Approximate- vs. full-Hessian in FWI: 1D analytical and numerical experiments

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ABSTRACT

For a 1D scalar medium, analytical derivations show that using the approximate-Hessian in FWI brings the model update to within a first order approximation of the scalar reflection coefficient for a single interface. Compared to the approximate-Hessian, we found that the full-Hessian provides additional scaling information at the depth of the interface, improving the accuracy of the inversion. These ideas were also tested using a numerical example displaying how both Hessians move very fast toward the actual velocity model. It is also shown that the full-Hessian leads to a very accurate inversion in the presence of large velocity contrasts superior to the approximate-Hessian. Hence, the full-Hessian may achieve a faster convergence and accurate inversion while providing amplitude information.

Full Waveform Inversion

The goal of FWI is to find the squared-slowness model, $s_0(\mathbf{r}) \equiv 1/c_{0,n}^2(\mathbf{r})$ which "most likely" generated, the recorded wavefield $P(\mathbf{r_g}, \mathbf{r_s}, \omega)$ by minimizing , after n iterations, the objective function $\Phi(s_0^{(n)})$ defined as

$$\Phi(s_0^{(n)}) \equiv \frac{1}{2}\int d\omega \left(\sum_{s,g} |\delta P|^2\right). \tag{1}$$

Here, δP is the residual wavefield defined as follows

$$\delta P(\mathbf{r}_g, \mathbf{r}_s, \omega | s_0^{(n)}) \equiv P(\mathbf{r}_g, \mathbf{r}_s, \omega) - G(\mathbf{r}_g, \mathbf{r}_s, \omega | s_0^{(n)}),$$

where $G\left(\mathbf{r_g}, \mathbf{r_s}, \omega | s_0^{(n)}\right)$ is the modelled field due to the n-th squared-slowness model iteration.

To achieve that goal the slowness model must be updated after each iteration. The model update $\delta s_0^{(n)}(\mathbf{r}'')$ can be computed as

$$\delta s_0^{(n)}(\mathbf{r}'') = -\int d\mathbf{r}' H^{(n)-}(\mathbf{r}'', \mathbf{r}') g^{(n)}(\mathbf{r}'), \tag{2}$$

where $H^{(n)-}(\mathbf{r}'', \mathbf{r}')$ is the inverse of the Hessian $H^{(n)}(\mathbf{r}', \mathbf{r})$, and $g^{(n)}(\mathbf{r}')$ is the gradient of the objective function, defined as follows

$$H^{(n)}(\mathbf{r}', \mathbf{r}) = \frac{\partial^2 \Phi(s_0^{(n)})}{\partial s_0^{(n)}(\mathbf{r}') \partial s_0^{(n)}(\mathbf{r})}, \qquad g^{(n)}(\mathbf{r}') = \frac{\partial \Phi(s_0^{(n)})}{\partial s_0^{(n)}(\mathbf{r}')}.$$

The Hessian can also be written as the sum of two terms

$$H^{(n)}(\mathbf{r}', \mathbf{r}) = H_1^{(n)}(\mathbf{r}', \mathbf{r}) + H_2^{(n)}(\mathbf{r}', \mathbf{r}). \tag{3}$$

In terms of Greens functions G and the residual wavefield δP, these terms can be written as

$$H_1^{(n)}(\mathbf{r}', \mathbf{r}) = -\sum_{s,g} \int d\omega \omega^4 [G(\mathbf{r}_g, \mathbf{r}, \omega) G(\mathbf{r}, \mathbf{r}', \omega) G(\mathbf{r}', \mathbf{r}_s, \omega) + G(\mathbf{r}_g, \mathbf{r}', \omega) G(\mathbf{r}', \mathbf{r}, \omega) G(\mathbf{r}, \mathbf{r}_s, \omega)] \delta P^*(\mathbf{r}_g, \mathbf{r}_s, \omega)], \tag{4}$$

$$H_2^{(n)}(\mathbf{r}', \mathbf{r}) = \sum_{s,g} \int d\omega \omega^4 G(\mathbf{r}_g, \mathbf{r}', \omega) G(\mathbf{r}', \mathbf{r}_s, \omega) G^*(\mathbf{r}_g, \mathbf{r}, \omega) G^*(\mathbf{r}, \mathbf{r}_s, \omega)). \tag{5}$$

Approximate-Hessian

When residuals are small the H_1 term in equation 3 may be neglected leading us to an approximate version of the Hessian which only depends on H_2.

The general expression for a causal homogeneous Green's function, in a 1D scalar medium is

$$G(z, z_s, \omega) = \frac{e^{ik|z - z_s|}}{i2k}. \tag{6}$$

For a source and a receiver on the surface we get

$$G(z, 0, \omega) = G(0, z, \omega) = \frac{e^{ik_0 z}}{i2k_0}. \tag{7}$$

Substituting equation 6 and its complex conjugate in equation 5 we get the following expression for H_2 and its inverse:

$$H_2(z', z) = \frac{c_0^5 \pi}{2^4} \delta(z' - z), \tag{8}$$

$$H_2^-(z, z'') = \frac{2^4}{c_0^5 \pi} \delta(z'' - z). \tag{9}$$

Quasi-Newton Step

For computing the model update $\delta s_0^{(n)}$, given by equation 2 we can use the following expression for the gradient

$$g(z) = \frac{R_1 c_0^3 \pi}{2^2} \mathrm{H}(z - z_1). \tag{10}$$

Substituting equations 9 and 10 in 2 we get

$$\delta s_0^{(n)}(z) = -\int dz' \left[\frac{2^4}{c_0^5 \pi} \delta(z' - z)\right] \left[\frac{R_1 c_0^3 \pi}{2^2} \mathrm{H}(z' - z_1)\right].$$

After integration, the model update given by taking a quasi-Newton step is

$$\delta s_0^{(n)}(z) = -\frac{4R_1}{c_0^2} \mathrm{H}(z - z_1). \tag{11}$$

Step Analysis

For understanding the magnitude of the model update let us write the scalar normal-incidence reflection coefficient R in term of slownesses and solve for the ratio s_1/s_0 as follows

$$R = \frac{s_0 - s_1}{s_0 + s_1}, \qquad \frac{s_1}{s_0} = \frac{(1 - R)}{(1 + R)}.$$

Expanding the denominator as a Taylor's series and keeping terms up to the second order

$$\frac{s_1}{s_0} = (1 - R)(1 - R + R^2 - R^3 ...) \approx (1 - R)(1 - R).$$

$$\frac{s_1}{s_0} \approx 1 - 2R + R^2.$$

Squaring the slowness ratio we get

$$\frac{s_1^2}{s_0^2} \approx (1 - 2R)^2 = 1 - 4R + R^2 \approx 1 - 4R.$$

Solving for the slowness-squared difference $\Delta S \approx s_1^2 - s_0 2$ results

$$\Delta S \approx -4 s_0^2 R = \frac{-4R}{c_0^2}. \tag{12}$$

Comparing equations 12 and 11 we can see that the update provided by the approximate-Hessian is in agreement with a first order approximation of the slowness contrast related to the reflection coefficient R.

Full-Hessian

For computing the residual dependent term H_2 in the full-Hessian we used the following expression for the complex conjugate of the residuals

$$\delta P^*(0, 0, \omega) = -R_1 \frac{e^{-i2k_0 z_1}}{i2k_0}. \tag{13}$$

Using equations 6, 7 and 13 in 4 we get

$$H_1(z', z) = -\int d\omega \omega^4 \left[\frac{e^{ik_0 z}}{i2k_0} \frac{e^{ik_0|z - z'|}}{i2k_0} \frac{e^{ik_0 z'}}{i2k_0} + \frac{e^{ik_0 z'}}{i2k_0} \frac{e^{ik_0|z' - z|}}{i2k_0} \frac{e^{ik_0 z}}{i2k_0}\right] \left(-R_1 \frac{e^{-i2k_0 z_1}}{i2k_0}\right). \tag{14}$$

After integration

$$H_1(z', z) = \frac{R_1 c_0^5 \pi}{2^3} \delta(Z + |z - z'|), \tag{15}$$

Where $Z = z + z' - 2z_1$.

Solving the absolute value term for the case when z > z' we get

$$H_1(z', z) = \frac{R_1 c_0^5 \pi}{2^4} \delta(z - z_1).$$

After substituting equations 8 and 15 in 3, we get the following expression for the full-Hessian

$$H(z', z) = \frac{R_1 c_0^5 \pi}{2^3} \delta(Z + |z - z'|) + \frac{c_0^5 \pi}{2^4} \delta(z' - z).$$

Numerical Analysis

Following figures show the numerical results for quasi- and full-Newton FWI's in a 1D scalar medium.

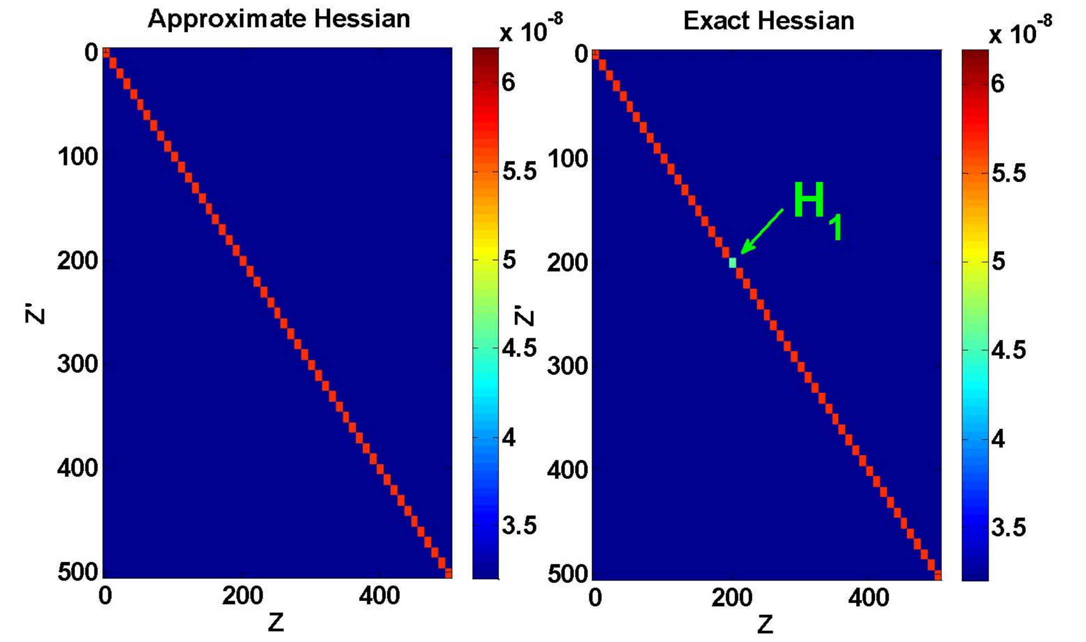

FIG. 1. Approximate- (left) and exact-Hessian (right) matrix. Note that the full Hessian just provide additional scaling information at the depth of the interface.

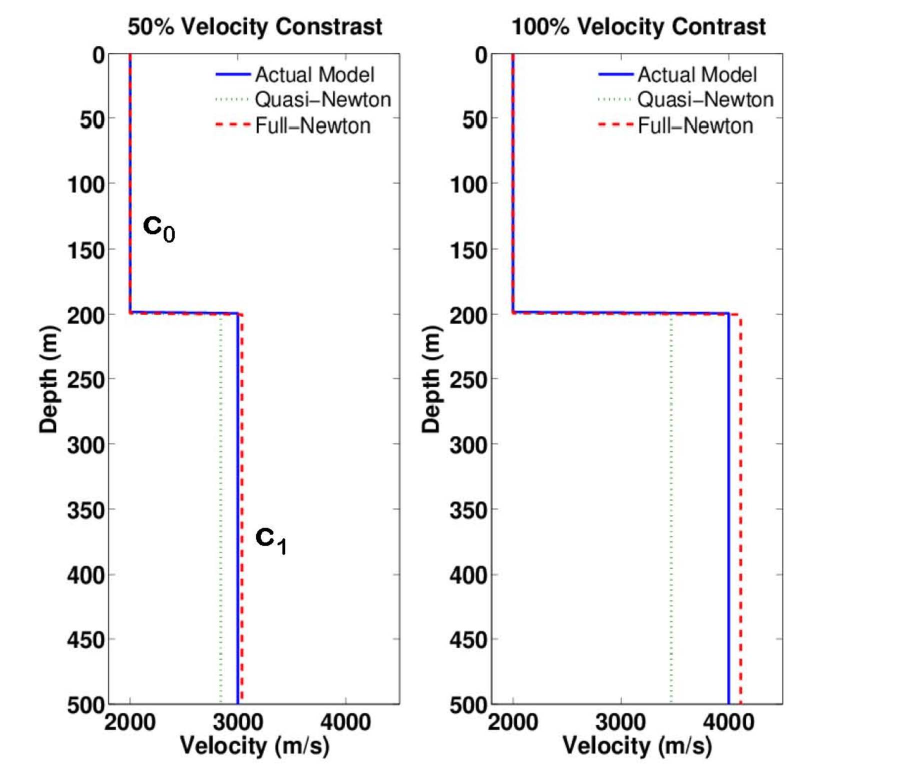

FIG. 2. FWI results for a meidum (left) and large (right) velocity contrasts. Results show that for a large velocity contrast the quasi-Newton FWI underestimate the velocity value for the second medium.

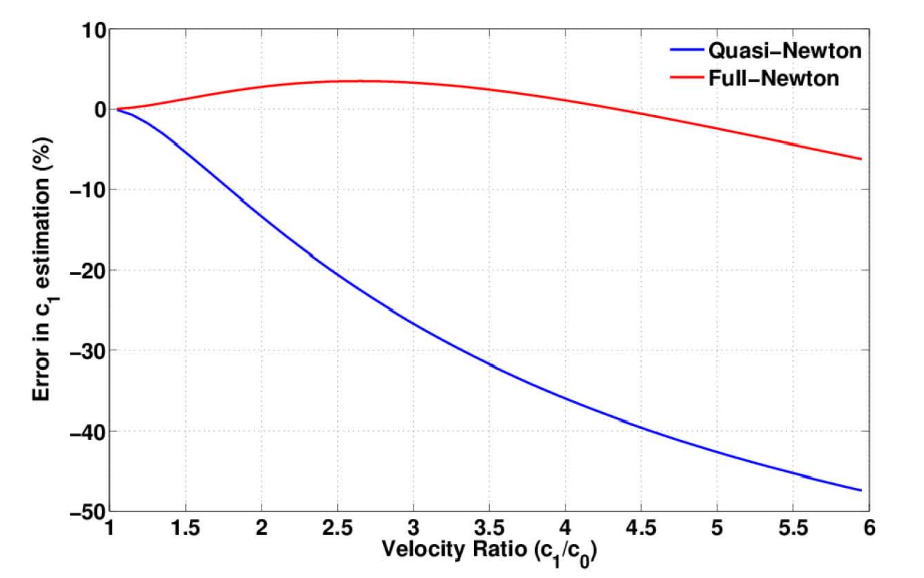

FIG. 3. Error in the estimation of c_1 for a wide range of velocity contrasts. Note that the quasi-Newton FWI increasingly underestimate the velocity of the second medium. On the other hand, the full-Newton FWI is always close to the actual c_1 value

CONCLUSIONS

- The effect of the Hessian in FWI is providing the proper scaling of the gradient for getting to the minimum of the objective function.
- The approximate Hessian gives a very accurate inversion for low velocity contrasts comparable to a first order approximation of the reflection coefficient.
- The full-Hessian improves the scaling of the gradient and give better results both for small and large velocity contrasts.
- Using the full-Hessian points toward a better inversion when strong AVO effects are present on the data. Studying its effect in a 2D sense is required to confirm this statement .