Partial wave analysis of seismic wave scattering

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Introduction

We study the partial wave analysis of elastic wave scattering in an isotropic radially heterogenous medium in the context of Born-approximation. We show that in the presence of scatterer there is a phase shift in the outgoing scattered spherical elastic wave. We also obtain the scattering amplitudes for scattering of P- and S-wave in terms of phase shift for P-, SV- and SH-waves. We show that the phase shifts can be calculated using the Lippman-Schwinger integral equation. Lippmann-Schwinger equation for scattered wave which in it's asymptotic behaviour

$$\psi_s(\mathbf{r}) \sim e^{ikz} + f(\theta)\frac{e^{ikr}}{r}. \tag{1}$$

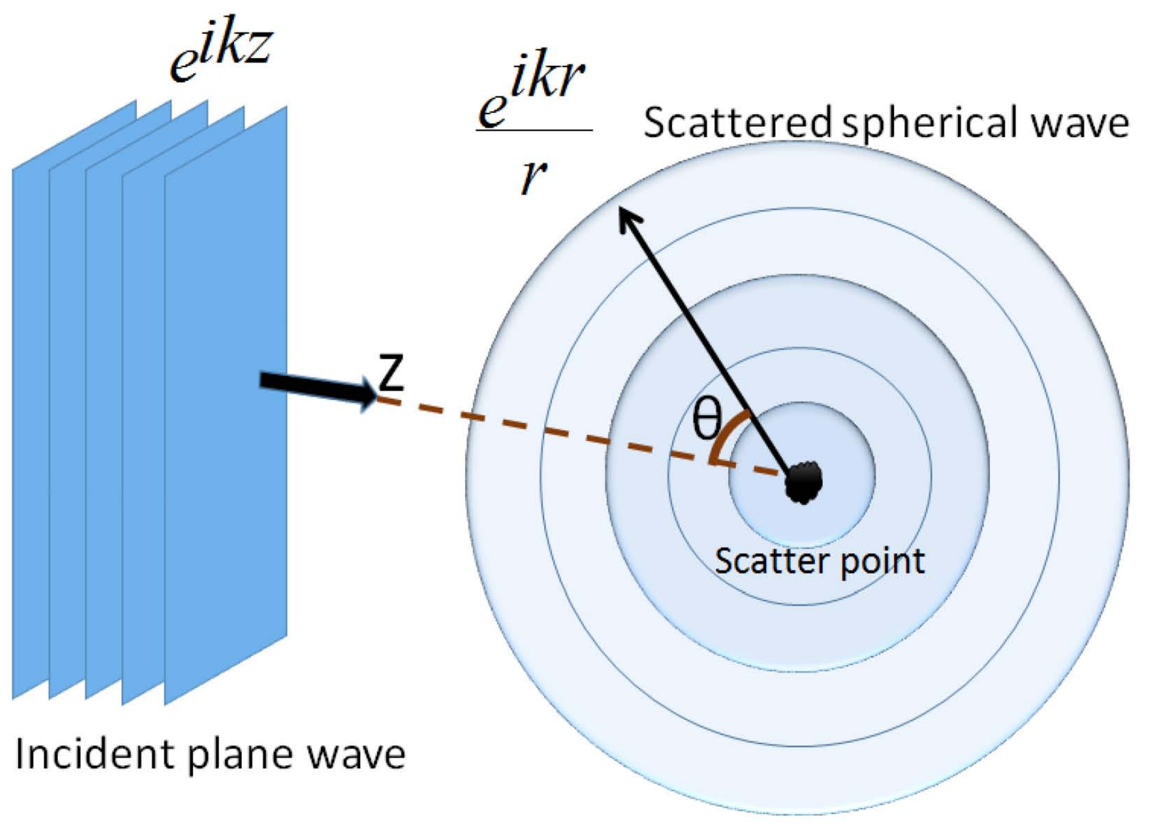

Figure 1: Scattering from a localized Scatterer. Incident plane wave after scattering changes to a spherical wave with an angle dependent distribution called scattering amplitude which modulates the outgoing wave according to direction. It carries all the physics information.

One-dimensional scalar wave scattering

The scalar wave equation in one-dimension

$$\phi''(x) + \omega^2 c^{-2}(x)\phi(x) = 0. \tag{2}$$

The essential assumption in perturbation theory is definition of reference medium with a constant velocity c_0 and a actual medium withe spacial dependent velocity $c(x)$. The relationship between the velocity in actual and reference medium is expressed by

$$c(x) = c_0 + \delta c(x) = c_0(1 + \xi(x)), \tag{3}$$

where $|\xi(x)| \ll 1$ is the fractional velocity. Wavefield in the reference medium is

$$\phi_0(x) = \sin(kx) = \frac{1}{2i}(e^{ikx} - e^{-ikx}). \tag{4}$$

In perturbed medium

$$\phi_p(x) = \frac{1}{2i}\left\{e^{2i\delta_k}e^{ikx} - e^{-ikx}\right\}, \tag{5}$$

where δ_k called phase shift and is related to the perturbation in the medium, and scattered wave

$$\phi_{sc}(x) = \phi_p(x) - \phi_0(x) = e^{i\delta_k}\sin\delta_k e^{ikx}. \tag{6}$$

For the anelastic scattering the total wave field is given by [1]

$$\phi_p^a(x) = \frac{1}{2i}\left\{\gamma_k e^{2i\delta_k}e^{ikx} - e^{-ikx}\right\}, \tag{7}$$

where γ_k is a real number such that $0 < \gamma_k < 1$ and scattered wave field

$$\phi_{sc}^a(x) = \frac{1}{2}e^{i\delta_k}(\gamma_k + 1)\sin(kx + \delta_k) + \frac{1}{2i}e^{i\delta_k}(\gamma_k - 1)\cos(kx + \delta_k). \tag{8}$$

One-dimensional scalar wave scattering continued

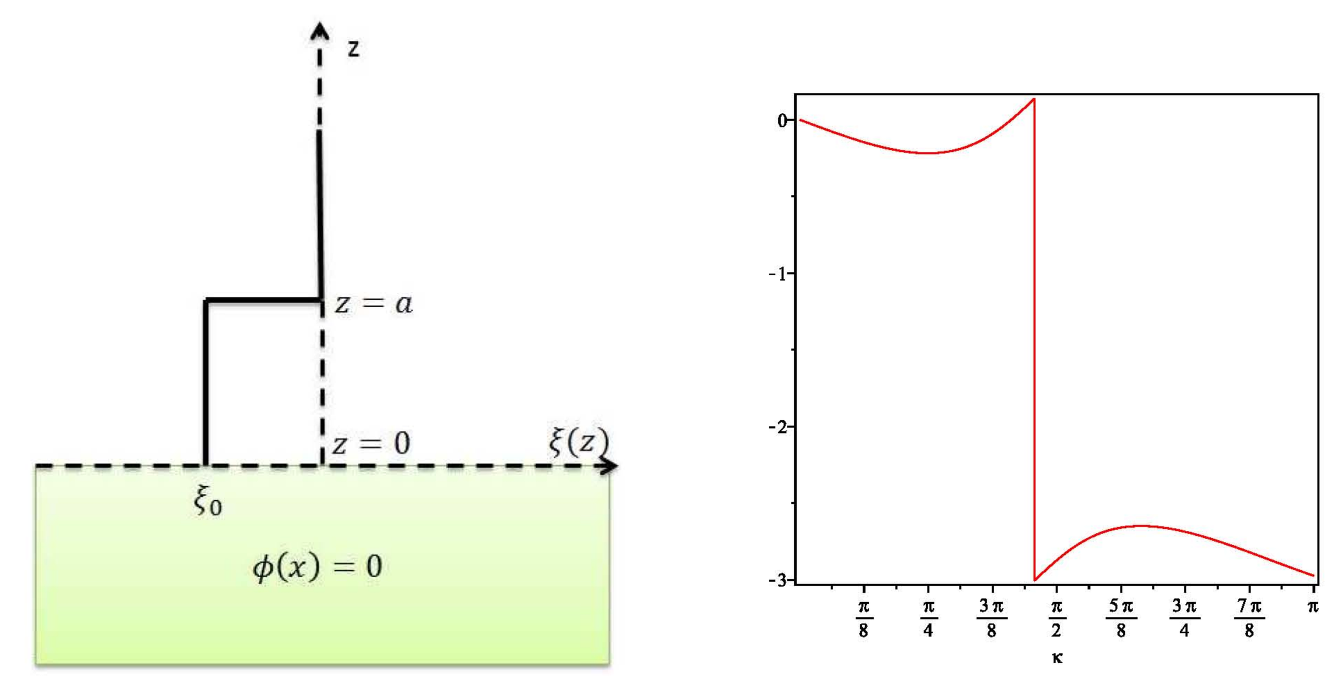

Figure 2: Model representing an actual medium in which the velocity is higher than the reference medium in the perturbation interval $0 < x < a$(left). Diagram illustrating the phase shift as a function of $\kappa = ka$ for perturbation factor $\xi_0 = 0.1$(right).

Using the continuity of wave function and it's first derivative we obtain the phase shift as

$$\delta_k = \tan^{-1}\left[\frac{\tan(ka\sqrt{1+2\xi_0})}{\sqrt{1+2\xi_0}}\right] - ka. \tag{9}$$

In fig.2(right), we can see that the shape of potential and phase shift are almost the same.

Elastic wave scattering

Incident P-wave with the polarization in the z-direction,

$$\mathbf{P} = \mathbf{z}e^{ikz} = \sum_l i^{l-1}(2l+1)\mathbf{L}_{l0}. \tag{10}$$

Incident S-wave wave can be either $\mathbf{x}e^{ikz}$ or $\mathbf{y}e^{ikz}$, which can be written as a superposition of *SH* and *SV* waves

$$\mathbf{SV} = \sum_l i^{l-1}\frac{2l+1}{2l(l+1)}\left[\mathbf{N}_{l1} - l(l+1)\mathbf{N}_{l,-1}\right], \tag{11}$$

$$\mathbf{SH} = \sum_l i^{l-1}\frac{2l+1}{2l(l+1)}\left[\mathbf{M}_{l1} + l(l+1)\mathbf{M}_{l,-1}\right]. \tag{12}$$

Where $\mathbf{L}_{lm}$, $\mathbf{N}_{lm}$ and $\mathbf{M}_{lm}$ are the Hanson vectors which are the solutions of the vector Helmholtz equation [2]. In the asymptotic region far from the scatterer

$$\mathbf{P} \approx \frac{1}{2ik_P r}\sum_l\left[e^{ik_P r} + (-1)^l e^{-ik_P r}\right]P_l\hat{\mathbf{r}}.$$

$$\mathbf{SH} \approx \frac{1}{ik_S r}\sum_l\frac{2l+1}{l(l+1)}\left[i^{-l}e^{ik_S r} - i^l e^{-ik_S r}\right]\left(\cos\varphi\frac{P_{l1}}{\sin\theta}\hat{\theta} - \sin\varphi\frac{dP_{l1}}{d\theta}\hat{\varphi}\right),$$

$$\mathbf{SV} \approx \frac{1}{ik_S r}\sum_l\frac{2l+1}{l(l+1)}\left[i^{-l}e^{ik_S r} + i^l e^{-ik_S r}\right]\left(\cos\varphi\frac{dP_{l1}}{d\theta}\hat{\theta} - \sin\varphi\frac{P_{l1}}{\sin\theta}\hat{\varphi}\right).$$

Where P_{lm} is Legender polynomial. If the initial wave is P-wave, the total wave field after scattering is a superposition of initial P-wave, scattered P, and S-waves

$$\mathbf{U} = \sum_l A_l^P\mathbf{L}_{l0} + B_l^P\mathbf{L}_{l0} + C_l^S\mathbf{N}_{l0}. \tag{13}$$

In asymptotic region the wave field has the following form

$$\mathbf{U} = \frac{1}{k_P r}\sum_l i^{l-1}\cos\left(k_P r - \frac{l\pi}{2}\right)P_l(\theta)\hat{\mathbf{r}} + \frac{e^{ik_P r}}{r}\sum_l\mathcal{F}_l^{PP}(\theta)\hat{\mathbf{r}} + \frac{e^{ik_S r}}{r}\sum_l\mathcal{F}_l^{PS}(\theta)\hat{\theta} \tag{14}$$

Where the first term is the initial P-wave in the z-direction, the second term is the scattered PP-wave and the third term is scattered PS-wave and $\mathcal{F}_l^{PS}(\theta)$ are the scattering patterns for PP and PS-wave modes. Inserting the asymptotic forms of *P* and *SV* waves in (13) and comparing

Elastic wave scattering: continued

to (14)

$$\mathcal{F}_l^{PP}(\theta) = \frac{P_l(\theta)}{k_P}e^{i\delta_l^P}\sin\delta_l^P, \tag{15}$$

$$\mathcal{F}_l^{PS}(\theta) = i^l\frac{C_l^S}{k_S}\frac{dP_l(\theta)}{d\theta}. \tag{16}$$

where δ_l^P, the phase shift for scattered P-wave, given by

$$\tan\delta_l^P = \frac{B_l^P}{A_l^P} \tag{17}$$

If the phase shift goes to zero, we don't have the scattered P-wave, namely the perturbation in the medium appears as a phase shift in scattering potential. The scattered wave satisfies in

$$\mu_0\nabla^2\mathbf{U}^s + (\lambda_0 + \mu_0)\nabla\nabla\cdot\mathbf{U}^s + \rho_0\omega^2\mathbf{U}^s + \mathbf{V}^B(\mathbf{U}^i) = 0 \tag{18}$$

Where Born vector potential is

$$\mathbf{V}^B = \delta\mu\nabla^2\mathbf{U}^i + (\delta\lambda + \delta\mu)\nabla\nabla\cdot\mathbf{U}^i + \delta\rho\omega^2\mathbf{U}^i - \hat{\mathbf{r}}\delta\lambda\frac{d}{dr}\nabla\cdot\mathbf{U}^i - \delta\mu\frac{d}{dr}\left(2\frac{\partial\mathbf{U}^i}{\partial r} + \hat{\mathbf{r}}\times\nabla\times\mathbf{U}^i\right). \tag{19}$$

Lippman-Schwinger integral equation which expresses the scattered wave field in terms of retarded green function and Born potential term is given by

$$\mathbf{U}^s = -\int d\Omega'\int r'^2dr'\mathbf{V}^B(\mathbf{r}')\cdot\mathrm{G}_>(\mathbf{r},\mathbf{r}'), \tag{20}$$

Using the above equation we can determine

$$e^{i\delta_l^P}\sin\delta_l^P = \frac{4\pi k_p}{2l+1}\int r'^2dr'\left[g_{rr}(r')v_r(r') + l(l+1)g_{\theta r}(r')v_\theta(r')\right], \tag{21}$$

$$C_l^S = \frac{4\pi k_S i^{-l}}{2l+1}\int r'^2dr'\left[g_{r\theta}(r')v_r(r') + l(l+1)g_{\theta\theta}(r')v_\theta(r')\right]. \tag{22}$$

Where, *g* and *v* are the radial components of dyadic green functions and vector potentials.

Conclusions

In born approximation, scattered wave field is spherical wave function undergoes a phase shift comparing to the case that perturbation is absence. This is the main idea of the Partial wave analysis. The analysis for scalar waves(quantum problems) and electromagnetic waves are well studied. However for elastic wave, the case is investigated rarely specially the phase shift interpretation of the scattering. In this research we applied the conventional partial wave analysis used in quantum theory and electrodynamics to the elastic wave scattering. We demonstrate that the scattering amplitude can be be expressed by one unknown parameter called phase shift obtained by the Lippmann-Schwinger equation.

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