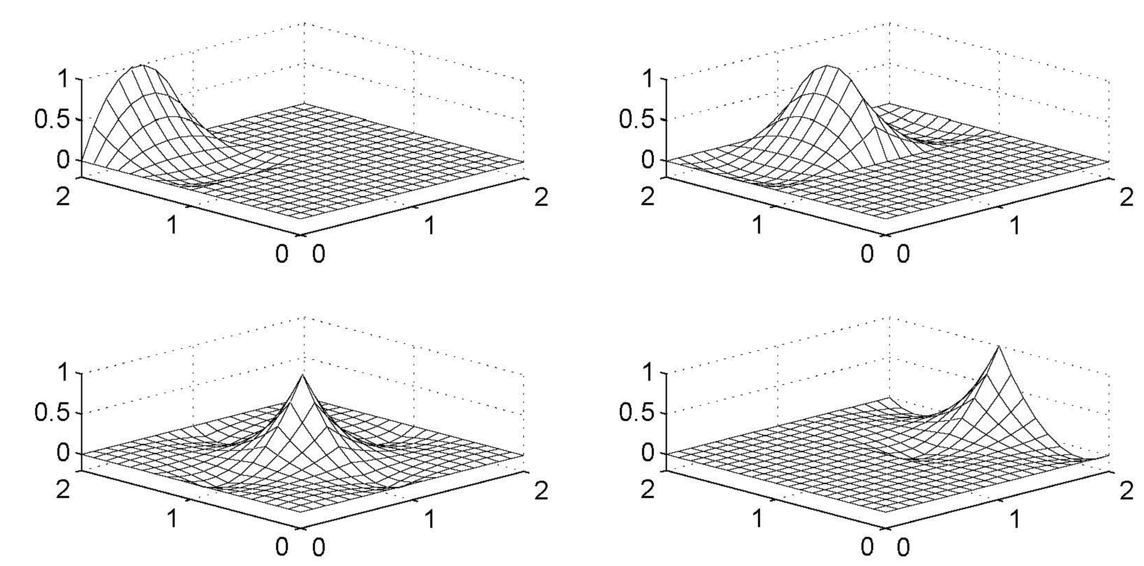

Figure: 2D spectral element basis functions defined on 4 elements.

- The second we get for free by deleting all interior surface integrals.

$$\sum_k \oint_{\Gamma_B} \sigma_{ij} \cdot \mathbf{n}|_{\Omega_k} v dS = 0.$$

- The resulting system is *very* sparse.

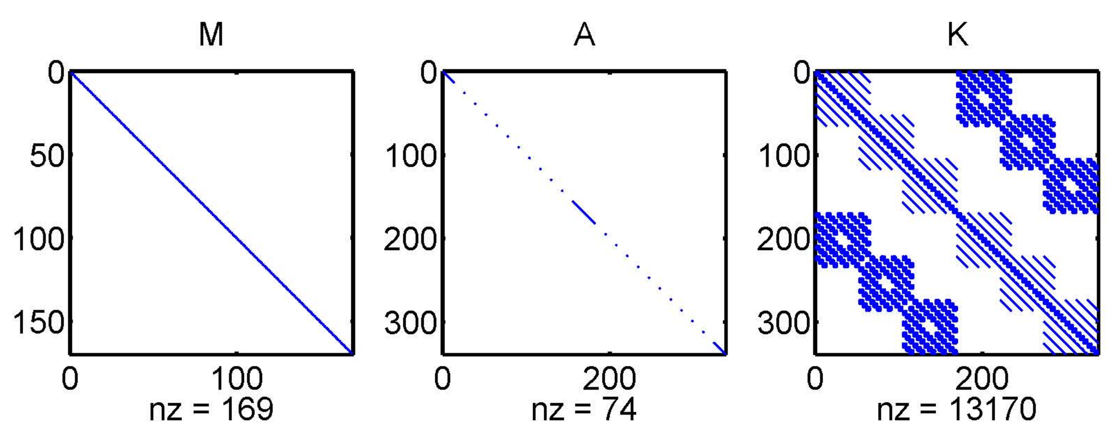

Figure: Sparsity patterns of M, A and K for 9 5-node elements.

- More complicated models can be built by using many smaller elements, akin to building an image from pixels.

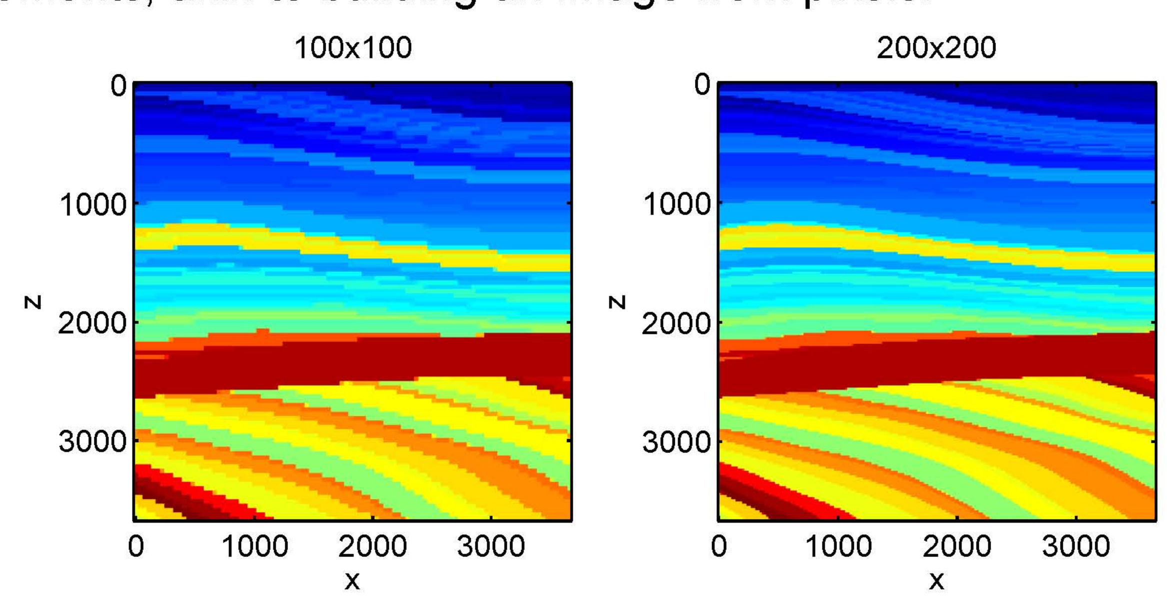

Figure: 2 simple meshes obtained by interpolating P wave velocities.

Example.

- Consider a simple two-layer medium with a free-surface and absorbing sides and bottom. The source is a Ricker wavelet applied at a single node.

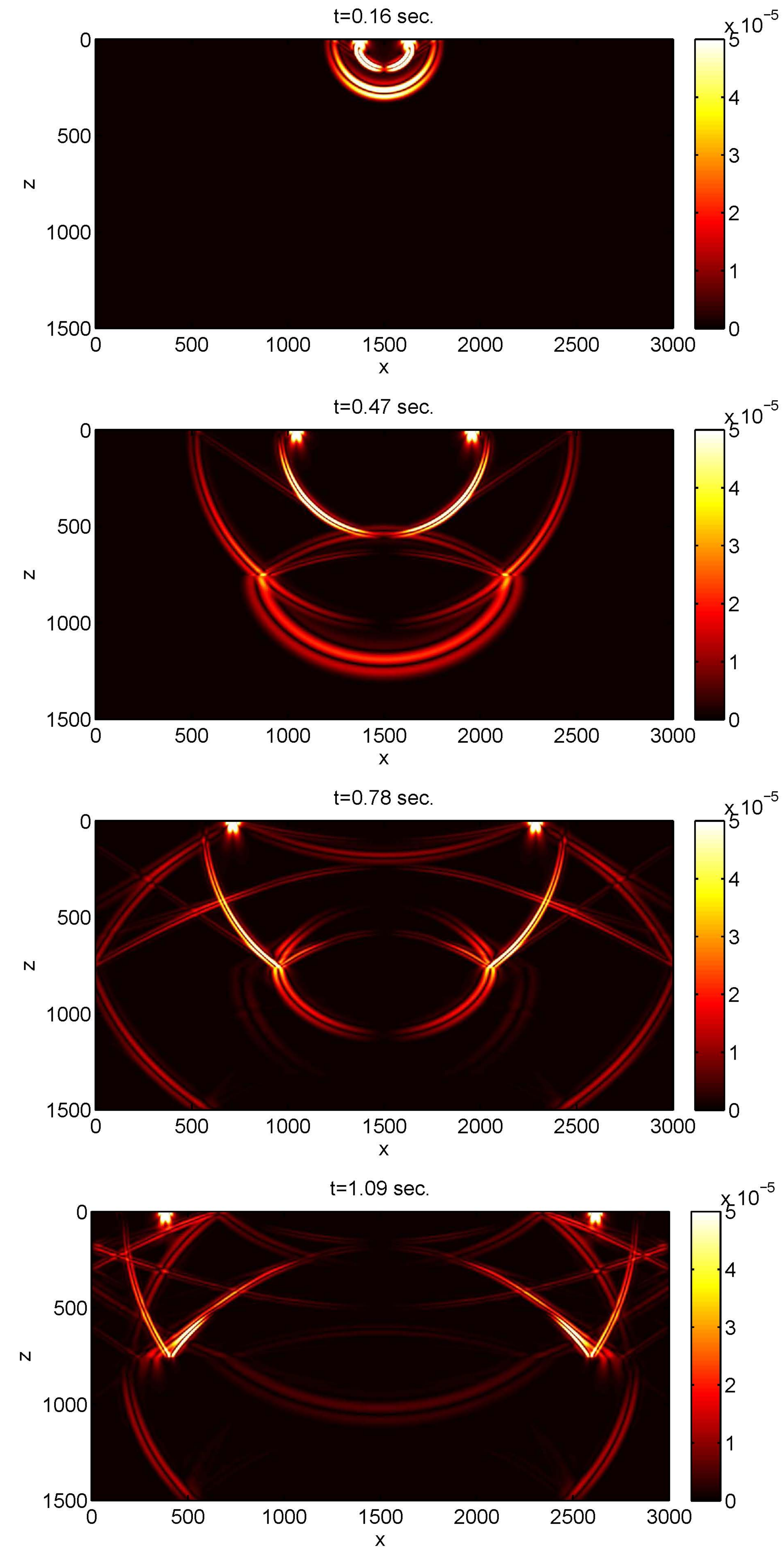

Figure: 2-norm of the displacement vector at various times.

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