Reflection/transmission coefficients in a viscoelastic medium: linearized and exact forms

Shahpoor Moradi* and Kris Innanen

*moradis@ucalgary.ca

Introduction

An explicit analytical form of the scattering matrix for Homogeneous Isotropic Linear Viscoelastic (HILV) continuum is obtained. The reflection and transmission are the complex function related to the P- and S-velocities and corresponding quality factors Q_P and Q_S. We also consider the problem of scattering of homogeneous and inhomogeneous waves from perturbations in five viscoelastic parameters (density, P- and S-wave velocities, and P- and S-wave quality factors), as formulated in the context of the Born approximation. In elastic media the scattering potential is real, but if dissipation is included through a viscoelastic model, the potential becomes complex and thus impacts the amplitude and phase of the outgoing wave.

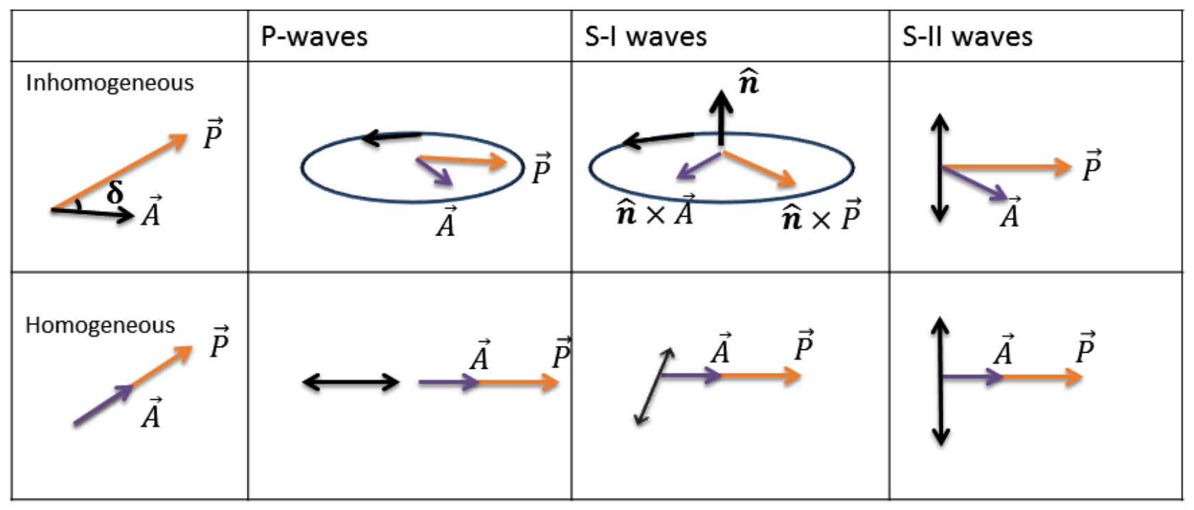

Figure 1: Diagram illustrating the particle motion orbit(polarization) of waves in a viscoelastic media. **P** and **A** respectively are propagation and attenuation vectors and δ refers to the attenuation angle.

Exact form of scattering matrix

Incident inhomogeneous P-wave

$$^{\downarrow}\mathbf{U}_p^i = \Phi_1{}^{\downarrow}(\xi_{1px}\mathbf{x} + \xi_{1pz}\mathbf{z}) \exp\left[i\omega\left(\mathbb{K}x + \frac{\xi_{1pz}}{\alpha_1}z\right)\right], \quad (1)$$

Reflected inhomogeneous P-wave

$$^{\uparrow}\mathbf{U}_p^r = \Phi_1{}^{\uparrow}(\xi_{1px}\mathbf{x} - \xi_{1pz}\mathbf{z}) \exp\left[i\omega\left(\mathbb{K}x - \frac{\xi_{1pz}}{\alpha_1}z\right)\right], \quad (2)$$

Transmitted inhomogeneous P-wave

$$^{\downarrow}\mathbf{U}_p^t = \Phi_2^{\downarrow}(\xi_{2px}\mathbf{x} + \xi_{2pz}\mathbf{z}) \exp\left[i\omega\left(\mathbb{K}x + \frac{\xi_{2pz}}{\alpha_2}z\right)\right]. \quad (3)$$

where we have defined the x- and z-components of polarization vectors as

$$\xi_x = \sin\theta + \frac{i}{2}Q^{-1}\tan\delta\cos\theta, \quad (4)$$

$$\xi_z = \cos\theta - \frac{i}{2}Q^{-1}\tan\delta\sin\theta. \quad (5)$$

In addition complex ray parameter $\mathbb{K}$ is given by

$$\omega\mathbb{K} = |\mathbf{P}|\sin\theta - i|\mathbf{A}|\sin(\theta-\delta), \quad (6)$$

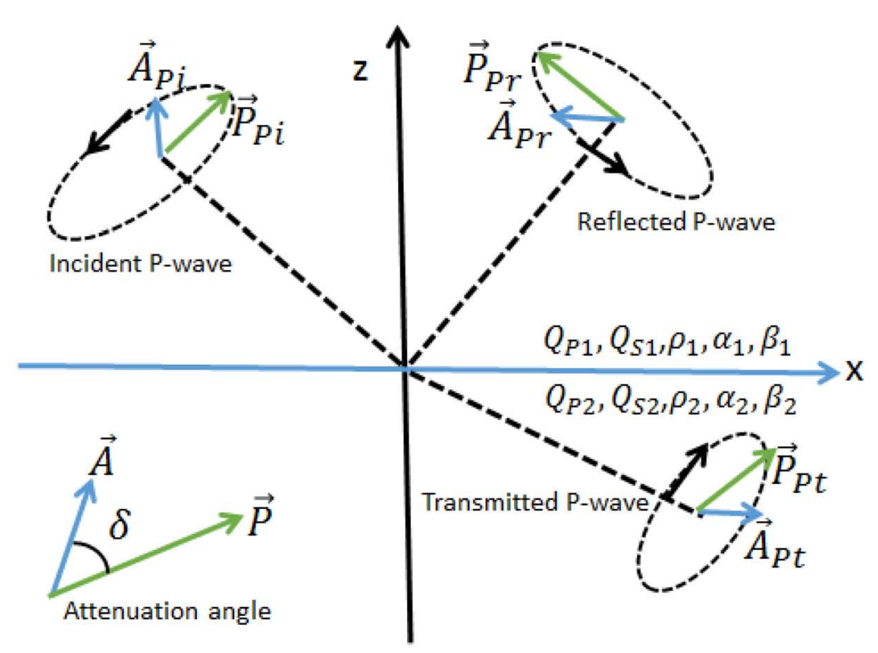

Figure 2: Diagram illustrating the scattering of inhomogeneous P-wave to inhomogeneous P-wave.

Exact form of scattering matrix continued

Accordingly the scattering matrix represents the all reflection transmission-coefficients is given by

$$S_{P,SI} = \begin{pmatrix} ^{\downarrow}PP^{\uparrow} & ^{\downarrow}SIP^{\uparrow} & ^{\uparrow}PP^{\uparrow} & ^{\downarrow}SIP^{\uparrow} \\ ^{\downarrow}PSI^{\uparrow} & ^{\downarrow}SISI^{\uparrow} & ^{\downarrow}SISI^{\uparrow} & ^{\uparrow}SISI^{\uparrow} \\ ^{\downarrow}PP^{\downarrow} & ^{\downarrow}SIP^{\downarrow} & ^{\uparrow}PP^{\downarrow} & ^{\uparrow}SIP^{\downarrow} \\ ^{\downarrow}PSI^{\downarrow} & ^{\downarrow}SISI^{\downarrow} & ^{\uparrow}PSI^{\downarrow} & ^{\uparrow}SISI^{\downarrow} \end{pmatrix} \quad (7)$$

The PP-mode is given by

$$^{\downarrow}PP^{\uparrow} = (^{\downarrow}PP^{\uparrow})_{elastic} + i(^{\downarrow}PP^{\uparrow})_{anelastic} = A\exp(i\Phi) \quad (8)$$

Amplitude and phase of reflected wave are

$$A = \sqrt{|(^{\downarrow}PP^{\uparrow})_{elastic}|^2 + |(^{\downarrow}PP^{\uparrow})_{anelastic}|^2}, \quad (9)$$

$$\Phi = \arctan\left\{\frac{(^{\downarrow}PP^{\uparrow})_{anelastic}}{(^{\downarrow}PP^{\uparrow})_{elastic}}\right\} \quad (10)$$

Linearized scattering matrix

The scattering potential in displacement space can be written as

$$^{R}_{I}\mathcal{V}_{ve} = \boldsymbol{\xi}_I^T V_{ve}\boldsymbol{\xi}_R. \quad (11)$$

Here $\boldsymbol{\xi}$ is the polarization vector, V_{ve} is the scattering operator and subscripts I and R respectively refer to the incident and reflected waves. The next task is to evaluate the scattering matrix in a system which naturally describes Borcherdt's viscoelastic modes $\mathrm{P}, \mathrm{SI}, \mathrm{SII}$, namely

$$\mathcal{V}_{ve} = \begin{pmatrix} ^{P}_{P}\mathcal{V}_{ve} & 0 & ^{P}_{SI}\mathcal{V}_{ve} \\ 0 & ^{SII}_{SII}\mathcal{V}_{ve} & 0 \\ ^{SI}_{P}\mathcal{V}_{ve} & 0 & ^{SI}_{SI}\mathcal{V}_{ve} \end{pmatrix}. \quad (12)$$

The scattering potential for PP mode is

$$^{P}_{P}\mathbb{V}_{ve} = {}^{P}_{P}\mathbb{V}_e + i{}^{P}_{P}\mathbb{V}_{ane} \quad (13)$$

where elastic scattering potential $\mathbb{V}_e^{PP}$ is given by

$$^{P}_{P}\mathbb{V}_e = \left(-1 + \cos\theta + \frac{2}{\gamma_0^2}\sin^2\theta\right)A_\rho + \left(\frac{4}{\gamma_0^2}\sin^2\theta\right)A_\beta - 2A_\alpha \quad (14)$$

and anelastic part of the scattering

$$^{P}_{P}\mathbb{V}_{ane} = {}^{P}_{P}\mathbb{V}^{\rho}_{ane}A_\rho + {}^{P}_{P}\mathbb{V}^{\beta}_{ane}A_\beta + {}^{P}_{P}\mathbb{V}^{Q_{hs}}_{ane}A_{Q_{hs}} + {}^{P}_{P}\mathbb{V}^{Q_{hp}}_{ane}A_{Q_{hp}} \quad (15)$$

with

$$^{P}_{P}\mathbb{V}^{\rho}_{ane} = \frac{1}{\gamma_0^2}\left\{-2\sin^2\theta(Q^{-1}_{hp_0} - Q^{-1}_{hs_0}) + \frac{Q^{-1}_{hp_0}}{2}(\sin\theta - \frac{2}{\gamma_0^2}\sin 2\theta)(\tan\delta_p^r - \tan\delta_p^i)\right\} \quad (16)$$

$$^{P}_{P}\mathbb{V}^{\beta}_{ane} = -\frac{2}{\gamma_0^2}\left\{2\sin^2\theta(Q^{-1}_{hp_0} - Q^{-1}_{hs_0}) + Q^{-1}_{hp_0}\sin 2\theta(\tan\delta_p^r - \tan\delta_p^i)\right\} \quad (17)$$

$$^{P}_{P}\mathbb{V}^{Q_{hs}}_{ane} = \frac{2Q^{-1}_{hp_0}}{\gamma_0^2}\sin^2\theta \quad (18)$$

$$^{P}_{P}\mathbb{V}^{Q_{hp}}_{ane}A_{Q_{hp}} = Q^{-1}_{hp_0} \quad (19)$$

The imaginary part is the term induced by the anelasticity of the medium. The scattering potential for P to SI is

$$^{P}_{SI}\mathbb{V}_{ve} = {}^{P}_{SI}\mathbb{V}_e + i{}^{P}_{SI}\mathbb{V}_{ane} \quad (20)$$

where the elastic part of the scattering potential $\mathbb{V}_e^{PSI}$ is given by

$$^{P}_{SI}\mathbb{V}_e = \left(\sin\theta - \frac{1}{\gamma_0}\sin 2\theta\right)A_\rho - \left(\frac{2}{\gamma_0}\sin 2\theta\right)A_\beta \quad (21)$$

Linearized scattering matrix continued

and anelastic part is given by

$$^{P}_{SI}\mathbb{V}_{ane} = {}^{P}_{SI}\mathbb{V}^{\rho}A_\rho + {}^{P}_{SI}\mathbb{V}^{\beta}A_\beta + {}^{P}_{SI}\mathbb{V}^{Q_{hs}}A_{Q_{hs}} \quad (22)$$

with

$$^{P}_{SI}\mathbb{V}^{\rho} = \frac{1}{\gamma_0}\left\{\sin 2\theta(Q^{-1}_{hp_0} - Q^{-1}_{hs_0}) + (\cos 2\theta - \frac{\gamma_0}{2}\cos\theta)(Q^{-1}_{hs_0}\tan\delta_s^r - Q^{-1}_{hp_0}\tan\delta_p^i)\right\} \quad (23)$$

$$^{P}_{SI}\mathbb{V}^{\beta} = \frac{2}{\gamma_0}\left\{\sin 2\theta(Q^{-1}_{hp_0} - Q^{-1}_{hs_0}) + \cos 2\theta(Q^{-1}_{hs_0}\tan\delta_s^r - Q^{-1}_{hp_0}\tan\delta_p^i)\right\} \quad (24)$$

$$^{P}_{SI}\mathbb{V}^{Q_{hs}} = \frac{Q^{-1}_{hs_0}}{\gamma_0}\sin 2\theta. \quad (25)$$

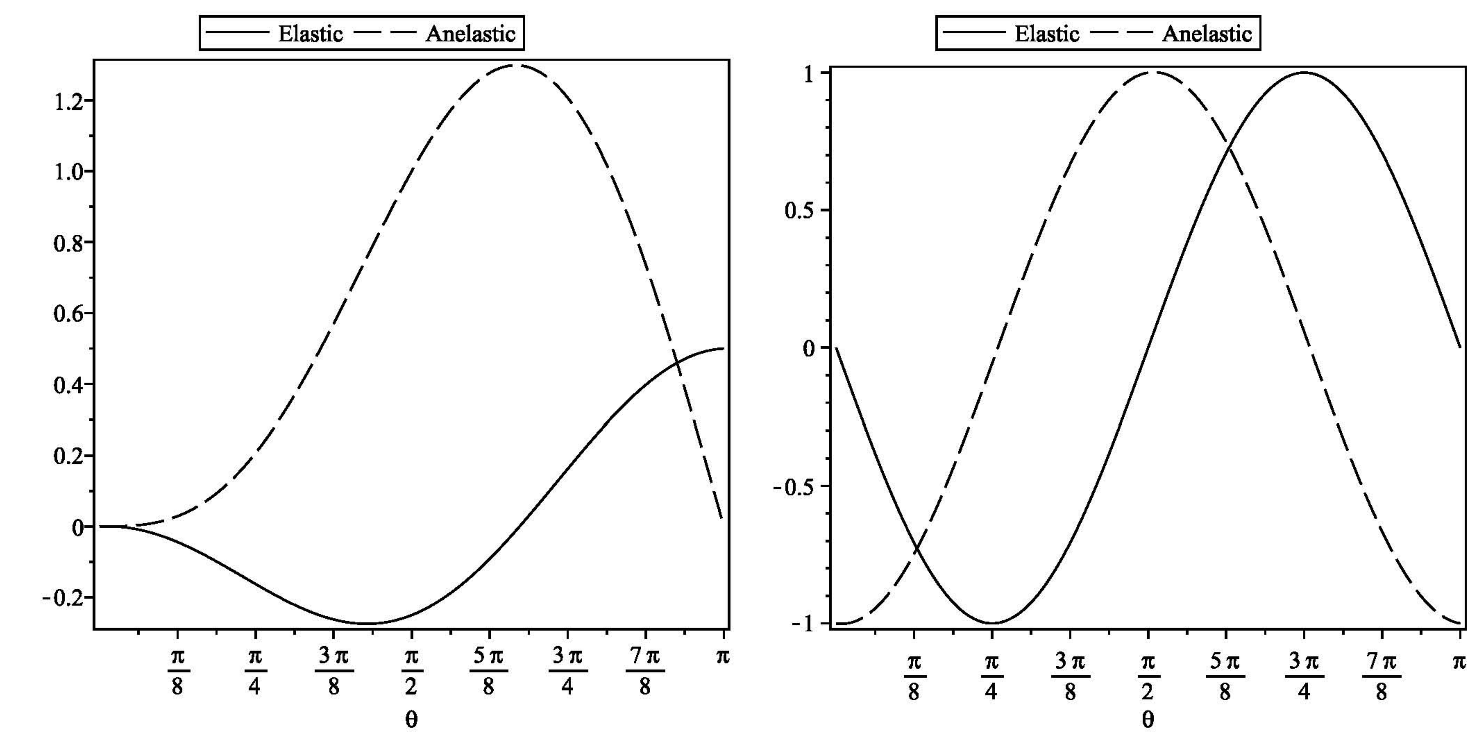

Figure 3: Elastic and anelastic density(left) and S-velocity(right) components of the viscoelastic potential for scattering of incident homogeneous P-wave to inhomogeneous reflected SI-wave versus of reflected wave angle θ, for $\delta_r = \frac{\pi}{3}$. Quality factor of P-wave for reference medium is to be 5 and for S-wave is 7. Also the S-to P-velocity ratio for reference medium is chosen to be $1/2$.

Conclusions

The scattering potential in displacement space is obtained by sandwiching the scattering operator between the incident and reflected polarization vectors. Since for the viscoelastic waves, polarizations are complex, the viscoelastic scattering potential we obtained is a complex function whose real part is elastic scattering potential and whose imaginary part is the related to the anelasticity of the medium. In contrast to the elastic scattering potential that only alters the amplitude of the outgoing field, the viscoelastic scattering potential alters both amplitude and phase of the outgoing field. Anelasticity appears to have more significant effect on converted waves than on conserved modes.

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Bibliography

Please see the reports for references.