Reflection coefficients through a linear velocity ramp

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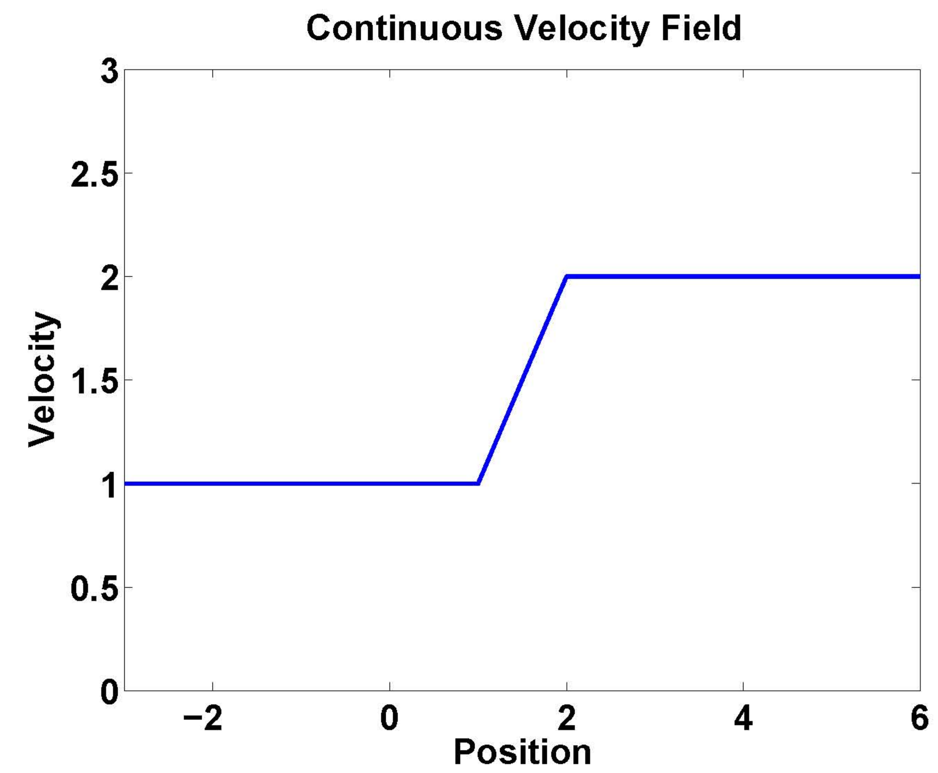
Overview

A seismic wave propagating from one region of constant velocity to another, through a smooth transition zone, will differentially reflect or transmit across the zone, depending on the relative sizes of the transition zone and the wavelength of the propagating wave. This work presents an exact analytic solution for the case of a linear ramp velocity in the transition zone,



Linear ramp velocity field, from $c = 1$ to $c = 2$.

We demonstrates that for long wavelengths, the ramp looks essentially like a jump discontinuity in the medium, with the corresponding reflection and transmission coefficients. For short wavelengths, the ramp provides essentially 100% transmission and no reflection. Energy conservation is verified for all wavelengths.

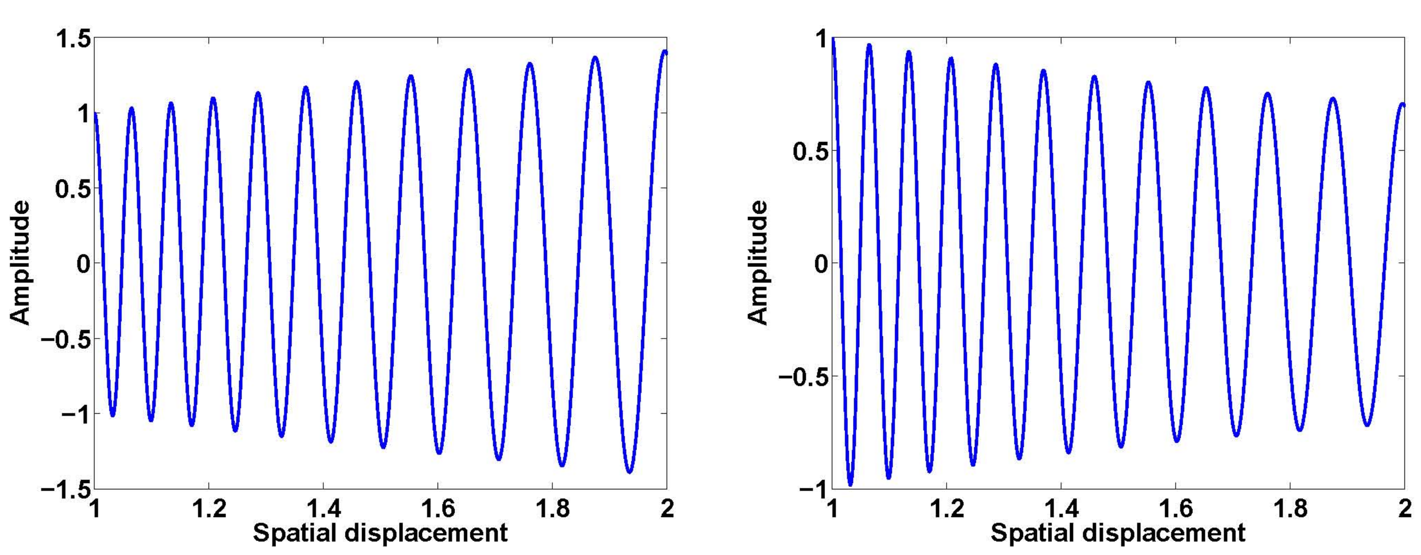
Variable density, elasticity

Careful consideration is given to the two cases of varying the velocity parameter, one via variations in the density of the propagation medium, the other in varying the modulus of elasticity.

In the transition zone, the analytic solution to the wave equation is in the form

$$u(x,t) = x^{\pm 1/2 \pm \sqrt{1/4-\omega^2}} e^{-i\omega t}$$

where the $+1/2$ is in the varying density case, $-1/2$ is in the varying elasticity case. A plot of the two cases:



Wavefield in the transition zone. Variable density, elasticity.

In both cases the wavelength expands in the transition from left to right, consistent with the increasing velocity of propagation, but the amplitude response is different.

Wave Equation

The one dimensional "elastic" wave equation for a displacement field $u(x,t)$, of a disturbance travelling along a weighted string under tension, with density ρ and modulus of elasticity (bulk modulus) K is given in the standard form

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right).$$

In the elastic equation, the modulus of elasticity appears inside the derivative, while the density is outside. For non-constant ρ, K, the position of these coefficients within the derivative has important physical consequences – in particular on the sign of reflection.

Analytic Solution

To compute reflection and transmission coefficients across the ramp, set an incoming wave on the left of the form $e^{i\omega(x-t)}$, and hypothesize a reflected wave of the form $Re^{i\omega(-x-t)}$ and transmitted wave on the right of the form $Te^{i\omega(x/2-t)}$. In the transition region, set the wave to a linear combination of the solutions $x^{\pm 1/2 \pm \sqrt{1/4-\omega^2}} e^{-i\omega t}$. This gives three regional solutions,

$$\begin{aligned} u_{left} &= e^{i\omega(x-t)} + Re^{i\omega(-x-t)}, \\ u_{trans} &= Ax^{n1}e^{-i\omega t} + Bx^{n2}e^{-i\omega t}, \\ u_{right} &= Te^{i\omega(x/2-t)}, \end{aligned}$$

with $n1, n2 = 1/2 \pm \sqrt{1/4-\omega^2}$ in the varying density case,
and $n1, n2 = -1/2 \pm \sqrt{1/4-\omega^2}$ in the varying elasticity case.
Setting physical continuity conditions across the interfaces leads to a system of equations for coefficients R, T, A, B

$$\begin{bmatrix} 1 & 1 & -e^{-i\omega} & 0 \\ 2^{n1} & 2^{n2} & 0 & -e^{i\omega} \\ n1 & n2 & i\omega e^{-i\omega} & 0 \\ n1\cdot 2^{n1-1} & n2\cdot 2^{n2-1} & 0 & -0.5i\omega e^{i\omega} \end{bmatrix} \begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} e^{i\omega} \\ 0 \\ i\omega e^{i\omega} \\ 0 \end{bmatrix}.$$

Reflection coefficients

With Cramer's rule, and Mathematica™, an exact solution for the reflection coefficient is obtained

$$R(\omega) = \frac{e^{2i\omega}(2^{n1}-2^{n2})(n1+n2)}{2^{n2}(2i\omega + 2\sqrt{1/4-\omega^2}) + 2^{n1}(-2i\omega + 2\sqrt{1/4-\omega^2})}.$$

In the varying density case, $n1 + n2 = 1$, and setting $a = \sqrt{1/4-\omega^2}$, yields

$$R(\omega) = \frac{e^{2i\omega}(2^a - 2^{-a})}{2^{-a}(2i\omega + 2a) + 2^a(-2i\omega + 2a)}.$$

Note that $R(0) = 1/3$, so there is a positive reflection in this varying density case, at low frequencies.

In the varying modulus case, $n1 + n2 = -1$, yielding exactly the negative of this previous solution,

$$R(\omega) = -\frac{e^{2i\omega}(2^a - 2^{-a})}{2^{-a}(2i\omega + 2a) + 2^a(-2i\omega + 2a)}.$$

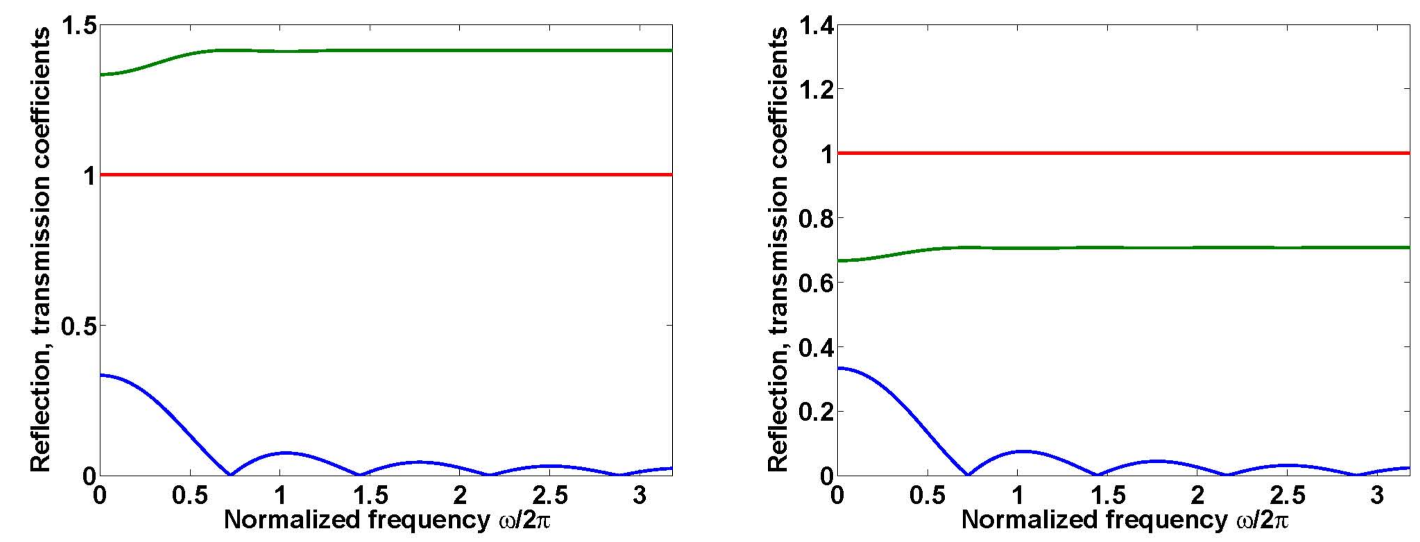
Here, $R(0) = -1/3$, so there is a negative reflection in this varying modulus case, at low frequencies.

A similar procedure will give the analytic solution for the transmission coefficients.

Reflection, transmission

We plot of the reflection and transmission coefficients R, T as a function of frequency, as well as a check on the conservation of energy, with

$$|R(\omega)|^2 + \frac{K}{c}|T(\omega)|^2 = 1.$$



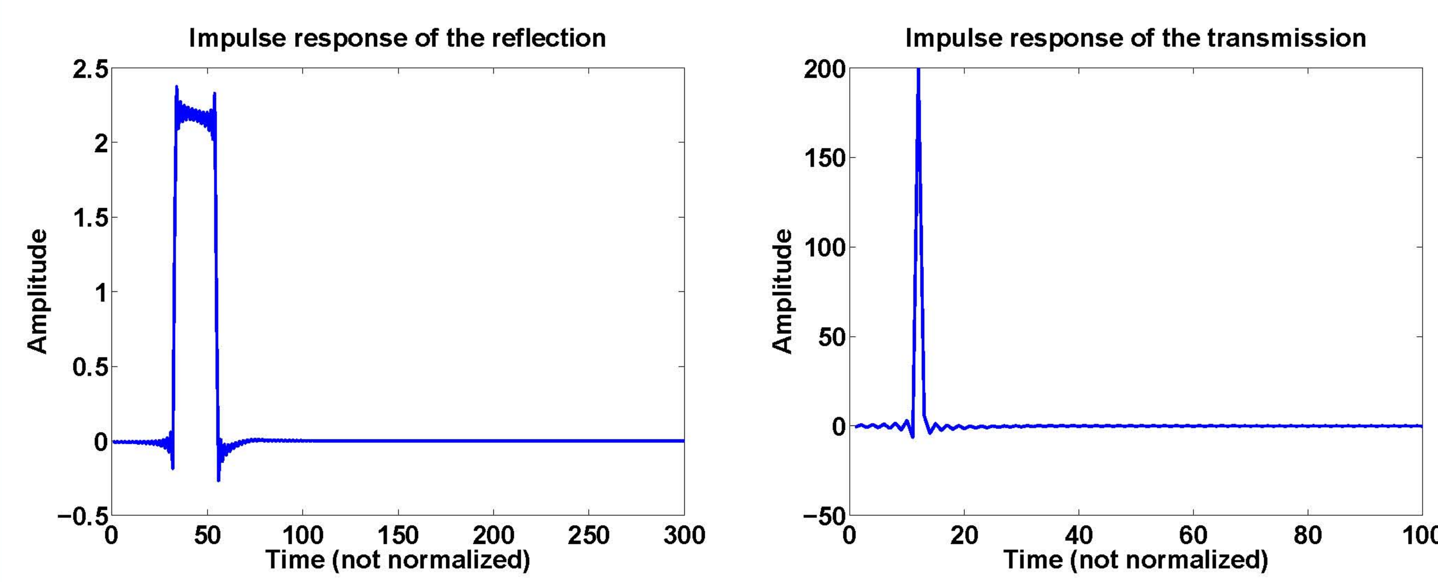
Reflection (blue), transmission (green), total energy (red).

The varying density (left) and elasticity (right) cases give different results, reflecting the physical differences in the setting. For low frequencies, the reflection is equivalent to a physical jump in velocity – that is, for long wavelength, the ramp looks like an abrupt physical jump.

Impulse response

An arbitrary waveform impinging on the velocity ramp is transformed into two resulting waveforms, the reflected waveform and the transmitted waveform. The filter response is simply the result of initiating a delta spike on the left of the velocity ramp, and allowing it to travel into the ramp, creating a reflected and a transmitted waveform. The filter response for the ramp in the frequency domain has been computed above; the inverse Fourier transform will give the filter response in space.

The numerical results are plotted, showing that the reflection from the ramp produces a broadened pulse (left), while the transmission is a delta spike (right). We can translate these results into typical physical parameters in a seismic experiment, to determine the significance of this frequency dependent response and broadening of the delta response.



Impulse response, reflection and transmission.

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