Energy partition at the boundary between anisotropic media; Part one: Generalized Snell's law.

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ABSTRACT

The mathematical description of phenomena related to wave propagation in anisotropic media differs significantly from that for isotropic media. In general, the expressions are more complicated and more difficult for intuitive understanding. In the anisotropic case the relationship between the angles of incident, reflected and refracted rays, i.e., Snell's law, cannot be reduced to such a simple form as in the isotropic case. This paper attempts, with aid of familiar, and thus rather intuitive, notions of vector calculus, to provide a framework for calculating these angles. In the anisotropic case there exist the concepts of both phase and group (ray, energy) velocities. The phase-slowness surface, i.e., the inverse of the phase-velocity surface, can be described as a function of three space variables: x, y, and z. By virtue of the continuity conditions across the planar, horizontal boundary between two media, the horizontal components of phase-slowness must be continuous across this boundary. The knowledge of the expression for phase-slowness surfaces in both the incidence and transmission media, the fact that all phase velocities and thus phase-slowness vectors must be coplanar and the enforcement of the continuity conditions form the core of the Snell's law in anisotropic media. The direction of the actual ray is perpendicular to the plane tangent to the phase-slowness surface at a given point and can be mathematically determined using gradients. One must also stipulate that the incident ray points towards the boundary while the reflected and transmitted rays point away from it. This condition can be mathematically described using the properties of the dot product. Implication for critical angles and Fermat's principle are discussed with the aid of analytically derived expressions.

There exist efficient, numerical schemes for calculating the angles of incidence, reflection and transmission. The merit of the present approach is believed to lie in the clarity of the analytical method. A method allowing the description of phase-slowness surfaces corresponding to various symmetry systems as a function of three space variables x, y and z would extend the usefulness of the presented approach.

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INTRODUCTION

The mathematical description of phenomena related to wave propagation in anisotropic media is considerably more complicated than that for isotropic media. This complexity stems from the many physical properties distinguishing the anisotropic and the isotropic medium. This paper presents a method of calculating the angles of reflection and transmission for a ray impinging on a boundary between two anisotropic media. For an interface between two isotropic media the relationship among all the angles is elegantly and concisely described by the classic form of Snell's law, i.e.,

$$\frac{\sin\theta_i}{v_1}=\frac{\sin\theta_r}{v_1}=\frac{\sin\theta_t}{v_2}. \tag{1}$$

where θ's with respective subscripts, correspond to incident, reflected and transmitted waves and v_1 and v_2 are the velocities in the two media separated by a planar interface.

In general, the inclusion of anisotropy renders the mathematical formulation quite complicated. Snell's law is not an exception and the calculation of reflection and transmission angles is not a trivial task. A graphical approach to calculating reflection and transmission angles for anisotropic media is presented by Auld (1973) and Rokhlin et al. (1986). Rokhlin et al. (1986) also outline a numerical scheme using tensor equations. Daley and Hron (1977, 1979) derive Snell's law in the particular cases of transversely isotropic and ellipsoidally anisotropic media. The present paper seeks to express the concept of incidence, reflection and transmission angles using the intuitively clear mathematical apparatus of vector calculus in a rather general case. Therefore, the foremost intention of this note is the clarity of the analytical formulation and its relation to physical phenomena, rather than the computational efficiency of the method.

Although providing a thorough overview of numerous physical phenomena in anisotropic media lies beyond the scope of this paper, it can be stated that two aspects of physical properties inherent in wave propagation in anisotropic media are responsible for the more complicated formulation than in the case of isotropic media. Firstly, the velocity of the ray depends on direction and thus, for instance, the velocity of the incident ray is, in general, unequal to the velocity of the reflected ray, although both rays propagate in the same medium. This means that, in general, the angle of incidence is not equal to the angle of reflection. This consequence should not be surprising since it is analogous to the phenomena observed in studying converted waves exhibiting different velocities for incident and reflected rays. Secondly, both group, $\mathbf{w}$, and phase, v, velocities have to be considered in studying wave propagation in anisotropic media. The two are related, e.g., Rokhlin et al. (1986) by the formula:

$$\mathbf{w}\cdot\mathbf{n}=v, \tag{2}$$

where **n** is the wave normal.

IMPORTANT CONCEPTS

In formulating the method for calculating reflected and transmitted angles at an interface between isotropic and anisotropic media, it is helpful to restate certain basic concepts. The intention is not to present those concepts in a rigorous and complete way but to invoke those aspects which are most useful for the task at hand. The concepts in question include: phase and group velocities, Snell's law and phase-slowness surfaces.

Phase and group velocities

Phase velocity is defined as the velocity with which plane-wave crests and troughs travel through a medium and is expressed as the ratio of the frequency of vibration and the wave number (i.e., the number of wavelengths per unit distance). Group velocity, also known as energy or ray velocity, is defined as a velocity with which the energy of a wave propagates. Direct measurements of traveltime usually yield the group velocity.

In dispersive media, e.g., an anelastic medium exhibiting frequency dispersion or an anisotropic medium exhibiting angular dispersion phase and group velocities are different; both in magnitude and direction. For an anisotropic medium, at the same point on the wavefront, the group velocity is higher than the phase velocity. Also, for an anisotropic medium the direction of the group velocity is perpendicular to the phase-slowness surface, i.e., to the surface representing the inverse of the phase-velocity surface (Rokhlin et al, 1986).

Snell's law

Snell's law is a direct consequence of Fermat's principle of stationary time. It can be conveniently restated as a requirement for the horizontal component of the wave number, k_x, to be continuous across the boundary. As a matter of fact, the horizontal component of the wave number remains constant for all layers and is analogous to the ray parameter. This property must be preserved for both isotropic and anisotropic media regardless of the type of the wave generated at the boundary, e.g., longitudinal or transverse, and serves as a kernel for the strategy of calculating reflected and transmitted angles.

Slowness surfaces

Phase-slowness is defined (e.g., Winterstein, 1990) as the reciprocal of the scalar phase velocity, and therefore can be expressed as the ratio of wave number, k, and angular frequency, ω. In an isotropic medium the phase-slowness surface is a sphere with radius equal to the inverse of the phase velocity (which does not vary with direction). In such a case, phase and group velocities are collinear since the normal to the surface of a sphere is collinear with its radius vector.

For an anisotropic medium the shape of the phase-slowness surface can form a much more complicated figure including concave and convex shapes. The number of symmetry planes decreases as the number of elastic constants necessary to describe the material increases. An infinite number of symmetry planes exist for an isotropic medium described by two elastic constants, i.e., Lamé parameters; no symmetry planes exist for a triclinic medium which requires twenty-one elastic constants to be uniquely characterized (see e.g., Crampin and Kirkwood, 1981).

There are, in general, three slowness surfaces, each corresponding to a given wave type: one for quasi-compressional and two for quasi-shear waves. The slowness surfaces can touch, thus forming singularities, i.e., points corresponding to orientations along which phase velocities become equal for two wave types. Interesting phenomena relating to polarization occur in the neighbourhood of those points.

GEOMETRICAL FORMULATION

Snell's law can be illustrated using phase-slowness surfaces for both incident and transmitted media (Auld, 1973). The geometrical construction is facilitated by the fact that the phase-slowness vectors of the incident, reflected and refracted waves are coplanar. Their being coplanar is guaranteed by the necessity to satisfy boundary conditions at all times and at every point of the interface. Therefore, it is convenient to choose a Cartesian coordinate system such that all the phase-slowness vectors lie in the *xz*-plane. Below we consider the familiar case of isotropic media.

Invoking the continuity of the horizontal component of the wave number, k_x, across a boundary, and by applying simple trigonometry to Figure 1, it is easy to obtain the usual form of Snell's law for an isotropic medium, i.e., Equation 1. Other concepts, such as total internal reflection, also have their geometrical interpretation. For a sufficiently large incidence angle one has $k_x \geq 1/v_2$, in which case no transmitted ray is possible. The equality $k_x=1/v_2$ yields the angle at which this first occurs, i.e., the critical angle. Note that if the radius of the phase-slowness sphere in the transmitted medium is larger than in the incident medium, there is always a transmitted ray and total internal reflection cannot occur. For anisotropic media, each phase-slowness surface is, in general, described by a different function and the phase- and group-velocity angles do not coincide. To deal with a more complex situation, a more complicated mathematical scheme has to be employed. Figure 1, without loss of generality, illustrates a generic case, i.e., the wave type is not specified. Consideration of mode conversion would yield, in an isotropic case, two concentric circles in both media, representing phase-slowness surfaces for compressional and shear waves. In an anisotropic case three geometrical figures would appear in each medium due to the bi-refringence of quasi-shear waves, i.e., due to different velocities of two types of quasi-shear waves.

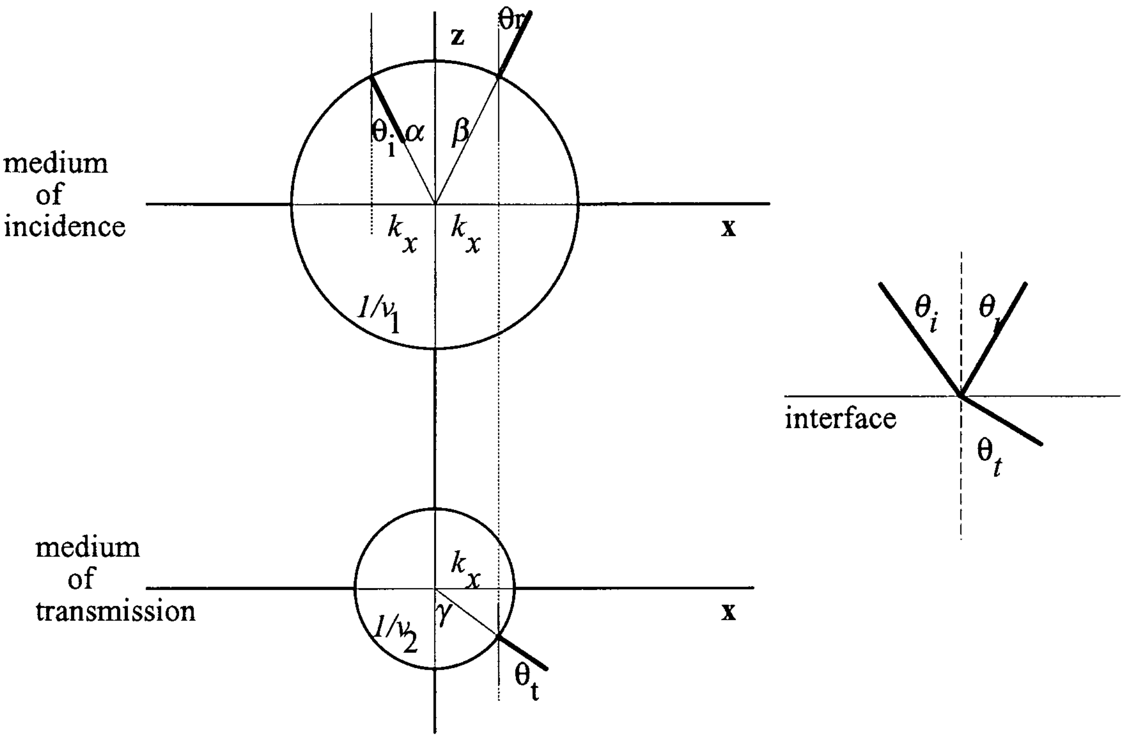

FIG. 1. The geometrical construction yielding reflection and transmission angles of slowness vectors in an isotropic medium using the phase-slowness surface. The same concept applies in an anisotropic medium except that the *xz*-plane cross-section of the phase-slowness surface does not, in general, form a circle. The thin lines within the circles (radii) are collinear with the phase-slowness vectors; the thick lines, normal to the phase-slowness surface correspond to the group-slowness vectors. α, β, and γ are the angles between phase-slowness vectors for incident, reflected and transmitted waves and the normal to the interface. θ_i, θ_r and θ_t are the angles between group-slowness vectors for incident, reflected and transmitted waves and the normal to the interface, i.e., ray angles. In the anisotropic case, α=θ_i, β=θ_r, γ=θ_t.

MATHEMATICAL FORMULATION

The geometrical approach described above for the isotropic case (i.e., spherical slowness surfaces) is easily extended to include more general scenarios, where the slowness surface is an arbitrary surface in slowness space. Although there exist more efficient computational schemes, e.g., Keith and Crampin (1976), Rokhlin et al.

(1986), the following analytical description provides an intuitive insight which is lost in various numerical methods.

Consider two anisotropic media separated by a planar, horizontal interface. Let the phase-slowness surface in the upper medium be given by the level surface of a function $f(x, y, z)$,

$$f(x,y,z)=a. \tag{3}$$

Similarly, let the phase-slowness surface in the lower medium be given by the level surface of a function $g(x, y, z)$,

$$g(x,y,z)=b. \tag{4}$$

A ray is incident on the boundary from above. Since all phase-slowness vectors (for incident, reflected and transmitted waves) must be coplanar, without loss of generality we take them to lie in the xz-plane (see Figure 2).

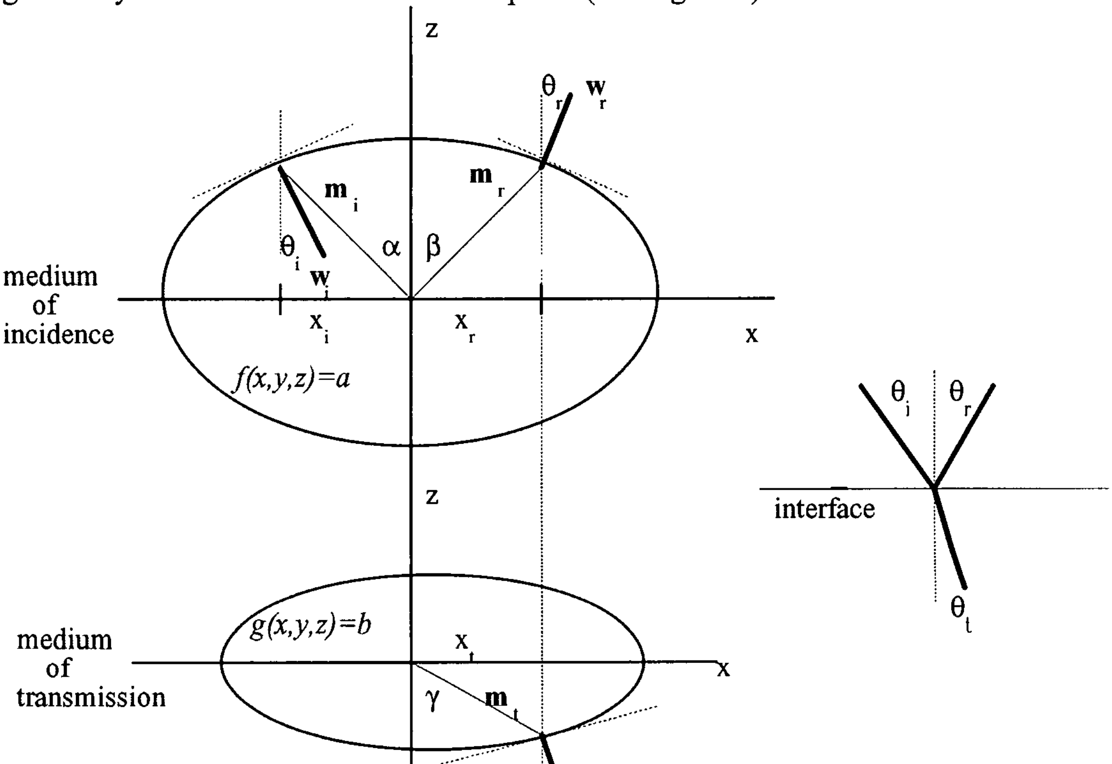

FIG.2. Illustration of ray angles for incident, reflected and transmitted rays in anisotropic media separated by a horizontal, planar interface using phase-slowness surfaces described by functions f and g. **m**'s correspond to phase-slowness vectors and **w**'s to group-slowness vectors and θ's to the ray angles for incident, reflected and transmitted waves. Note that $\alpha\neq\theta_i$, $\beta\neq\theta_r$, and $\gamma\neq\theta_t$; cf. Figure 1.

Denoting the phase-slowness vector as **m**, the continuity conditions require that:

$$\mathbf{m}_i \cdot \bar{\mathbf{x}} = \mathbf{m}_r \cdot \bar{\mathbf{x}} = \mathbf{m}_t \cdot \bar{\mathbf{x}}, \tag{5}$$

where $\mathbf{x}$ is a unit vector in the x-direction and subscripts i, r, and t refer to incident, reflected and transmitted waves respectively. Recall that the group (ray)-slowness vector, $\mathbf{w}$, is normal to the phase-slowness surface at the point where the phase-slowness vector intersects the surface. Using properties of the gradient, i.e., its pointing in the direction along which f is increasing the fastest and its being normal to the surface on which f is constant, gives:

$$\mathbf{w}_i \propto \nabla f(x,y,z)\big|_{(x_i,y_i,z_i)}. \tag{6}$$

Normalizing, and choosing the function f to have a minimum at the origin $O(0,0,0)$ and be monotonicaly increasing outwards, yields

$$\bar{\mathbf{w}}_i = -\frac{\nabla f(x,y,z)\big|_{(x_i,y_i,z_i)}}{\left|\nabla f(x,y,z)\right|\big|_{(x_i,y_i,z_i)}}, \tag{7}$$

where the negative sign ensures that the incident unit ray vector points towards the boundary. The angle of incidence, i.e., the angle between the ray vector and the normal to the interface is given by:

$$\cos\theta_i = \bar{\mathbf{w}}_i \cdot (-\bar{\mathbf{z}}) = \frac{\bar{\mathbf{z}} \cdot \nabla f(x,y,z)\big|_{(x_i,y_i,z_i)}}{\left|\nabla f(x,y,z)\right|\big|_{(x_i,y_i,z_i)}} = \frac{\left.\frac{\partial f}{\partial z}\right|_{(x_i,y_i,z_i)}}{\left|\nabla f(x,y,z)\right|\big|_{(x_i,y_i,z_i)}} \tag{8}$$

Now by choice of the coordinate system, $y_i = 0$; given x_i, z_i is determined by Equation (3). Thus Equation (3) provides an expression for θ_i as a function of x_i. Typically θ_i is taken as an independent parameter; however, it may not always be possible to invert Equation (3) to obtain a closed-form expression for x_i as a function of θ_i. Thus as already emphasized, the present approach is presented chiefly for the intuitive understanding it provides, rather than for its computational convenience.

Similarly, the normalized reflected ray vector can be expressed as:

$$\bar{\mathbf{w}}_r = \frac{\nabla f(x,y,z)\big|_{(x_r,y_r,z_r)}}{\left|\nabla f(x,y,z)\right|\big|_{(x_r,y_r,z_r)}}, \tag{9}$$

and the angle of reflection, i.e., the angle between the ray vector and the normal is given by:

$$\cos\theta_r = \overline{\mathbf{w}}_r \cdot \overline{\mathbf{z}} = \frac{\overline{\mathbf{z}} \cdot \nabla f(x,y,z)\big|_{(x_r,y_r,z_r)}}{\left\|\nabla f(x,y,z)\right\|_{(x_r,y_r,z_r)}} = \frac{\left.\frac{\partial f}{\partial z}\right|_{(x_r,y_r,z_r)}}{\left\|\nabla f(x,y,z)\right\|_{(x_r,y_r,z_r)}} \tag{10}$$

In evaluating the above expression one uses the fact that by continuity $x_r = -x_i$; y_r is zero by the choice of the coordinate system, and z_r can be found by substituting into Equation (3).

Physically, we require that the reflected ray be directed back into the incident medium; thus the physical solutions must have $\overline{\mathbf{w}}_r \cdot \overline{\mathbf{z}} \geq 0$. Note that the reflected ray need not be unique: given the slowness surface in Figure (3), for instance, we have two physical reflected solutions and one non-physical solution.

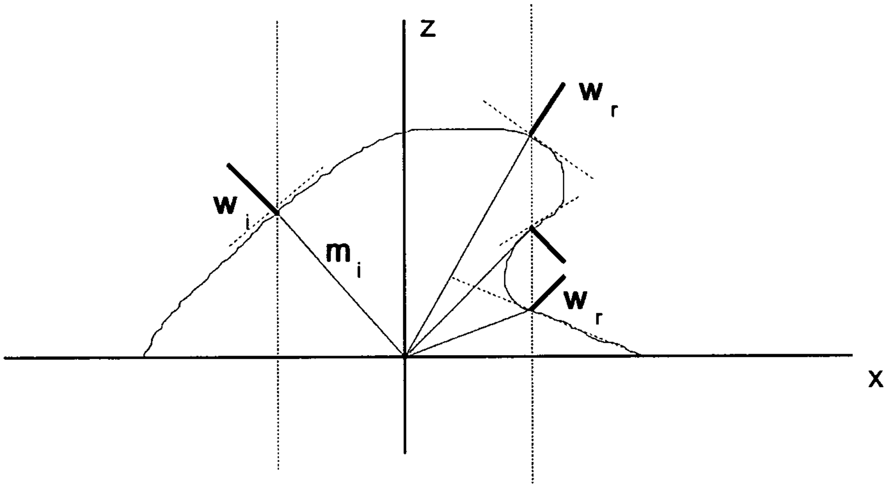

FIG.3. Illustration of the mathematical solution giving three reflected ray vectors. The one pointing towards the interface is not physically realizable. The existence of the remaining two rays depends on the existence of the symmetry system giving such a slowness surface.

The normalized transmitted ray vector is given by:

$$\overline{\mathbf{w}}_t = \frac{\nabla g(x,y,z)\big|_{(x_t,y_t,z_t)}}{\left\|\nabla g(x,y,z)\right\|_{(x_t,y_t,z_t)}}, \tag{11}$$

and thus the angle of the transmitted ray measured between the transmitted ray vector and the normal is given by:

$$\cos\theta_t = (-\overline{\mathbf{z}}) \cdot \overline{\mathbf{w}}_t = -\frac{\overline{\mathbf{z}} \cdot \nabla g(x,y,z)\big|_{(x_t,y_t,z_t)}}{\left\|\nabla g(x,y,z)\right\|_{(x_t,y_t,z_t)}} = \frac{\left.\frac{\partial g}{\partial z}\right|_{(x_t,y_t,z_t)}}{\left\|\nabla g(x,y,z)\right\|_{(x_t,y_t,z_t)}}. \tag{12}$$

In evaluating the above expression one uses the fact that $x_t = -x_i$; y_t is zero by the choice of the coordinate system and z_t can be found by substituting into Equation (4). Similar comments concerning physical realizability and uniqueness apply to transmitted rays; the only difference here is that in the transmitted case, we require that rays be directed into the transmitted medium, i.e., physical solutions must have $\overline{\mathbf{w}}_t \cdot (-\overline{\mathbf{z}}) \geq 0$.

It must be emphasized that, although the phase-slowness vectors, **m**, are coplanar for the incident, reflected and transmitted waves, the ray vectors, **w**, need not lie in the same plane. Their direction is determined by that of the normal to the plane tangent to the phase-slowness surface. They will, however, remain in the same plane if the phase-slowness surfaces are, for instance, rotationally symmetric about the *x*-axis. In all cases the magnitude of the ray velocity is obtained from Equation (2).

Other concepts, such as total internal reflection, also emerge naturally from the present formalism. Although for complicated slowness surfaces it may be impossible to characterize total internal reflection by a single critical angle as in the isotropic case, the general approach remains as described above.

EXAMPLES

The approach described above can be illustrated by several examples. Some cases allow a simple analytical solutions leading to valuable physical insight.

General case of elliptical anisotropy

Let us consider the ellipsoidal case where the velocities are different in the *x*, *y* and *z* directions. Considering the *xz*-plane, the two phase-slowness surfaces can be written as:

$$f(x,z) = (v_x x)^2 + (v_z z)^2 = 1, \tag{13}$$

and

$$g(x,z) = (v'_x x)^2 + (v'_z z)^2 = 1, \tag{14}$$

for the media of incidence and transmission respectively. Again, without loss of generality, we treat a generic case ignoring the mode conversions between different wave types.

Using Equation (7) we can write:

$$\overline{\mathbf{w}}_i = -\frac{[v_x^2 x_i, v_z^2 z_i]}{\sqrt{(v_x^2 x_i)^2 + (v_z^2 z_i)^2}}, \tag{15}$$

and by Equation (8):

$$\cos\theta_i = \frac{v_z^2 z_i}{\sqrt{(v_x^2 x_i)^2 + (v_z^2 z_i)^2}}, \tag{16}$$

Solving for z in Equation (13) yields:

$$z_i = \frac{\sqrt{1-(v_x x_i)^2}}{v_z} \tag{17}$$

in the case of the incident ray. Substituting Equation (17) into Equation (16) gives:

$$\cos\theta_i = \frac{v_z\sqrt{1-(v_x x_i)^2}}{\sqrt{v_x^4 x_i^2 + v_z^2[1+(v_x x)^2]}}, \tag{18}$$

which can be explicitly solved for x_i^2:

$$x_i^2 = \frac{v_z^2(1-\cos^2\theta_i)}{v_x^4\cos^2\theta_i + v_x^2 v_z^2(1-\cos^2\theta_i)}. \tag{19}$$

Analogous expressions can be derived for reflected and transmitted rays. Furthermore, since $x_i^2 = x_r^2 = x_t^2$, the three expressions can be equated, thus giving Snell's law for ellipsoidally anisotropic media. Upon some algebraic manipulation this can be expressed as:

$$v_x^2[\frac{v_x^2}{v_z^2}\cot^2\theta_i + 1] = v_x^2[\frac{v_x^2}{v_z^2}\cot^2\theta_r + 1] = v_x'^2[\frac{v_x'^2}{v_z'^2}\cot^2\theta_t + 1], \tag{20}$$

where primed quantities refer to the medium of transmission. Equivalent expressions for Snell's law with elliptical anisotropy were obtained by Dunoyer de Segonzac and Laherrere (1959). Examining Equation (20) we see that $\theta_r = \theta_i$. Also, the critical angle can be obtained by setting $\theta_t = \pi/2$. After some algebraic manipulation one gets:

$$\cot\theta_c = \frac{v_z}{v_x}\sqrt{\frac{v_x'^2}{v_x^2} - 1} \tag{21}$$

For isotropic media one can write:

$$v_x = v_z = v_1, \tag{22}$$

and:

$$v_x' = v_z' = v_2. \tag{23}$$

Equation (20) then reduces to Equation (1), i.e., the standard form of Snell's law in isotropic media and Equation (21) to the standard expression for the critical angle in the isotropic medium.

A particular case of isotropic/anisotropic interface

Let us consider a planar boundary between the isotropic and elliptically anisotropic media. Let the velocities be so chosen that:

$$v_x = v_z \equiv v = v_x', \tag{24}$$

i.e., the horizontal velocity in the anisotropic medium equals the velocity in the isotropic medium. Using Equation (20) gives:

$$\tan\theta_t = \frac{v}{v_z'}\tan\theta_i. \tag{25}$$

An interesting phenomenon can be observed by examining Equation (25). For small angles one can write:

$$\theta_t \approx \frac{v}{v_z'}\theta_i. \tag{26}$$

Thus if $v_z' > v$, the transmitted ray is bent towards the normal, which is the opposite of what happens in the isotropic case. This phenomenon is related to the complicated form Fermat's principle takes in the anisotropic case, as discussed below.

DISCUSSION

Numerous physical consequences can be described using the approach presented above. First of all, however, mathematical solutions stemming from this formulation must be examined in the light of physical realizability. The ray vector for an incident ray must be pointing towards the boundary, while for reflected and transmitted rays must point away from it. This physically intuitive requirement is not satisfied naturally by the mathematical formalism. Employing either the numerical approach stemming from tensor analysis (Rokhlin et al., 1986) or the analytical approach described above, one must select correct ray vectors and reject the ones which fail to satisfy the obvious physical requirements.

Since, in general, the slowness surface is not represented by a single-valued function, i.e., several points on the slowness surface can correspond to the same value of x, there is a non-uniqueness by which a given incident ray could generate several reflected or transmitted rays of the same wave type, i.e., originating from the same slowness surface. This apparent or actual non-uniqueness has to be investigated in the light of the existence of such slowness surfaces and the consequences for reflection and transmission coefficients.

As already mentioned above, an interesting phenomenon related to Fermat's principle of stationary time can be observed by applying the small-angle approximation to Equation (20):

$$\theta_t \approx \frac{v_x'^2}{v_x^2}\frac{v_z}{v_z'}\theta_i . \tag{27}$$

In the isotropic case if, say, the velocity in the medium of transmission is greater than in the medium of incidence, we obtain the familiar result that the transmitted ray is bent away from the normal. However in the particular case of isotropic/anisotropic interface considered in Equation (26), we obtain the opposite result. Equation (27) gives the behavior in the case of general ellipsoidal anisotropy.

The behaviour in the isotropic case may be intuitively understood as being the consequence of Fermat's principle of stationary time; the behaviour in the particular anisotropic case, considered in Equation (26), appears counterintuitive when viewed from this point of view. However, the form of Fermat's principle in the general anisotropic case can be formulated as (Helbig, 1994):

$$\delta(\int \mathbf{m}\cdot \mathbf{dw}) = \delta(\int \mathbf{m}_w \cdot \mathbf{dw}) = 0 . \tag{28}$$

where $\mathbf{m}$ is the phase-slowness vector, $\mathbf{m}_w$ is the ray-slowness vector and $\mathbf{dw}$ is a length element along the ray. Thus in deriving Snell's law we are minimizing the traveltime along the ray with respect to the group (ray) velocity, rather than the phase velocity. As a consequence the group- (ray-) slowness surface as well as the phase-slowness surface must be considered in order to understand the behaviour of the ray at the interface. Therefore, in the general anisotropic case, ray bending does not lend itself to such an intuitive understanding as in the isotropic case. In the latter case, the phase and group velocities are collinear, and the ray bending away from the normal when it passes from a slower to a faster medium is easily understood as a consequence of Fermat's principle which favours shortening the distance traveled by the ray in the slower medium.

CONCLUSIONS

An analytical way for determining the angles of incidence, reflection and transmission rays was derived. This analytical formulation provides a description of some physical phenomena including the concepts of the critical angle and implications of Fermat's principle. A case of ellipsoidal anisotropy is considered explicitly and formulæ for Snell's law and hence the critical angle are derived. The usefulness of the above approach would be extended if one could easily express the slowness surfaces for various materials as functions of three variables x, y and z. Most commonly the slowness surfaces are derived from the Christoffel equation which is a form of a wave equation for plane waves in elastic media (Auld, 1973).

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