Poroelastic Scattering Potentials and Inversion Sensitivities

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Summary

Estimating the seismic wavefields response corresponding to the small model parameters' perturbations is a classical problem in inverse scattering problem of exploration geophysics. The Fréchet derivatives play a crucial role in least-squares inverse problems. The scattering potentials indicating the perturbations of model parameters can be considered as engines for seismic wave scattering. In this research, we reviewed the Biot's theory for poroelastic wave equations and derived the poroelastic scattering potentials represented by different field variables firstly. And then we derived the coupled poroelastic Fréchet derivatives with respect to 9 poroelastic parameters, namely, the Lamé coefficients of the dry frame λ_{dry} and μ, porosity/fluid term f, density of saturated medium ρ_{sat}, fluid density ρ_f, C, M, $\tilde{\rho}$ and mobility of the fluid m using perturbation method and non-perturbation method. The porosity/fluid term f involved by Russell et al. (2011) for linearized AVO analysis is considered as a poroelastic parameter for sensitivity analysis. The explicit expressions for these Fréchet derivatives with respect to different poroelastic parameters are provided. When wave propagating in poroelastic media, there are two kinds of compressional waves: the fast compressional wave and the slow compressional wave. In this research, we also derived the P-SV Fréchet derivatives in which the fast compressional wave and slow compressional wave are coupled together.

Biot's Theory for Poroelastic Wave Equations

Recall the poroelastic wave equations represented by solid displacements $\mathbf{u}$ and relative fluid-solid displacements $\mathbf{w}$:

$$\nabla \cdot (\lambda_{sat} \nabla \cdot \mathbf{u} + C \nabla \cdot \mathbf{w}) \mathbf{I} + 2\mu \nabla^2 \mathbf{u} + \mathbf{F} = -\omega^2 (\rho_{sat} \mathbf{u} + \rho_f \mathbf{w}),$$

$$\nabla \cdot (C \nabla \cdot \mathbf{u} + M \nabla \cdot \mathbf{w}) \mathbf{I} + \mathbf{f} = -\omega^2 (\rho_f \mathbf{u} + \tilde{\rho} \mathbf{w} + m \mathbf{w}),$$

The poroelastic wave equations can be expressed in matrix form:

$$\boldsymbol{\mathcal{L}}_P(\mathbf{r}, \omega) \cdot \begin{pmatrix} \mathbf{u}(\mathbf{r}, \omega) \\ \mathbf{w}(\mathbf{r}, \omega) \end{pmatrix} = -\begin{pmatrix} \mathbf{F} \\ \mathbf{f} \end{pmatrix}$$

where the poroelastic wave modeling operator $\boldsymbol{\mathcal{L}}_P$ is a 6×6 matrix and it can be written as:

$$\boldsymbol{\mathcal{L}}_P(\mathbf{r}, \omega) = \begin{pmatrix} \boldsymbol{\mathcal{L}}^{s1}(\mathbf{r}, \omega) & \boldsymbol{\mathcal{L}}^{f1}(\mathbf{r}, \omega) \\ \boldsymbol{\mathcal{L}}^{s2}(\mathbf{r}, \omega) & \boldsymbol{\mathcal{L}}^{f2}(\mathbf{r}, \omega) \end{pmatrix}$$

where $\boldsymbol{\mathcal{L}}^{s1}$, $\boldsymbol{\mathcal{L}}^{s2}$, $\boldsymbol{\mathcal{L}}^{f1}$ and $\boldsymbol{\mathcal{L}}^{f2}$ are all 6×6 matrices and each element can be found in the research report.

Inverse Scattering Theory

We can review the unperturbed and perturbed wave equations:

$$\boldsymbol{\mathcal{L}}_0(\mathbf{r}_g, \omega) \mathbf{G}_0(\mathbf{r}_g, \mathbf{r}_s, \omega) = \delta(\mathbf{r}_g - \mathbf{r}_s),$$

$$\boldsymbol{\mathcal{L}}(\mathbf{r}_g, \omega) \mathbf{G}(\mathbf{r}_g, \mathbf{r}_s, \omega) = \delta(\mathbf{r}_g - \mathbf{r}_s),$$

The **Scattering Potential** indicates the perturbations in the model properties:

$$\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{V}}$$

Substituting it into the perturbed wave equation forms the famous **Lippmann-Schwinger** equation:

$$\mathbf{G} = \mathbf{G}_0 + \mathbf{G}_0 \boldsymbol{\mathcal{V}} \mathbf{G}$$

And it can be expanded as a Born series in the quantity of $\mathbf{G}_0 \boldsymbol{\mathcal{V}}$:

$$\delta \mathbf{G} = \sum_{n=1}^{\infty} \mathbf{G}_0 (\boldsymbol{\mathcal{V}} \mathbf{G}_0)^n = \mathbf{G}_0 (\boldsymbol{\mathcal{V}} \mathbf{G}_0)^1 + \mathbf{G}_0 (\boldsymbol{\mathcal{V}} \mathbf{G}_0)^2 + \mathbf{G}_0 (\boldsymbol{\mathcal{V}} \mathbf{G}_0)^3 + \dots,$$

According **Born Approximation** the nonlinear terms can be ignored:

$$\delta \mathbf{G} \simeq \mathbf{G}_0 \boldsymbol{\mathcal{V}} \mathbf{G}_0,$$

And the **Fréchet Derivative** can be expressed as:

$$\frac{\delta \mathbf{G}}{\delta \mathbf{s}} = \mathbf{G}_0 \frac{\boldsymbol{\mathcal{V}}}{\delta \mathbf{s}} \mathbf{G}_0,$$

Coupled Poroelastic Fréchet Derivatives

According to the poroelastic wave equations, the poroelastic scattering potentials can be expressed as:

$$\boldsymbol{\mathcal{V}}_P = \begin{pmatrix} \boldsymbol{\mathcal{V}}^{s1} & \boldsymbol{\mathcal{V}}^{sf} \\ \boldsymbol{\mathcal{V}}^{sf} & \boldsymbol{\mathcal{V}}^{f2} \end{pmatrix}$$

Each element in the poroelastic scattering potential matrix can be expressed as:

$$V_{ii}^{s1} = \rho_{11}^0 a_{\rho_{11}} \omega^2 + \lambda_{dry}^0 \partial_i a_{\lambda_{dry}} \partial_i + 2\mu^0 \partial_i a_\mu \partial_i + \mu_0 \sum_{i \neq j} \partial_j a_\mu \partial_j, i, j = x, y, z;$$

$$V_{ij}^{s1} = \lambda_{dry}^0 \partial_i a_{\lambda_{dry}} \partial_j + f^0 \partial_i a_f \partial_j + \mu_0 \partial_i a_\mu \partial_j, i \neq j.$$

$$V_{ii}^{sf} = \rho_{12}^0 a_{\rho_{12}} \omega^2 + Q_0 \partial_i a_Q \partial_i, i, j = x, y, z;$$

$$V_{ij}^{sf} = Q_0 \partial_i a_Q \partial_j, i \neq j.$$

$$V_{ii}^{f2} = \rho_{22}^0 a_{\rho_{22}} \omega^2 + Q_0 \partial_i a_R \partial_i, i, j = x, y, z;$$

$$V_{ij}^{f2} = R_0 \partial_i a_R \partial_j, i \neq j.$$

Through a perturbation derivation method, we can the expressions of the poroelastic Fréchet derivatives:

$$\begin{pmatrix} \frac{\delta G_{il}^{s1}}{\delta s} & \frac{\delta G_{il}^{sf}}{\delta s} \\ \frac{\delta G_{il}^{sf}}{\delta s} & \frac{\delta G_{il}^{f2}}{\delta s} \end{pmatrix} \simeq \int_\Omega dV \begin{pmatrix} {}^0G_{ij}^{s1} & {}^0G_{ij}^{sf} \\ {}^0G_{ij}^{sf} & {}^0G_{ij}^{f2} \end{pmatrix} \cdot \begin{pmatrix} \frac{V_{jk}^{s1}}{\delta s} & \frac{V_{jk}^{sf}}{\delta s} \\ \frac{V_{jk}^{sf}}{\delta s} & \frac{V_{jk}^{f2}}{\delta s} \end{pmatrix} \cdot \begin{pmatrix} {}^0G_{kl}^{s1} & {}^0G_{kl}^{sf} \\ {}^0G_{kl}^{sf} & {}^0G_{kl}^{f2} \end{pmatrix},$$

where $i, j, k, l = x, y, z$.

Coupled Poroelastic Fréchet Derivatives

To get the Fréchet Derivatives for different poroelastic parameters, firstly, we introduced the poroelastic scatting potentials for different parameters $\boldsymbol{\mathcal{V}}_p$, where p indicates different poroelastic parameters. Each element of the single parameter-dependent scattering potentials can be found in APPENDIX A of the research report.

So, the Fréchet derivatives for different poroelastic parameters can be expressed as:

$$\frac{\delta G_{il}^{s1}}{a_p} = \sum_{i,j,k,l=x,y,z} \left({}^0G_{ij}^{s1} \left(\frac{V_{jk,p}^{s1}}{a_p} \right) {}^0G_{kl}^{s1} + {}^0G_{ij}^{sf} \left(\frac{V_{jk,p}^{sf}}{a_p} \right) {}^0G_{kl}^{sf} + {}^0G_{ij}^{sf} \left(\frac{V_{jk,p}^{sf}}{a_p} \right) {}^0G_{kl}^{s1} + {}^0G_{ij}^{sf} \left(\frac{V_{jk,p}^{f2}}{a_p} \right) {}^0G_{kl}^{sf} \right),$$

$$\frac{\delta G_{il}^{sf}}{a_p} = \sum_{i,j,k,l=x,y,z} \left({}^0G_{ij}^{s1} \left(\frac{V_{jk,p}^{s1}}{a_p} \right) {}^0G_{kl}^{sf} + {}^0G_{ij}^{s1} \left(\frac{V_{jk,p}^{sf}}{a_p} \right) {}^0G_{kl}^{f2} + {}^0G_{ij}^{sf} \left(\frac{V_{jk,p}^{sf}}{a_p} \right) {}^0G_{kl}^{sf} + {}^0G_{ij}^{sf} \left(\frac{V_{jk,p}^{f2}}{a_p} \right) {}^0G_{kl}^{f2} \right),$$

$$\frac{\delta G_{il}^{f2}}{a_p} = \sum_{i,j,k,l=x,y,z} \left({}^0G_{ij}^{sf} \left(\frac{V_{jk,p}^{f1}}{a_p} \right) {}^0G_{kl}^{sf} + {}^0G_{ij}^{sf} \left(\frac{V_{jk,p}^{sf}}{a_p} \right) {}^0G_{kl}^{f2} + {}^0G_{ij}^{f2} \left(\frac{V_{jk,p}^{sf}}{a_p} \right) {}^0G_{kl}^{sf} + {}^0G_{ij}^{f2} \left(\frac{V_{jk,p}^{f2}}{a_p} \right) {}^0G_{kl}^{f2} \right).$$

where a_p indicate the normalized model perturbations.

P-SV Fréchet Derivatives

There are two kinds of compressional waves when seismic wave propagating in poroelastic media: the fast compressional wave and slow compressional wave. The following figure shows the scattering scheme in poroelastic media.

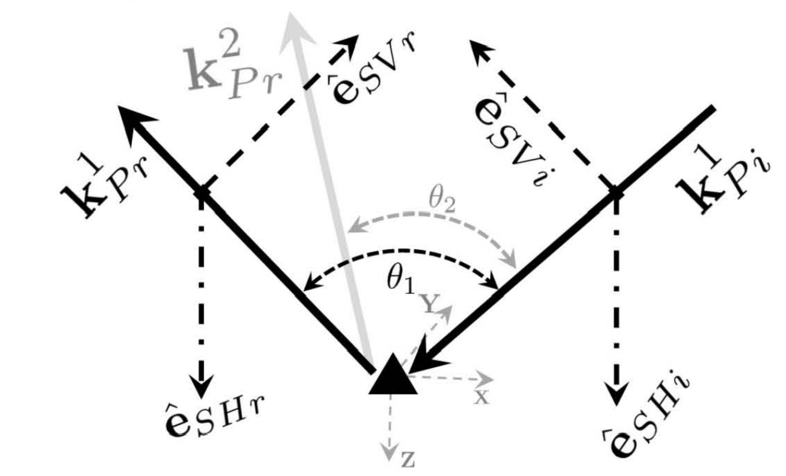

Fig.1 Scattering scheme in poroelastic medium.

Similarly, we can get the P-SV Fréchet Derivatives according to the P-SV governing equations. Here, I show the P-SV scattering potential and Fréchet derivatives for parameter f:

$$\boldsymbol{\mathcal{V}}_f^{PSV} = \begin{pmatrix} \partial_z f^0 a_f \partial_z & -\omega p f^0 a_f \partial_z & 0 & 0 \\ f^0 a_f \omega p \partial_z & -p^2 \omega^2 f^0 a_f & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\delta \boldsymbol{U}^{PSV}}{a_f} = -\int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV} \left(\frac{\delta \boldsymbol{\mathcal{L}}_f^{PSV}}{a_f} \boldsymbol{U}_0^{PSV} \right) dz$$

$$\frac{\delta \mathbf{U}_{ij}}{a_f} = -f^0 \partial_z \mathbf{U}_{sz}^0 G_{ij}^{sz} \partial_z + f^0 \omega p \mathbf{U}_{sr}^0 \partial_z G_{ij}^{sz} + f^0 \omega p \partial_z G_{ij}^{sr} \mathbf{U}_{sz}^0 - f^0 \omega^2 p^2 \mathbf{U}_{sr}^0 G_{ij}^{sr},$$

The scattering potentials and Fréchet Derivatives for other parameters can be found in the research report.

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