Wave propagation in elastic, linear, homogeneous (inhomogeneous) and anisotropic media: A proposal

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ABSTRACT

This paper constitutes, in essence, the Ph.D. thesis proposal of the first author, the research is to be carried out under the supervision of the second author. It is proposed to study certain aspects of wave propagation in elastic, linear, homogeneous (inhomogeneous), and anisotropic media. This formidable subject will be broken into three basic interrelated parts; these are: basic concepts, forward modeling, and inversion. Some areas of research interest in these parts are: the inclusion of the rotational term in the constitutive relations; determinations of macroscopic anisotropic discriptions from preferentially oriented microscopic heterogeneities; the elastodynamic Green's function in anisotropic media and/or its approximations; the statistical approach to determine elastic constants (sparse-matrix techniques); the determination of axes of symmetry by use of the Maxwell multipole representation; the use of (τ-p) (plane-wave) methods in modeling and inversion, the effects of singularities on the quasi-shear velocity sheet, the extension of point-source decomposition to anisotropic media, and applying the topics above in an inversion scheme. A brief review of work already done and possible areas of further research of the topics above will be given. Not all these points will be brought to fruition, but at this point it would be unwise to limit myself too severely, as I am still surveying this vast subject. Though the area of thesis research is at present concerned mostly with the theoretical aspects of anisotropy, there are many practical aspects of this subject - such as fracture detection, influences on amplitude-verses-offset and general traveltime effects that profoundly effects seismological data - that can also be explored.

INTRODUCTION

The theory of wave propagation in anisotropic media appears to be well documented in certain areas and lacking in others. I will attempt to make a brief survey of the literature with which I (D.T.E.) have familiarized myself.

The basic theory is well developed in a myriad texts (e.g., Aki and Richards, 1980; Hudson, 1980; Achenbach, 1973; Musgrave, 1970; and Fedorov, 1968); the last two mentioned are especially relevant in terms of anisotropy. An overview of more recent works can be found in the paper by Crampin et al. (1984). The basic theoretical developments which will be considered consist of defining the linear constitutive relations, developing the dynamic relationships from these and deriving the resulting expressions for fundamental properties that can be observed (e.g., velocities, polarizations, etc...).

The forward problem is a complex one in general anisotropic media. I have broken this into three general interrelated areas; they are:

Approximation for anisotropic situations
- Elliptical anisotropy (Daley and Hron, 1979)
- Fourier truncations (Crampin, 1981)

Traveltime techniques
- Asymptotic ray theory (Cerveny, 1972)
- Geometric construction (Helbig, 1990)
- Field discontinuities (Vlaar, 1968)
- Riemannian space and geodesics (Eby, 1969)
- Contact transformations (Gassman, 1964; Baker and Copson, 1953)

Wave-equation techniques
- Plane waves - modeling
- reflection, transmission (Keith and Crampin, 1976)
- decomposition (Rommel, 1990)
- Finite difference/element
- Green's functions (Aki and Richards, 1980),
  the form of which has not been determined except for the case of transverse isotropy (Ben Menahem and Sena, 1990) .

Finally, the inverse problem must be addressed. At present the main thrust seems to be in using a forward modeling scheme to generate a data set which in some esthetic sense matches observed data (Campden et al., 1990). This is a widely used methodology for solving the inverse problem, but is somewhat an art as shown by the serendipitous discovery made in the above mentioned paper. There are some mention of techniques (Hake, 1986) to aid in this regard, but by and large this area is still in its infancy. Therefore, this area provides much ground for further exploration. This, of course, can only proceed if the first two subjects are more fully explored. The following gives a more expanded description of areas I have concentrated on.

THEORY

To avoid too lengthy a paper, I will only touch upon some of the more relevant relationships and refer the reader to appropriate literature. Einstein notation will be used throughout this paper.

First, the equation of motion is derived for a volume of material subjected to body and surface forces. With the assumption that all displacements are small the equation of motion will have the form

$$\Sigma_{ij,j} + \rho F_i = \rho \ddot{U}_i \qquad (1)$$

where $\Sigma_{ij} \equiv$ total stress tensor,
$\rho F_i \equiv$ body force per unit volume,
$U_i \equiv$ displacement vector.

Since we are interested in relationship (1) within a medium with an intrinsic internal resistance to deformation and a tendency to return to the original undeformed state; we need a relationship between a description of deformation (strain and rotation) and a description of deforming forces acting on surface elements (stress and surface couple). Note, we have used the terms stress and strain as generally used in the literature, to represent the symmetrical and irrotational components of total strain and total stress respectively. A useful description of deformation is the ratio of change of displacement (δU_i) to difference in position (δx_i) of two points, This will be defined as total strain (Aki and Richards, 1980). Refer also to Figure 1.

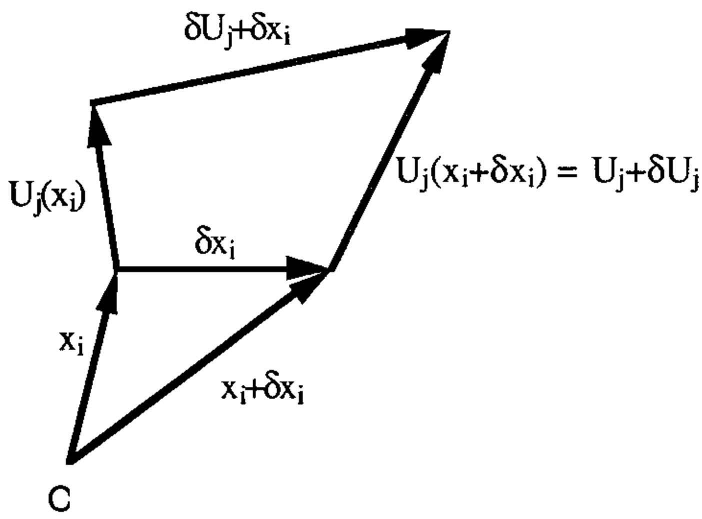

Figure 1. Description of Deformation

This ratio can be written as:

$$\frac{\delta U_j}{\delta x_i} = \mu_{ji} = U_{j,i} = \frac{1}{2}(U_{j,i} + U_{i,j}) + \frac{1}{2}(U_{j,i} - U_{i,j})$$

$$\Rightarrow \mu_{ij} = \varepsilon_{ij} + \omega_{ij} \qquad (2)$$

where

$\varepsilon_{ij} = (U_{i,j} + U_{j,i})/2 \equiv$ the infinitesimal strain tensor, which expresses the deformation of an elemental volume, and is symmetric by definition,

$\omega_{ij} = (U_{i,j} - U_{j,i})/2 \equiv$ the infinitesimal rotation tensor, which is related to the solid body rotation of an elemental volume and is by definition antisymmetric.

In' general, the total stress tensor (Σ_{ij}) will also have a symmetric and anti-symmetric part; these corresponds to a stress (σ_{ij}) and a surface couple (τ_{ij}) respectively (Symon, 1971). This will be written in the form:

$$\Sigma_{ij} = \sigma_{ij} + \tau_{ij} \, . \qquad (3)$$

Assuming that body forces do not add to the deformation of the elemental volume, the most general linear relationship between the forces represented by total stress (Σ_{ij}) and deformation represented by total strain (μ_{kl}) will be:

$$\Sigma_{ij} = A_{ijkl} \, \mu_{kl} \, . \qquad (4a)$$

This relationship can be seen to violate the principle of material frame indifference (Epstein, 1990), but in materials with body moments this is inevitable as will be seen in subsequent developments. Substitution of (3) and (2) into (4) yields

$$\sigma_{ij} + \tau_{ij} = C_{ijkl}\,\varepsilon_{kl} + B_{ijkl}\,\omega_{kl} \qquad (4b)$$

where $C_{ijkl} = A_{ijkl} + A_{ijlk}$

and $B_{ijkl} = A_{ijkl} - A_{ijlk}$,

giving conditions $C_{ijkl} = C_{ijlk}$ and $B_{ijkl} = -B_{ijlk}$.

By noting that C_{ijkl} and B_{ijkl} are symmetrical and antisymmetrical respectively in the indices l and k, one can derive the form of equation (4b).

This is one area of investigation I (D.T.E.) wish to pursue. I would like to find out the difference between the standard development based on the constitutive relation,

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{ij}\,, \qquad (5)$$

and the general relationships (4a and 4b). Of course equations (4a and 4b) will reduce to equation (5) if certain symmetry conditions are enforced; for instance, if there are no body moments (refer to appendix C) then

$$\Sigma_{ij} = \Sigma_{ji} = \sigma_{ij} \qquad (6)$$

which implies

$$A_{ijkl} = A_{jikl}, \qquad (7)$$

and if there exists a strain-energy function Φ defined by

$$\Sigma_{ij} = \frac{\partial \Phi}{\partial \mu_{ij}}, \qquad (8)$$

then

$$\frac{\partial^2 \Phi}{\partial \mu_{ij}\,\partial \mu_{kl}} = A_{ijkl}, \qquad (9)$$

which implies

$$A_{ijkl} = A_{klij}. \qquad (10)$$

With both (7) and (10) satisfied the most general constitutive relation will be given by equation (5). These results are summarized in the flow chart of Figure 2. The simple case of isotropic material has been looked at cursorily. By using standard analytical techniques for body waves (Bullen, 1965) I was able to show that under conditions appropriate for equations (4a and 4b) the P-wave equation remains the same whereas the S-wave equations are modified in such a way that the velocity includes a factor introduced by the rotational term (Appendix A).This will still be true if we have conditions represented by Box 3 in Figure 2.

By substitution of the constitutive relation (4a) into the equation of motion (1) we get the displacement equation:

$$A_{ijkl}U_{k,lj} - \rho\ddot{U}_i = 0 \qquad (11)$$

where I have ignored body forces; of course, if the medium is inhomogeneous equation (11) would be of the form:

$$(A_{ijkl}U_{k,l})_{,j} - \rho\ddot{U}_i = 0\ . \qquad (12)$$

PLANE WAVES

Substitution of trial plane-wave solutions of the form

$$U_k = \alpha p_k e^{i\omega(s_r x_r - t)} = \alpha p_k e^{i(k_r x_r - \omega t)} \qquad (13)$$

(where $\alpha \equiv$ amplitude scalar,
$p_k \equiv$ unit particle motion vector,
$\omega \equiv$ angular frequency,
$s_r \equiv$ slowness vector,
$x_r \equiv$ space position vector,
$t \equiv$ time scalar,
and $k_r \equiv$ propagation (wave-number) vector)

in equation (11) results in the following condition, for a plane wave solution to exist,

$$(A_{ijkl}s_j s_l - \rho\delta_{ik})p_k = 0. \qquad (14)$$

Equivalently, one may write:

$$(A_{ijkl}n_j n_l - \rho v^2\delta_{ik})p_k = 0, \qquad (15)$$

where $v \equiv$ phase velocity,
and $s_i = n_i/v$.

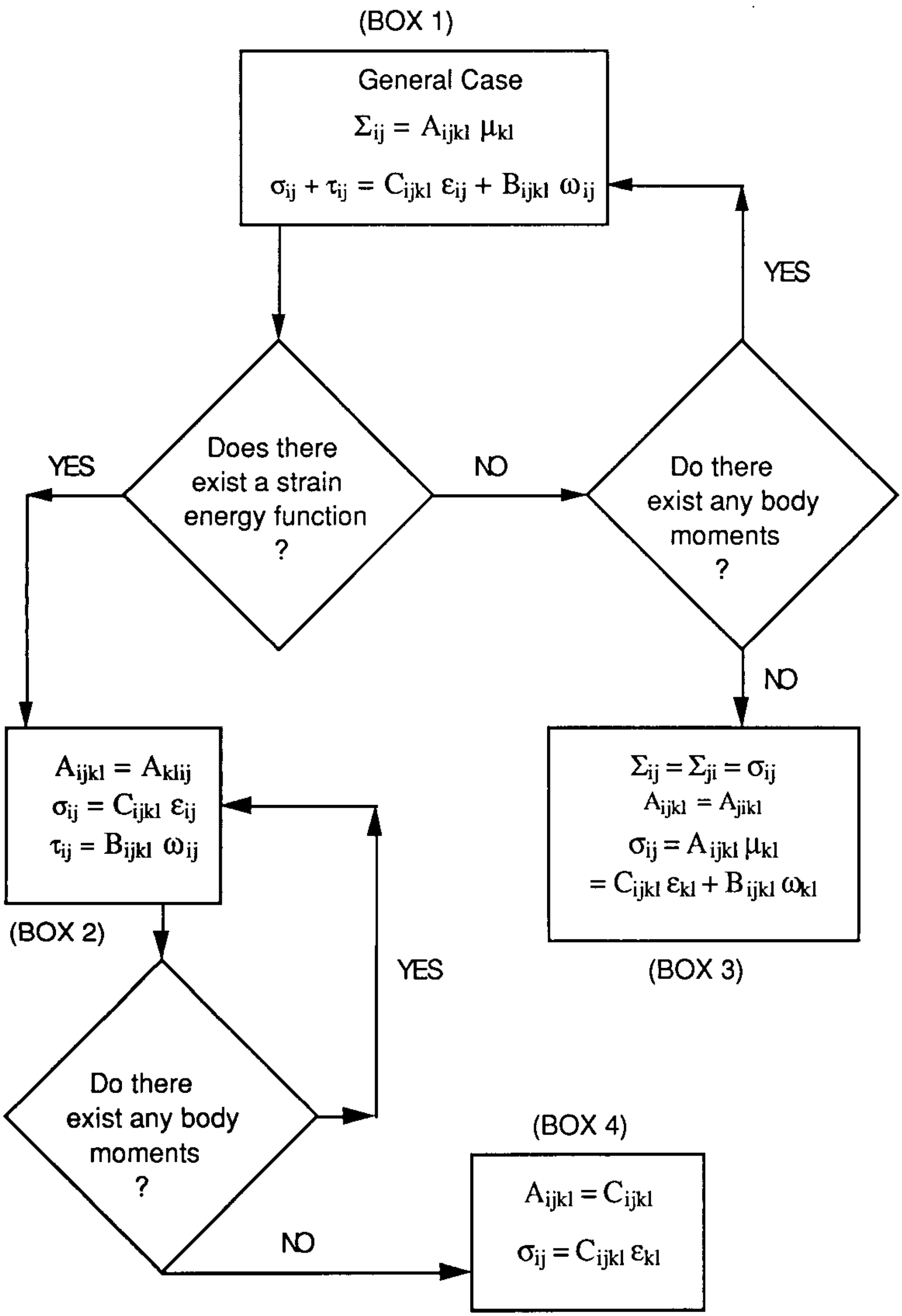

Figure 2. Flow Chart of Constitutive Relationships

The only way for either equations (14) or (15) to have a non-trivial solution for p_k is to have the determinant of the bracketed terms equal to zero. This will give the equations of both the slowness and velocity surfaces. Substituting back the velocity will then yield the particle-motion polarization vector p_k .

When the slower and faster velocity sheets of the quasi-S waves meet, there can exist singularities in terms of curvature of these sheets; this will produce fast variations on the propagating wave surfaces (which propagate normal to the slowness surface, see Appendix D), or group-velocity sheets (eg. cusps and holes) (Crampin and Yedlin, 1981). This provides another area which has possibilities for further research, since present studies are concentrated in ray modeling and graphical presentations, other techniques may be brought to bear on this problem.

MACRO-ANISOTROPY

An assumption that is generally made (Crampin, 1981) is that "no analytical distinction can be made between behavior of what might be called inherent anisotropy such as aligned crystals ... and oriented two-phase materials when the seismic wavelengths are sufficiently large ..." . I (D.T.E.) feel that in geological situations mixtures of multiphase and multicomponent materials are not restricted to standard symmetries in the same way as crystals are by space filling and point group constraints, and may exhibit more general forms of anisotropic symmetries in a statistical sense. It may be possible to generate anisotropic macroscopic descriptions which describe waves propagating in random heterogeneous material with statistical alignments. A sparse matrix technique may then be employed to determine simple symmetrical representations of the constitutive relations in such materials. This type of description has been made in materials with fine layering. The resulting long-wavelenth description turns out to be transversely isotropic (Backus, 1962). Though this is a satisfying analysis I find it difficult to generalize directly to the general case mentioned above. Another approach is to treat the wavefield ψ as propagating through a random media then try to describe how the expected value $\langle\psi\rangle$ and relevant moments such as $\langle\psi(\vec{r}_1)\psi(\vec{r}_2)\rangle$ propagates through the medium (Uscinski, 1977). A finite-difference or finite-element modeling technique may also be employed to determine viability for further analysis.

ELASTIC CONSTANTS AND VECTOR BOUQUETS

From a seismogram one can only easily deduce the traveltime surface $\Gamma(\hat{n})$ from which one could calculate the group velocity V and if observations were dense enough the direction $\hat{n}$ of the group velocity. With this information in hand it is the possible to obtain the phase velocity $v(\hat{k})$ either through analytic techniques (Synge, 1957) or geometric arguments (Helbig, 1990). With enough phase-velocity determinations one can then use deterministic techniques (Backus, 1970) to determine the elastic constants or, in the case of inaccuracies giving rise to inconsistent equations, an optimizing algorithm is preferable. Even if there are no errors in one's velocities the possibility of not being oriented along axes of symmetry is great; this will give rise to an elastic tensor with up to twenty one non-zero coefficients. This makes it difficult to determine which symmetry system may be represented. One method which allows one to determine this uses Maxwell multipole representation of the elastic tensor. The multipoles can be seen as unique vector bouquets which by direct inspection give the symmetry axes and symmetry group represented by the particular elastic tensor (Backus, 1970). With errors in the velocities the derived elastic tensor $\tilde{C}_{ijkl}$ may always tend towards a triclinic system. To counter this problem one could use the vector bouquets of the symmetry systems as templates, and rotate them until the difference between the vector bouquets of $\tilde{C}_{ijkl}$ and the template is minimized. Then one constructs the tensor $\varepsilon^{\alpha}_{ijkl}$ from the template tensor C^{α}_{ijkl} as follows:

$$\varepsilon^{\alpha}_{ijkl}= \tilde{C}_{ijkl}-C^{\alpha}_{ijkl} ,$$

where α signifies the particular template used. One then calculates $\varepsilon^{\alpha}_{ijkl}\varepsilon^{\alpha}_{ijkl}$ or some other indicator of smallness. Some measure of randomness of the elements may also be necessary to determine the best symmetry system which can be used to represent the original elastic tensor. Since the purpose is to discriminate against the triclinic case it is excluded from the discussion above.

APPROXIMATIONS

Approximations to specific anisotropic situations exist, such as elliptical anisotropy for transversely isotropic cases (Daley and Hron, 1979) and Fourier truncations for general anisotropic media (Crampin, 1981). Even though these approximations do not give the full picture of wave propagation in anisotropic media, they do provide simplifications that allow some understanding of underlying principles. This area has room for further investigation into other forms of approximation and estimation of error introduced by the approximations. This last point was partially addressed for elliptical anisotropy by Helbig (1983).

TRAVELTIME

The traveltime problem in anisotropic media is well studied and has been subjected to analysis by varying techniques; In this area I see little room for me to make a contribution. There is one area which may be linked to this however; that is the construction of an approximate Green's function from the techniques above. One very interesting development in terms of field discontinuities (Vlaar, 1968) is very akin to the discontinuous nature of the Green's problem.

ELASTODYNAMIC GREEN'S FUNCTION

The elastodynamic Green's function G_{in} is defined as the solution to the following equation

$$(A_{ijkl}G_{kn,l})_{,j} - \rho\ddot{G}_{in} = -\delta_{in}\delta(x_i-\xi_i)\delta(t-\tau) \qquad (16)$$

where: $A_{ijkl} \equiv$ elastic tensor,
$\rho \equiv$ density,
$\delta_{in} \equiv$ Kroneker delta,

and $\delta(t-\tau) \equiv$ Dirac delta,

usually with homogeneous boundary conditions (e.g. free-space Green's function) (Aki and Richards, 1980). This, at present, has no analytical form except for the case of transverse isotropy (Ben-Menahem and Sena, 1990); therefore, this would be an interesting topic to investigate. Ben-Menahem and Sena's (1990) analysis of the source function showed the far field is more profoudly effected by the nearfield signature then the concavity of the slowness surface. This analysis shows the usefulness of the Green's function in studying wave propagation problems. The use of Green's functions in the solutions and elucidation of boundary-value problems is well known (Stakgold, 1979) and the powerful techniques available which use this (e.g. Born's approximation) makes the solution of equation (16) a very desirable objective.

PLANE WAVES

Some work in plane-wave techniques for the decomposition of point sources into plane-waves in anisotropic media is being done at present (Rommel, 1990), but is still very much in the preliminary research stage. It would be interesting to see if the technique for plane-wave decomposition introduced by Tygel and Hubral (1984) for representing the Sommerfeld-Weyl integral may be brought to bear on this problem. The simplicity of the phase velocity representation of plane waves should pay dividens when used in a modeling scheme.

DISCRETE GRID TECHNIQUES

Another approach towards forward modeling is to find a numerically tractable finite-difference or finite-element scheme that approximates equation (11) or (12). At present, I have not read a paper on this approach. Another area of interest is to explore the possibility of useing cellular automata concepts in this area.

CONCLUSION

Inversion in anisotropic media is really in its infancy at this time. There are many techniques which can be tried including: generalized linear inversion, Born inversion, other approximate inverse methods and (τ-p) methods. We feel it is better thesis research strategy to investigate fully the previously mentioned research areas: (i) basic theory (ii) approximations for anisotropic situations, (iii) traveltime techniques (subject to the reservations stated above) and (iv) wave-equation techniques. With a firm foundation in these techniques, the anisotropic inverse problem, with its many intertwined and interesting complexities may then be faced.

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APPENDIX A

ISOTROPIC WAVES WITH ROTATION

Given the general form of an isotropic tensor of order four from appendix B we can write:

$$A_{ijkl} = \lambda(\delta_{ij}\delta_{kl}) + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \quad \text{(A-1)}$$

The form of the general constitutive relations will be of the form

$$\Sigma_{ij} = A_{ijkl}(\varepsilon_{kl}+\omega_{kl}) \quad \text{(A-2)}$$

where $\varepsilon_{kl} \equiv$ the symmetrical strain tensor
and $\omega_{kl} \equiv$ the antisymmetrical rotation tensor.

Substitution of (A-2) into (A-1) results in:

$$\Sigma_{ij} = \lambda\delta_{ij}\theta + 2\mu\varepsilon_{ij} - 2\nu\omega_{ij} \quad \text{(A-3)}$$

where $\theta = \varepsilon_{kk} \equiv$ the dilatation.

The equation of motion, ignoring body forces, is given by

$$\rho\ddot{U}_i = \Sigma_{ij,j} \quad \text{(A-4)}$$

where $U_i \equiv$ the usual displacement vector
and $\rho \equiv$ the density.

Taking the spatial derivative of equation (A-3) gives

$$\begin{aligned}\Sigma_{ij,j} &= (\lambda\delta_{ij}\theta)_{,j} + 2\mu\varepsilon_{ij,j} - 2\nu\omega_{ij,j}\\ &= \lambda\theta_{,i} + \mu(U_{i,jj} + U_{j,ij}) - \nu(U_{i,jj} - U_{j,ij})\\ &= \lambda\theta_{,i} + (\mu-\nu)U_{i,jj} + (\mu+\nu)\theta_{,i}.\end{aligned} \quad \text{(A-5)}$$

Substitution of (A-5) into (A-4) results in:

$$\rho\ddot{U}_i = (\lambda+\mu+\nu)\theta_{,i} + (\mu-\nu)U_{i,jj}, \quad \text{(A-6a)}$$

which in vector notation is:

$$\rho\ddot{\vec{U}} = (\lambda+\mu+\nu)\vec{\nabla}\theta + (\mu-\nu)\vec{\nabla}^2\vec{U}. \tag{A-6b}$$

Now we use the standard procedure and take the divergence ($\vec{\nabla}\cdot$) of equation (A-6b) to get:

$$\rho\vec{\nabla}\cdot\ddot{\vec{U}} = (\lambda+\mu+\nu)\vec{\nabla}\cdot\vec{\nabla}\cdot\theta + (\mu-\nu)\vec{\nabla}\cdot\vec{\nabla}^2\vec{U},$$

but $\vec{\nabla}\cdot\ddot{\vec{U}} = \ddot{\theta},$

$\vec{\nabla}\cdot\vec{\nabla}\cdot\theta = \vec{\nabla}^2\theta$

and $\vec{\nabla}\cdot\vec{\nabla}^2\vec{U} = \vec{\nabla}^2\vec{\nabla}\cdot\vec{U} = \vec{\nabla}^2\theta$

therefore,

$$\rho\ddot{\theta} = (\lambda+2\mu)\vec{\nabla}^2\theta. \tag{A-7a}$$

Analogous to the method above, we can take the curl ($\vec{\nabla}\times$) of equation (A-6b) and get

$$\rho\vec{\nabla}\times\ddot{\vec{U}} = (\lambda+\mu+\nu)\vec{\nabla}\times\vec{\nabla}\cdot\theta + (\mu-\nu)\vec{\nabla}\times\vec{\nabla}^2\vec{U}$$

but $\vec{\nabla}\times\vec{\nabla}\cdot\theta = 0$

and $\vec{\nabla}\times\vec{\nabla}^2\vec{U} = \vec{\nabla}^2\vec{\nabla}\times\vec{U} = \vec{\nabla}^2\vec{\psi},$

where I have let $\vec{\nabla}\times\ddot{\vec{U}} = \vec{\psi},$

therefore,

$$\rho\ddot{\vec{\psi}} = (\mu-\nu)\vec{\nabla}^2\vec{\psi}. \tag{A-7b}$$

Equations (A-7a and A-7b) are wave equations in the standard forms seen in seismological texts and are related to the propagation of P and S waves respectively. The difference from the standard equations is only in the S-wave velocity, where in the standard development we have

$$v_s = \beta = \sqrt{\frac{\mu}{\rho}},$$

while here the velocity becomes

$$v_s = \beta = \sqrt{\frac{\mu - \nu}{\rho}}. \qquad \text{(A-8a)}$$

The P-wave velocity on the other hand remains the same and is of the form

$$v_p = \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \qquad \text{(A-8b)}$$

APPENDIX B

ISOTROPIC/CUBIC TENSORS OF ORDER FOUR

For a rotation about the origin "O" from one frame of reference (designated by O123) to another frame (designated as O1'2'3') an arbitrary tensor will have the following relationship:

$$C'_{ijkl} = l_{mi} l_{nj} l_{ok} l_{pl} C_{mnop} \qquad \text{(B-1a)}$$

where $C_{ijkl} \equiv$ tenor of order four

and $l_{mi} \equiv$ direction cosines between the two frames.

If furthermore the tensor is isotropic (ie., same from any frame of reference) then we would have the relation

$$C'_{ijkl} = l_{mi} l_{nj} l_{ok} l_{pl} C_{mnop} = C_{ijkl} \qquad \text{(B-1b)}$$

(Note: With suffices allowed to take on values of 1,2 and 3 only and there being four suffices then at least one pair of the suffices must be the same. Some possible forms are: 1111, 1112, 1122, 1123.) Rotation about $\hat{n} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ (the vector from the origin to the barrycenter) will bring about cyclic interchange of suffices as shown in figure B-1a and B-1b. This rotation plus equations (B-1a and b) implies cyclic permutation of suffices leaves the tensor unchanged, as exemplified by relations (B-1c) below:

$$C_{1111} = C_{2222} = C_{3333},\ C_{1112} = C_{2223} = C_{3331}, \qquad \text{(B-1c)}$$
$$C_{1122} = C_{2233} = C_{3311},\ C_{1123} = C_{2231} = C_{3312}.$$

Now consider a ninety degree rotation along the O3 axis as in figure B-2

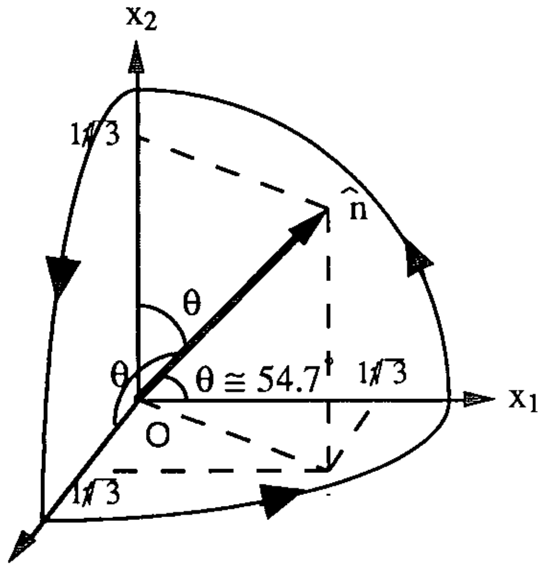

Figure B-1a

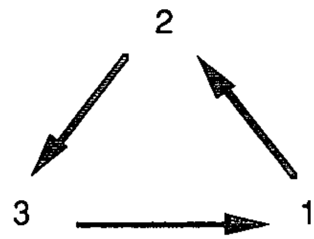

Figure B-1b

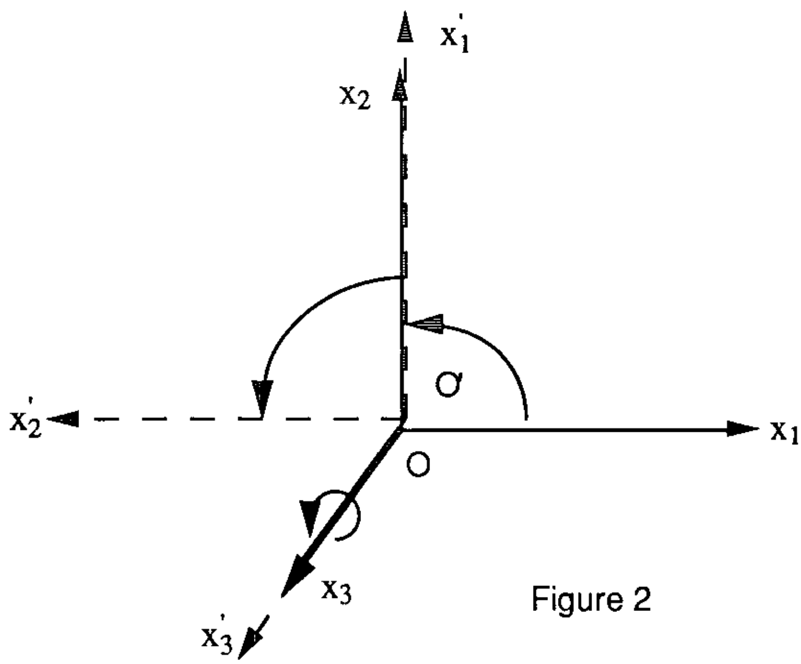

Figure 2

This rotation will have two major consequences. First consider the direction cosines

$$l_{12} = 1,\ l_{12} = -1,\ l_{12} = 1\ ,$$

all the rest will equal zero since for these

$$\theta_{ij} = 90^{\circ} \Rightarrow \mathrm{Cos}\ \theta_{ij} = 0 = l_{ij}.$$

Now consider the case where i=3 and j=k=l=1 then from (1) we can write

$$C_{3111} = l_{m3}l_{n1}l_{o1}l_{p1}C_{mnop} = -C_{3222},$$

but by the same argument we can state

$$C_{3222} = C_{3111},$$

which implies that

$$C_{3111} = -C_{3111} = 0;$$

therefore, by cyclic permutation and ninety degree rotations about the other two axes, all elements with 3 suffices equal will also be zero

$$(\text{i.e., } C_{ijjj} = C_{jijj} = C_{jjij} = C_{jjji} = 0\ \forall\ i \neq j = 1,2,3\ ).$$

By the same arguments above we can consider the case where i=3, j=k=1 and l=2 giving the following

$$C_{3112} = -C_{3221},$$

but

$$C_{3221} = C_{3112} \Rightarrow C_{3112} = -C_{3112} = 0,$$

again by cyclic permutation and ninety-degree rotation about the other two axes. This would imply that all elements with two suffices equal and the other two different are zero.

$$(\text{i.e. } C_{ijjk} = C_{jijk} = C_{jjik} = C_{jijk} = 0\ \forall\ i \neq j \neq k = 1,2,3.)$$

Now let us look at some non-zero consequences of these rotations. Consider the following:

$$i=j=1,\ k=l=2\ \&\ m=n=2,\ o=p=1 \Rightarrow C_{1122} = C_{2211},$$
$$i=k=2,\ j=l=3\ \&\ m=o=1,\ n=p=3 \Rightarrow C_{2323} = C_{2211},$$
$$i=l=1,\ j=k=2\ \&\ m=p=2,\ n=o=1 \Rightarrow C_{1221} = C_{2112},$$
$$i=j=k=l=1\ \ \&\ \ m=n=o=p=2 \Rightarrow C_{1111} = C_{2222}.$$

The upshot of all this is that the only non-zero elements must have either all suffices equal or equal in pairs. Armed with this and using cyclic permutations we can write:

$$C_{1111} = C_{2222} = C_{3333} = \hat{\kappa},$$
$$C_{1122} = C_{2211} = C_{2233} = C_{3322} = C_{3311} = C_{1133} = \hat{\lambda},$$
$$C_{2323} = C_{1313} = C_{3131} = C_{2121} = C_{1212} = C_{3232} = \hat{\mu},$$
$$C_{1221} = C_{2112} = C_{2332} = C_{3223} = C_{3113} = C_{1331} = \hat{\nu}.$$

Using the Kroneker delta the above relations can be written in the compact form

$$C_{ijkl} = \hat{\lambda}(\delta_{ij}\delta_{kl}) + \hat{\mu}(\delta_{ik}\delta_{jl}) + \hat{\nu}(\delta_{il}\delta_{jk}) + \hat{\kappa}(\delta_{ij}\delta_{jk}\delta_{kl}). \quad \text{(B-2)}$$

Equation (B-2) is the form taken by a tensor with cubic symmetry. To see that this is not isotropic, consider four arbitrary tensors w_i, x_j, y_k and z_l, then form the tensor product

$$C_{ijkl}w_ix_jy_kz_l = \hat{\lambda}(w_ix_iy_kz_k) + \hat{\mu}(w_iy_ix_jz_j) + \hat{\nu}(w_iz_iy_jx_j)$$

$$+ (\hat{\kappa}-\hat{\lambda}-\hat{\mu}-\hat{\nu})(x_1y_1z_1w_1 + x_2y_2z_2w_2 + x_3y_3z_3w_3).$$

Note that the first three terms on the right-hand side are scalars while the last term depends on the selection of the four arbitrary tensors and only has cubic symmetry. To see that this is indeed true we let $w_i=y_i=z_j=x_j$; then the last term will have the form $\omega(x_1^4 + x_2^4 + x_3^4)$ where we have allowed $\omega=\hat{\kappa}-\hat{\lambda}-\hat{\mu}-\hat{\nu}$. Now let

$x_1 = x_2 = x_3 = 1/\sqrt{3}$ this would imply $x_1^4 + x_2^4 + x_3^4 = 1/3$, but by a simple rotation we change the form of the vector to $x_1 = 1, x_2 = x_3 = 0$ then $x_1^4 + x_2^4 + x_3^4 = 1$. Therefore, for the fourth-order tensor to be isotropic we must have:

$$\omega = \hat{\kappa}-\hat{\lambda}-\hat{\mu}-\hat{\nu} = 0.$$

Substituting this into equation (B-2) results in:

$$C_{ijkl} = \hat{\lambda}(\delta_{ij}\delta_{kl}) + \hat{\mu}(\delta_{ik}\delta_{jl}) + \hat{\nu}(\delta_{il}\delta_{jk}), \qquad \text{(B-3)}$$

which upon substitution of $\lambda = \hat{\lambda}$, $\mu + \nu = \hat{\mu}$ and $\mu - \nu = \hat{\nu}$ results in:

$$C_{ijkl} = \lambda(\delta_{ij}\delta_{kl}) + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \qquad \text{(B-4a)}$$

If the anti-symmetric term is neglected then:

$$C_{ijkl} = \lambda(\delta_{ij}\delta_{kl}) + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad \text{(B-4b)}$$

which is the form most commonly seen in elastic wave theory.

APPENDIX C

MOMENT OF INERTIA AND THE SYMMETRY OF STRESS

We will begin with Newton's second law applied to a volume "V" bounded by a surface "S", as indicated in figure C-1.

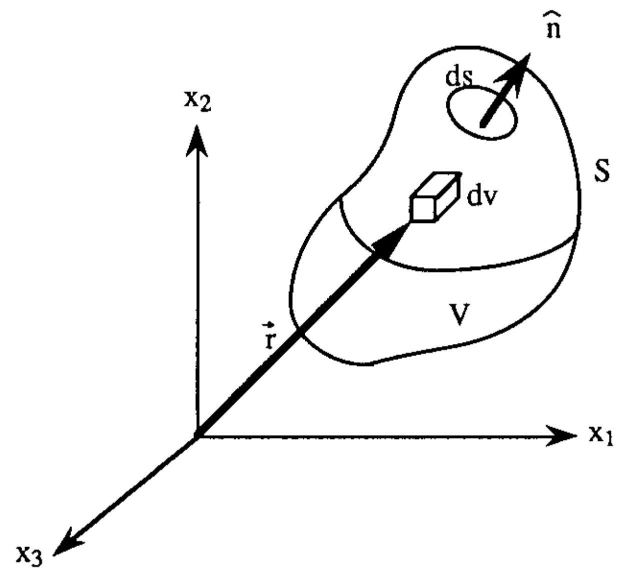

Figure C-1

This will give rise to the following integral equation,

$$\int_v F_i \rho \, dv + \int_s \Sigma_{ij} n_j \, ds = \int_v \rho \ddot{U}_i \, dv \qquad \text{(C-1)}$$

where:

$F_i \equiv$ body forces,

$\rho \equiv$ mass density,

$\Sigma_{ij} \equiv$ total stress,

$n_j \equiv$ unit normal to surface element ds

and $U_i \equiv$ displacement vector.

We now apply Gauss's theorem to the surface integral and rearrange terms to get

$$\int_v [(F_i + \ddot{U}_i)\rho + \Sigma_{ij,j}] \, dv = 0 \quad , \qquad \text{(C-2)}$$

since the volume "V" is arbitrary the integrand must be zero, as shown below

$$(F_i + \ddot{U}_i)\rho + \Sigma_{ij,j} = 0 \quad . \qquad (C\text{-}3)$$

Armed with equation (C-3) we proceed to look at the moments about the origin;

$$\int_V \varepsilon_{ijk} x_j F_k \rho \, dv + \int_S \varepsilon_{ijk} x_j \Sigma_{kl} n_l \, ds + \int_V M_j \rho \, dv = \int_V \rho \varepsilon_{ijk} x_j \ddot{U}_k \, dv \quad , \qquad (C\text{-}4)$$

where;

$\varepsilon_{ijk} \equiv$ permutation symbol,

$x_j \equiv$ position vector

and $M_j \equiv$ some possible body moment per unit mass.

Useing Gauss's theorem we can rewrite the surface integral in equation (C-4) as

$$\int_S \varepsilon_{ijk} x_j \Sigma_{kl} n_l \, ds = \int_V \varepsilon_{ijkl} [x_{j,i} \Sigma_{kl} + x_j \Sigma_{kl,l}] \, dv \quad . \qquad (C\text{-}5)$$

Now by substitution of equation (C-5) into equation (C-4) and regrouping terms we get

$$\int_V \varepsilon_{ijk} \Sigma_{kj} + M_i \rho \, dv = \int_V \varepsilon_{ijkl} x_j [\Sigma_{kl,l} + \rho(F_k - \ddot{U}_k)] \, dv = 0 \qquad (C\text{-}6)$$

By direct comparison of Equation (C-3) to the second integrand in equation (C-6) we can set the second integral to zero, and since the volume "V" is arbitrary we can set the integrand of the first integral to zero giving

$$\varepsilon_{ijk} \Sigma_{jk} = M_i \rho , \qquad (C\text{-}7)$$

and if we have no body moments (refer to Appendix E) then (C-7) becomes

$$\varepsilon_{ijk} \Sigma_{kj} = 0$$

or

$$\Sigma_{kj} = \Sigma_{jk} = \sigma_{jk} . \qquad (C\text{-}9)$$

Thus in the absence of body moments we must require

$$A_{ijkl} = A_{jikl} . \qquad (C\text{-}10)$$

APPENDIX D

ENERGY PROPAGATION IN ANISOTROPIC MEDIA

This section will be dealing with the propagation of energy in an anisotropic medium. The path traced out along the wavefront as the energy propagates defines the ray path. Even though the existance of a strain energy function Φ , as defined below is not proven, it has proven to be such a useful mathematical tool it behooves us to define it with the following property:

$$\Sigma_{ij} = \frac{\partial \Phi}{\partial \mu_{ij}} . \tag{D-1}$$

Expanding the strain energy function about an equilibrium point 0 in a Taylor series gives:

$$\Phi = \Phi(0) + \left[\frac{\partial \Phi}{\partial \mu_{ij}}\right]_0 \mu_{ij} + \frac{1}{2}\left[\frac{\partial^2 \Phi}{\partial \mu_{ij}\partial \mu_{kl}}\right]_0 \mu_{ij}\mu_{kl} + \dots . \tag{D-2}$$

Since we have defined 0 to be an equilibrium point then the first derivative with respect to strain must be zero, as shown below:

$$\left[\frac{\partial \Phi}{\partial \mu_{ij}}\right]_0 = 0 .$$

This implies

$$\Phi - \Phi(0) = \frac{1}{2}\left[\frac{\partial^2 \Phi}{\partial \mu_{ij}\partial \mu_{kl}}\right]_0 \mu_{ij}\mu_{kl} + O(\mu_{ij}^3) . \tag{D-3}$$

Now we make the assumption that the system in the equilibrium configuration has zero energy (though not absolutely necessary), and noteing that equation (D-1) gives us:

$$\frac{\partial^2 \Phi}{\partial \mu_{ij}\partial \mu_{kl}} = A_{ijkl} , \tag{D-4}$$

allows equation (D-3) to be written in the form:

$$\Phi = \frac{1}{2} A_{ijkl}\mu_{ij}\mu_{kl} + O(\mu_{ij}^3) . \tag{D-5}$$

We now drop all terms of order three and higher and substitute the stress strain relation of equation (4a) into equation (D-5) to get:

$$\Phi = \frac{1}{2}A_{ijkl}\mu_{ij}\mu_{kl} = \frac{1}{2}\Sigma_{ij}\mu_{ij} , \tag{D-6}$$

which is an expression of the potential energy stored in the elastic field. The expression for kenetic energy density has the form:

$$W = \frac{1}{2}\rho \dot{U}_i \dot{U}_i ; \tag{D-7}$$

With the expression of potential and kenetic energy in hand we proceed to write down the expression for total energy in a volume V as:

$$E = \int_v (W+\Phi)\, dv . \tag{D-8}$$

We can then write the expression of the change in energy with time in the volume as:

$$\frac{dE}{dt} = \int_v (\frac{\partial W}{\partial t} + \frac{\partial \Phi}{\partial t})\, dv \int_v \rho \dot{U}_i \ddot{U}_i + \frac{\partial \Phi}{\partial \mu_{ij}}\dot{\mu}_{ij}\, dv, \tag{D-9}$$

but

$$\frac{\partial \Phi}{\partial \mu_{ij}} = \Sigma_{ij} \ \& \ \dot{\mu}_{ij} = \dot{U}_{i,j} ; \tag{D-10}$$

therefore,

$$\int_v \frac{\partial \Phi}{\partial t}\, dv = \int_v \Sigma_{ij}\dot{U}_{i,j}\, dv = \int_?^? (\Sigma_{ij}\dot{U}_i)_{,j} - \dot{U}_i\Sigma_{ij,j}\, dv$$
$$= \int_s \Sigma_{ij}\dot{U}_i n_j\, ds - \int_v \dot{U}_i\Sigma_{ij,j}\, dv . \tag{D-11}$$

Substitution of equation (D-11) into (D-9) results in:

$$\frac{dE}{dt} = \int_v \dot{U}_i(\rho\ddot{U}_i - \Sigma_{ij,j})\, dv + \int_s \Sigma_{ij}\dot{U}_i n_j\, ds \quad . \tag{D-12}$$

From equation (1) we can see that, if body forces are ignored, the integrand of the volume integral is identically zero, leaving us with:

$$\frac{dE}{dt} = \int_s \Sigma_{ij}\dot{U}_i n_j\, ds = -\int_s F_j n_j\, ds \quad . \tag{D-13}$$

The physical significance of equations (D-12) and (D-13) is that the rate of change of energy in a volume V is equal to the flux of the vector F through the enclosing surface S; this prompts us to demine F as the energy-density flux vector:

$$F_j = -\Sigma_{ij}\dot{U}_i \qquad \text{(D-14)}$$

We will now use the definition of a plane wave given by equation (13) to study the energy propagation problem. Since we are now dealing with energy we need to consider the real part of displacement only, namely:

$$(U_k + \overline{U}_k)/2 \,. \qquad \text{(D-15)}$$

Substitution of equation (D-15) into the energy flux density equation (D-14) results in:

$$\begin{aligned} F_i &= -\Sigma_{ij}(\dot{U}_j + \dot{\overline{U}}_j)/2 = -\frac{1}{4}A_{ijkl}(U_{k,l} + \overline{U}_{k,l})(\dot{U}_j + \dot{\overline{U}}_j) \\ &= -\frac{1}{4}A_{ijkl}[(U_{k,l}\dot{\overline{U}}_j + \overline{U}_{k,l}\dot{U}_j) + (U_{k,l}\dot{U}_j + \overline{U}_{k,l}\dot{\overline{U}}_j)] \,. \end{aligned} \qquad \text{(D-16)}$$

Note that the first term in parenthesis has no complex exponential dependence while the second does. The oscillatory nature of the complex exponential causes it to average out to zero over a span of time long in comparison to its period. Useing this fact we will develope a better idea of how energy propagates by averaging equation (D-16) as follows:

$$\xi_i = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} F_i \, dt = -\frac{1}{4}A_{ijkl}(U_{k,l}\dot{\overline{U}}_j + \overline{U}_{k,l}\dot{U}_j) \,. \qquad \text{(D-17)}$$

We shall now substitute equation (13) inot equation (D-17) to get

$$\xi_i = -\frac{1}{4}A_{ijkl}(\overline{p}_j p_k s_l + p_j \overline{p}_k \overline{s}_l) \,, \qquad \text{(D-18a)}$$

and if and are real, as if commonly assumed, equation (D-18) takes the special form:

$$\xi_i = \frac{1}{4}A_{ijkl}p_j p_k s_l \,. \qquad \text{(D-18b)}$$

We now have the necessary machinery to draw a connection between the averaged energy flux density vector and the normal to the slowness surface.

The slowness surface as discussed in relation to equation (14) will have the following form:

$$\Omega = \det[S_{ik}] = \det[A_{ijkl}s_j s_l - \rho\delta_{ik}] = 0 \,. \qquad \text{(D-19)}$$

Letting C_{ik} be the cofactor of element S_{ik} in equation (D-19), allows us to write the following relationship:

$$C_{ik}S_{il} = \delta_{kl}\Omega = 0; \qquad \text{(D-20)}$$

this comes from the definition of cofactor expansion of a determinant. The plane wave equation (14) can also be written as:

$$S_{ik}p_k = 0 = S_{ik}p_i. \qquad \text{(D-21)}$$

By direct comparison of equation (D-21) and (D-20) we can come to the conclusion that:

$$C_{ik} = C_{ik} = fp_ip_k, \qquad \text{(D-22)}$$

where f is some constant. We now proceed to calculate the gradient of the slowness surface which is the normal to the surface which will have the form:

$$\frac{\partial\Omega}{\partial s_j} = C_{ik}\frac{\partial S_{ik}}{\partial s_j} = C_{ik}A_{ijkl}s_l \quad . \qquad \text{(D-23)}$$

Substitution of equation (D-22) into equation (D-23) yields:

$$\frac{\partial\Omega}{\partial s_j} = fA_{ijkl}p_ip_ks_l \qquad \text{(D-24a)}$$

or

$$\frac{\partial\Omega}{\partial s_j} \propto A_{ijkl}p_ip_ks_l \,. \qquad \text{(D-24b)}$$

By makeing a direct comparison of equations (D-24a and b) to equation (D-18b), we would find that they are identical up to a multiplicative constant, which represented symbolically is:

$$\xi_i \propto \frac{\partial\Omega}{\partial s_i} \,. \qquad \text{(D-25)}$$

Note the constant of proportionality is constant for all components. The physical significance of equation (D-25) is that energy flows in the direction normal to the slowness surface.

APPENDIX E

SHEAR WAVES AND BODY MOMENTS

The existence of shear waves suggests that the wavefield is in general is not irrotational in an elastic medium. This can be easily demonstrated by a simple closed circuit integration of particle motion in the presence of plane S-waves for instance. We can use the formula:

$$\dot{\omega}_i = e_{ijk}\dot{U}_{j,k} \,, \qquad \text{(E-1)}$$

where $\omega_i \equiv$ the rotation vector,

$e_{ijk} \equiv$ the permutation symbol,

and $U_i \equiv$ the displacement vector,

this provides a description of local infinitesimal angular velocity, and is in general not zero.

Consider the following relationship in a small volume ΔV,

$$\Delta L_i = \Delta I_i \, \dot{\theta}_{(i)} \quad \text{(no sum on i)} \qquad \text{(E-2)}$$

where $\Delta L_i \equiv$ angular momentum,

$\Delta I_i \equiv$ moment of inertia,

and $\dot{\theta} \equiv$ angular velocity .

We have from the above discussion an expression for angular velocity therefore substituting equation (E-1) into (E-2) yields:

$$\Delta L_i = \Delta I_{(i)} e_{ijk}\dot{U}_{j,k} \,. \qquad \text{(E-3)}$$

Now we can write:

$$\Delta I_i = \int_v r^2 \rho \qquad \text{(E-4)}$$

where $r \equiv$ radius in a plane perpendicular to the axis x_i ,

and $\rho \equiv$ density .

If we further allow:

$$Mk^2 = \Delta I_i = \int_v r^2 \rho \, dv \qquad \text{(E-5)}$$

such that $M \equiv$ total mass in V,

and $k \equiv$ radius of gyration .

The density distribution will greatly influence the integral (E-5), specially if there are distributed hard cores in the volume. If we assume that the density is constant in the volume ΔV we can write $M = \rho \Delta V$ and

$$Mk^2 = \rho \int_v r^2 \, dv ,$$

therefore

$$k^2 = \frac{\int_v r^2 \, dv}{\Delta V} . \tag{E-6}$$

In a strict continuum equation (E-6) will tend toward zero; however, in the small scale this may be a poor representation and thus the possibility of equation (E-6) not being zero aries (probably quite small). Then we can write:

$$\frac{\Delta L_i}{\Delta V} = \rho k^2 e_{ijk} \dot{U}_{j,k} , \tag{E-7}$$

which is a body moment density.