Theory of nonstationary linear filtering in the Fourier domain with application to time variant filtering

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ABSTRACT

A general linear theory is presented which describes the extension of the convolutional method to nonstationary processes. Two alternate extensions are explored. The first, called *nonstationary convolution*, corresponds to the linear superposition of scaled impulse responses of a nonstationary filter. The second, called *nonstationary combination*, does not correspond to such a superposition but is shown to be a linear process capable of achieving arbitrarily abrupt temporal variations in the output frequency spectrum. Both extensions have stationary convolution as a limiting form.

The theory is then recast into the Fourier domain where it is shown that stationary filters correspond to a multiplication of the input signal spectrum by a diagonal filter matrix while nonstationary filters generate off-diagonal terms in the filter matrix. The width of significant off-diagonal power is directly proportional to the degree of nonstationarity. Both nonstationary convolution or combination may be applied in the Fourier domain, and for quasi-stationary filters, efficiency is improved by using sparse matrix methods.

Unlike stationary theory, a third domain which combines time and frequency is also possible. Here, nonstationary convolution expresses as a generalized forward Fourier integral of the product of the nonstationary filter and the time domain input signal. The result is the spectrum of the filtered signal. Nonstationary combination reformulates as a generalized inverse Fourier integral of the product of the spectrum of the input trace and the nonstationary filter which results in the time domain output signal. The mixed domain is an ideal domain for filter design which proceeds by specifying the filter as an arbitrary complex function on a time-frequency grid. Explicit formulae are given to move nonstationary filters expressed in any one of the three domains into any other.

INTRODUCTION

A common occurrence in geophysical research and data processing is the need to apply convolutional operators which somehow depend on both variables of a Fourier transform pair. Time variant filtering is a typical example. Filters are convolutional operators which shape the spectrum of a time series therefore a time variant filter must both shape the spectrum and change with time. Another example is the vertical extrapolation of a wavefield through a laterally variable velocity structure. The kinematics of wavefield extrapolation can be handled by a phase shift which depends on horizontal wavenumber and velocity, or equivalently by a spatial convolution over the lateral coordinate. Thus, when velocity varies laterally, a convolution is desired which depends on both the lateral coordinate and the horizontal wavenumber.

Ordinary convolutional filters are incapable of directly handling these and other similar situations since they assume a "stationary" impulse response. By stationary it is meant that the filter's properties do not change with time or space. Since the convolution theorem (see any good text on signal processing, for example Karl, 1989, p 88, or Brigham ,1974, p 58) states that stationary convolution is a multiplication of Fourier spectra, it is commonly assumed that Fourier methods are also incapable of handling nonstationary filters. However, a more fundamental result is that a continuous function's Fourier transform is a complete description of the function. It follows that if nonstationary filtering can be done at all, it can be done in the Fourier domain.

Nonstationary filtering is done routinely in seismic data processing (and elsewhere). Wavefield extrapolation through laterally varying media (i.e. depth migration) is typically done with either spatially varying finite difference techniques (see Claerbout 1985) or spatially variant convolutional operators (Berkhout 1985, section X). Gazdag and Squazzero (1984) have accomplished wavefield extrapolation in this setting with the PSPI method which amounts to lateral interpolation between wavefields extrapolated with stationary Fourier phase shifts.

Time variant filtering has been implemented in various ways. The simplest method is to apply stationary filters to different overlapping trace segments and to interpolate them into a unified result (Yilmaz, 1986, p25-26). Pann and Shin (1976) show that the interpolation results in an embedded spectrum which is not the desired one in the overlap zones. Pann and Shin (1976) and later Scheuer and Oldenburg (1988) implemented time varying bandpass filters with an efficient algorithm which uses the theory of complex signals (Taner et al. 1979) as a basis. These methods achieve a filter with a continuously variable pass band and a possible nonstationary phase rotation. However, they are limited in the shape of the amplitude and phase spectra and the rate of time variation. Park and Black (1995) present an excellent summary of previous work as well as a new method based upon Fourier transform scaling laws. Their method is able to achieve a more rapid and flexible temporal variation of the bandpass characteristics. A third class of time variant filters are implemented as recursive algorithms (Stein and Bartley, 1983). These methods can be quite strongly nonstationary but have only very limited phase options and, for short recursions, may produce amplitude spectral responses far from the desired ones (see Park and Black, 1995, or Scheuer and Oldenburg, 1988 for summaries).

Also important in this context are nonstationary filters based on nonstationary transforms such as the wavelet transform (Chakraborty and Okaya, 1994, Kabir et al., 1995, Chakraborty and Okaya, 1995), the short time Fourier transform, or the Gabor transform (Gabor, 1946). (For a general discussion on nonstationary transforms see Kaiser, 1994). These methods are capable of achieving quite general filtering effects but their acceptance has been slow for a number of reasons including: complex mathematics, slow or unavailable software, and difficulty relating the theory to ordinary stationary theory. These problems are gradually lessening and there will certainly be increased usage of nonstationary transforms in the next few years. Nonstationary transform techniques are perhaps most important when the nonstationary filter must be determined from a spectral analysis of the data. However, there are many situations when the filter parameters are either known apriori (as in depth migration) or they are easily estimated without a nonstationary transform (as is the often case with time variant filtering). Given such an apriori nonstationary filter specification, it is shown here that the filter can be efficiently applied in the time domain or the ordinary (stationary) Fourier domain by a generalization of convolutional concepts. It may be sensible to use nonstationary transforms to design filters which are then applied with the techniques developed here.

This paper proposes a general mathematical theory for nonstationary filtering. The basis for the theory is a new extension of the stationary convolutional integral such that it contains stationary filtering as an obvious limit and that it forms the scaled superposition of nonstationary impulse responses. This process is termed *nonstationary convolution*. In addition, an alternate extension for the stationary convolution integral is developed which is called *nonstationary combination* and, though it does not correspond to the scaled superposition of impulse responses, it still has stationary convolution as a limiting form. Unlike nonstationary convolution, nonstationary combination is found to be capable of producing an output spectrum which varies in time with arbitrary abruptness. The distinction between nonstationary convolution and combination is that the former considers the temporal variation of the filter to be in terms of "input time" while the latter assumes "output time".

Both processes are then reformulated in the Fourier domain. When represented as matrix operations appropriate for digital filters, stationary filters are shown to be achieved as the multiplication of the input signal spectrum by a diagonal matrix which has the filter spectrum on the diagonal. As the filter becomes nonstationary, the spectral filter matrix generates off diagonal terms to describe the filter variation. The width of the significant off diagonal power is directly proportional to the degree of nonstationarity. Quasi-stationary filters may be implemented efficiently in the Fourier domain by sparse matrix methods which keep only the "significant" spectral components.

In addition to these two traditional domains for filter application, two mixed domains of time and frequency emerge. Nonstationary convolution is recast as a generalized forward Fourier integral of the product of the nonstationary filter and the time domain signal which yields the spectrum of the filtered trace. Alternatively, nonstationary combination may be expressed as a generalized inverse Fourier integral of the product of the nonstationary filter and the spectrum of the input trace. This results in a time domain output signal.

This theory is capable of applying any nonstationary filter with arbitrary time and frequency variation of the amplitude and phase spectra. The three possible application domains as well as the two possible application methods (i.e. convolution and combination) allow considerable latitude in filter optimization for both performance and efficiency. The simple and natural connection of this theory with ordinary stationary filter theory allows stationary filter design techniques and concepts such as minimum phase to be easily extended into the nonstationary realm.

PRINCIPLES OF NONSTATIONARY LINEAR FILTERING

Generalization of stationary convolution

It is well understood that stationary convolutional filters are completely described by their impulse response (Papoulis, 1984, p. 15-18). This means that if the response of a linear, stationary process to a unit impulse input at any particular time is known, then the response to an impulse at any other time is identical, except for a causal delay (principle of stationarity), and a scale factor. The response to more complicated inputs is the scaled superposition of many identical impulse responses. The process of forming the scaled superposition is called convolution. If $h(\tau)$ represents an input signal and $a(u)$ is an arbitrary linear stationary filter, then $g(t)$, the filtered output, is given by the convolutional integral

$$g(t) = \int_{-\infty}^{\infty} a(t-\tau)h(\tau)d\tau \equiv a(t) \bullet h(t) \qquad (1)$$

Certainly this is a familiar expression to most readers; however, it is presented again here so that the nonstationary results can be seen as reasonable generalizations of the stationary case. The presence of $a(t-\tau)$ in (1) means that the actual response of the filter to an impulse at time τ_0 is $a(t-\tau_0)$ which is a delayed version of the filter; the delay being a consequence of a causality constraint. Thus, though a(u) is called the filter impulse response, the actual response to a general impulse is $a(t-\tau_0)$. (It is a simple matter to prove that $h(t) \bullet a(t) = a(t) \bullet h(t)$ and thus h(t) could equally well be delayed in (1); though, the form of (1) is better suited to generalization for nonstationary filters.)

The generalization of (1) to nonstationary systems can be done in a number of ways though some are more intuitively appealing than others. Pann and Shin (1976) and Scheuer and Oldenburg (1988) have chosen to replace $a(t-\tau)$ with $\hat{a}(t,\tau)$ which refers to an arbitrary real function of t and τ. Though correct, this formalism is perhaps overly general and does not suggest a simple relation to the stationary form (1). Instead, consider replacing $a(t-\tau)$ with $a(t-\tau,\tau)$. This notation preserves the concept of delaying the filter response to account for causality and incorporates explicit τ dependence to describe the filter variation with time. Also the stationary limit is simply obtained by letting the τ dependence become constant. Alternatively, $a(t-\tau)$ could also be replaced with $a(t-\tau,t)$ with similar appeal. Intuitively, $a(t-\tau,\tau)$ and $a(t-\tau,t)$ differ in that the former prescribes the temporal variation of the nonstationary filter as a function of input time, τ, while the latter uses output time, t, for the same purpose. The choice of which form to use and the comparison of results which follow from both is a central theme of this paper.

Light can be cast on this issue by considering the matrix equivalent to discrete convolution. Given sampled versions of g(t), h(t), and a(t), then the convolutional process can be represented as

$$g_k = \Delta t \sum_j a_{k-j} h_j \qquad (2)$$

Here Δt is the temporal sample interval and the range of summation is left implied and is assumed to be over all appropriate values. This expression can be recast as a matrix operation by representing g and h as column vectors and building a special "convolutional matrix" with a(t) (see Strang, 1986, for an excellent discussion). As shown in equation (3), such convolutional matrices have a high degree of symmetry and are known as "Toeplitz" matrices. There are a number of equivalent ways to construct a Toeplitz matrix. First, each sample of a(t) can be replicated along the appropriate diagonal. Alternately, a(t) can be replicated in each column with zero time always being shifted to the main diagonal. Lastly, a(t) can be time reversed and replicated in each row again with zero time shifted to the diagonal. Each of these methods leads to the construction of the same Toeplitz matrix; however, all three symmetries cannot be retained in a nonstationary generalization. Figure 1 is a graphical representation of equation (3) for the case of the convolution of a minimum phase wavelet with a reflectivity sequence to produce a simple "seismogram". (In this figure and all other similar ones in this paper, a gray scale is used which maps black to a large positive number, white to the negative of the same number, and medium gray to zero.)

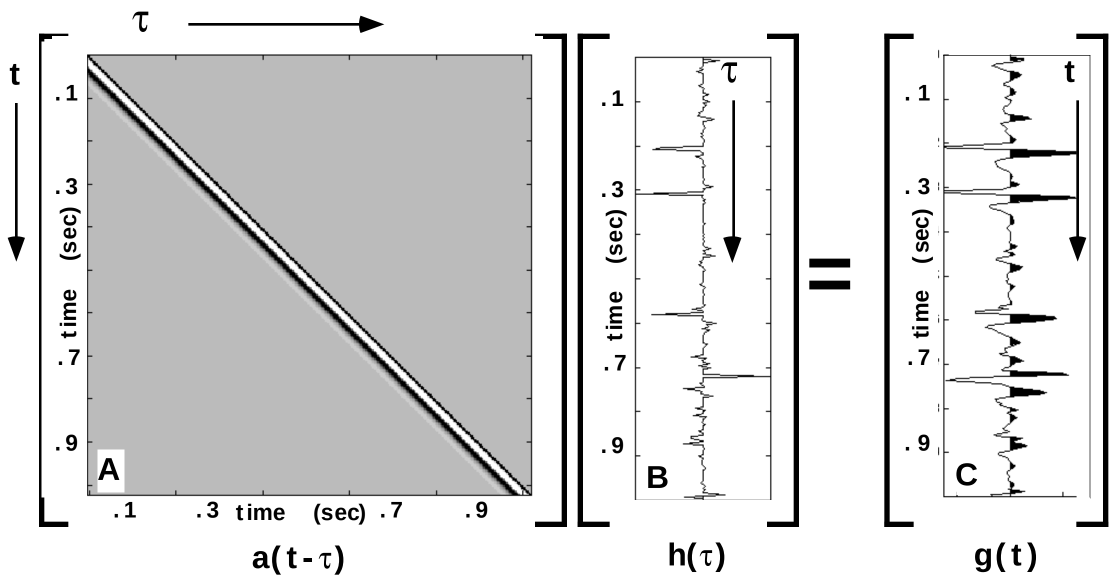

Fig. 1. An illustration of stationary convolution as a time domain matrix operation. (A) is the stationary convolution matrix for a particular minimum phase bandpass filter. The matrix displays Toeplitz symmetry meaning that each column contains the filter impulse response, each row contain the time reverse of the impulse response, and any diagonal is constant. (B) is a reflectivity series in time to which the convolution matrix is applied. (C) is the output stationary seismogram.

$$\begin{bmatrix} \vdots \\ g_0 \\ g_1 \\ g_2 \\ g_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_{-1} & a_{-2} & \vdots \\ \vdots & a_1 & a_0 & a_{-1} & \vdots \\ \vdots & a_2 & a_1 & a_0 & \vdots \\ \vdots & a_3 & a_2 & a_1 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ h_0 \\ h_1 \\ h_2 \\ h_3 \\ \vdots \end{bmatrix} \tag{3}$$

In order to understand how to generalize convolution such that it models physical nonstationary systems, it is instructive to examine the matrix multiplication in equation (3). Beginning with g_0 and performing the matrix multiplication for the first few terms

$$\begin{aligned} g_0 &= \cdots + a_0 h_0 + a_{-1} h_1 + a_{-2} h_2 + \cdots \\ g_1 &= \cdots + a_1 h_0 + a_0 h_1 + a_{-1} h_2 + \cdots \\ g_2 &= \cdots + a_2 h_0 + a_1 h_1 + a_0 h_2 + \cdots \\ &\vdots \end{aligned} \tag{4}$$

Examination of these equations shows that g_k is computed by multiplying the kth row of the convolution matrix times [h] in an element by element fashion and adding the products. This is the familiar process of matrix multiplication "by rows". Also, it is a simple matter to check that equation (2) evaluates to the set (4). Less familiar, but fundamental to this analysis, is matrix multiplication "by columns" (Strang, 1986). Examination of the set (4) shows that each sample of h_j multiplies a column of the convolution matrix and the set is equivalent to

$$\begin{bmatrix} \vdots \\ g_0 \\ g_1 \\ g_2 \\ g_3 \\ \vdots \end{bmatrix} = \cdots + \begin{bmatrix} \vdots \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} h_0 + \begin{bmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} h_1 + \begin{bmatrix} \vdots \\ a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{bmatrix} h_2 + \cdots \tag{5}$$

Thus, each sample of h(t) is used to scale a time shifted version of a(t) and the set of scaled and shifted waveforms are then superimposed. This is matrix multiplication "by columns" and it is equivalent to the familiar process of convolution by replacement which is the numerical analog of the scaled superposition of impulse responses. This is a desirable property to preserve in a nonstationary generalization since it can be regarded as a direct consequence of Green's function analysis for linear partial differential equations (Morse and Feshbach, 1953).

The fundamental difference between stationary and nonstationary linear filters is that the impulse response of the latter must be allowed to vary arbitrarily with time. The complete description of a general nonstationary filter requires that its impulse response be known for any and all times. Given such a description, it is easy to see how to modify equations (5) to apply it as the scaled superposition of time varying impulse responses. Each column vector on the right hand side becomes the impulse response of the filter at the time corresponding to value of h_j which scales the column. For the stationary filter, these impulse responses are all the same except for a time shift, while for the nonstationary filter they vary as prescribed in the filter description. If a second subscript is attached to a, the nonstationary filter application can be written

$$\begin{bmatrix} \vdots \\ g_0 \\ g_1 \\ g_2 \\ g_3 \\ \vdots \end{bmatrix} = \cdots + \begin{bmatrix} \vdots \\ a_{0,0} \\ a_{1,0} \\ a_{2,0} \\ a_{3.0} \\ \vdots \end{bmatrix} h_0 + \begin{bmatrix} \vdots \\ a_{-1,1} \\ a_{0,1} \\ a_{1,1} \\ a_{2,1} \\ \vdots \end{bmatrix} h_1 + \begin{bmatrix} \vdots \\ a_{-2,2} \\ a_{-1,2} \\ a_{0,2} \\ a_{1,2} \\ \vdots \end{bmatrix} h_2 + \cdots \tag{6}$$

Obviously, equations (6) can be written as a single matrix operation similar to (4) in which the impulse response of the filter is contained in each column. A graphical representation of this matrix product is shown in figure 2 for the case of the application of a forward Q filter. The Q filter was constructed using the constant Q theory of Kjartansson (1979) and has been further bandlimited by the minimum phase waveform used in figure 1. Inspection of the nonstationary convolution matrix shows a waveform in each column which is progressively losing overall amplitude and high frequency content while undergoing a phase rotation. In the case of a Q filter, the frequency decay and phase rotation are linked by the minimum phase condition (Kjartansson, 1979 and Futterman, 1962).

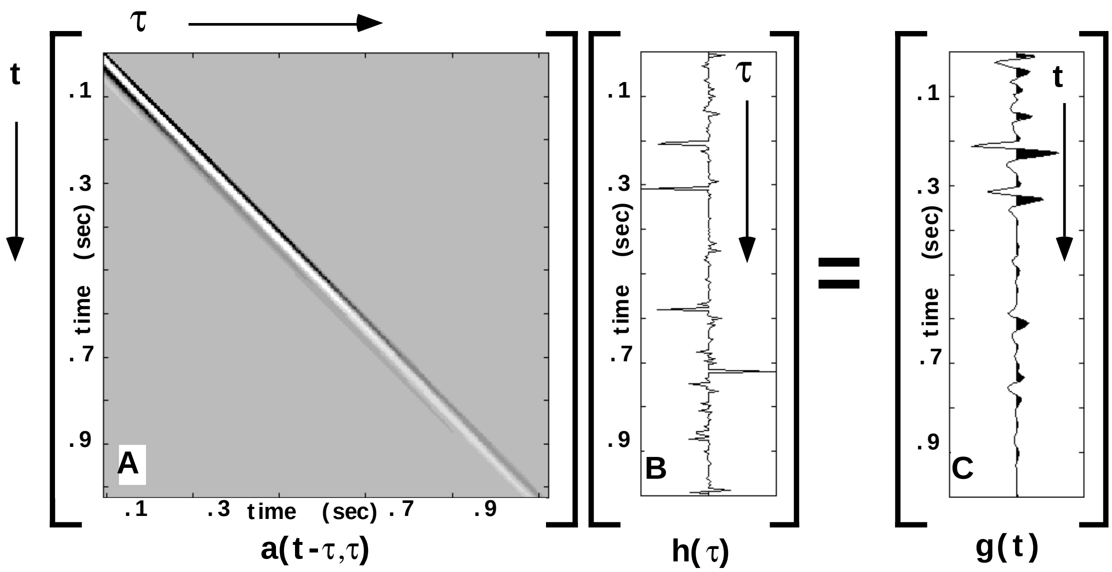

Fig. 2. An illustration of nonstationary convolution as a time domain matrix operation. (A) is the nonstationary convolution matrix for a particular forward Q filter bandlimited by the stationary waveform of figure 1. Each column contains the convolution of the minimum phase waveform of figure 1 and the minimum phase impulse response of a constant Q medium for a traveltime equal to the column time. (B) is a reflectivity series in time to which the convolution matrix is applied. (C) is the output constant Q seismogram. Compare to figure 1.

Nonstationary convolution and combination

Thus, conceptually at least, it is a simple matter to apply an arbitrary nonstationary filter via a matrix multiplication in the time domain. This operation is a direct extension of stationary convolution and will be referred to as a generalized or nonstationary convolution. Equation (2) can be modified to express (6) as

$$g_k = \Delta t \sum_j a_{k-j,j} h_j \tag{7}$$

Equation (7) suggests that the convolution integral of equation (1) should be generalized to

$$g(t) = \int_{-\infty}^{\infty} a(t-\tau,\tau)\, h(\tau)\, d\tau \tag{8}$$

Thus replacing $a(t-\tau)$ with $a(t-\tau,\tau)$ in equation (1) is seen to preserve the notion of a scaled superposition of impulse responses. The other possible form alluded to previously is

$$\bar{g}(t) = \int_{-\infty}^{\infty} a(t-\tau,t)\, h(\tau)\, d\tau \tag{9}$$

The matrix equivalent to (9) places the filter impulse response (time reversed) in the rows of the "convolution" matrix and the result does not correspond to the desired scaled superposition. However; it will be shown that (9) has interesting properties which may be of considerable utility in a data processing scheme. Equation (8) is called *nonstationary convolution* while (9) will be termed *nonstationary combination* . Both are a type of *nonstationary filtering.*

The nonstationary impulse response function

Equations (8) and (9) imply the existence of a nonstationary impulse response function which is a generalization of a(u). Let this function be called a(u,v) with u symbolizing the time axis of a particular impulse response and v denoting the time axis tracking the variation of the impulse form. If a(u,v) has no v dependence, then the stationary limit is obtained. Like a(u) it is conceptualized without the causal delay as shown in figure 3. For any input time, v, the matrix equivalent to a(u,v) contains the non-delayed impulse response as a function of u in the v^{th} column. Equation (8) employs a(u,v) by letting $u = t - \tau$ and $v = \tau$ while (9) uses $u = t - \tau$ and $v = t$. Thus insertion of a(u,v) into (8) or (9) incorporates the causal delay and achieves a nonstationary convolution or combination.

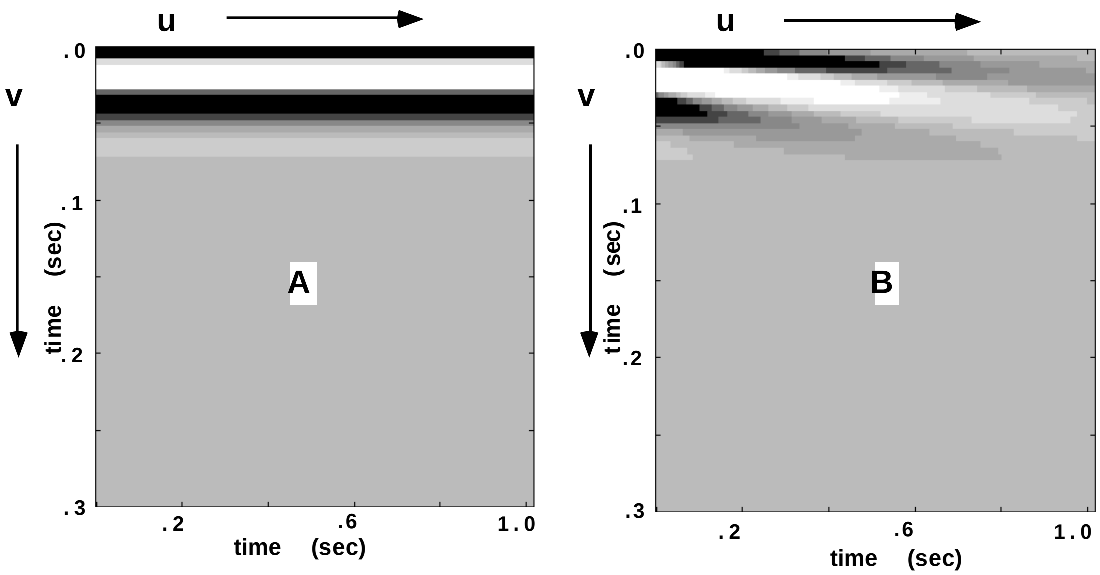

Fig. 3. The impulse response matrices, a(u,v), for (A) figure 1 and (B) figure 2. (A) is a stationary minimum phase bandpass filter and is therefore invariant with u. (B) is a minimum phase constant Q impulse response matrix (bandlimited by the waveform in (A)) that varies strongly with u in both amplitude and phase. Note that the vertical timing scales are considerably enlarged relative to figures 1 and 2. The matrices in figures 1 and 2 are formed by delaying each column of (A) and (B) respectively such that the time zero is on the main diagonal.

An advantage of this formulation over that of Pan and Shin (1976) or Scheuer and Oldenburg (1988) is that a(u,v) is a well defined filter response whose properties follow immediately from stationary filter theory. Since a(u,v = constant) is an ordinary impulse response, it can be dealt with using the stationary theory. For example, if it is desirable that a(u,v) have the minimum phase property, then it is sufficient to ensure that the phase and log amplitude spectra of a(u,v = constant) are related by the Hilbert transform (Karl, 1989).

Reformulation in the Fourier domain

Though easily formulated, the time domain methods may not always be optimal for reasons of computational efficiency and filter design. Given the speed of the fast Fourier transform, it is often advantageous to perform stationary filtering in the frequency domain so it is reasonable to expect that a large class of "quasi-stationary" processes will benefit from a frequency domain formulation. Since there are two times, t and τ, in equations (8) and (9) there will be two corresponding frequencies, f and F.

The time and frequency of the input signal h are denoted (τ,F) and those of the output signal are (t,f).

The fundamental results of the reformulation of equations (8) and (9) into the (f,F) domain and the two mixed domains (t,F) and (τ,f) are summarized and discussed below. The detailed derivations are presented in the appendix.

When transformed into the dual frequency domain of (f,F), the nonstationary convolution operation (8) becomes

$$G(f) = \int_{-\infty}^{\infty} H(F)\, A(f, f-F)\, dF \tag{10}$$

where G(f) and H(F) are the ordinary Fourier spectra of g(t) and h(τ) respectively and are given by

$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-2\pi i f t} dt \tag{11}$$

$$H(F) = \int_{-\infty}^{\infty} h(\tau)\, e^{-2\pi i F \tau} d\tau \tag{12}$$

and A(p,q) is the *frequency connection function* which is the 2-D Fourier transform (spectrum) of a(u,v)

$$A(p,q) = \iint_{-\infty}^{\infty} a(u,v) e^{-2\pi i p u} e^{-2\pi i q v} du\, dv \tag{13}$$

Similarly, nonstationary combination (9) can be moved into the Fourier domain giving

$$\bar{G}(f) = \int_{-\infty}^{\infty} H(F)\, A(F, f-F)\, dF \tag{14}$$

where H(F) and A(p,q) are as given in (12) and (13) and $\bar{G}(f)$ is the Fourier transform of $\bar{g}(t)$ in analogy with (11).

Comparing equation (10) with the time domain expression of nonstationary convolution (8) shows that they are formally similar. Also (14) is similar to (9). In fact, when the nonstationary filter is a nonstationary convolution in one domain it is a nonstationary combination in the other domain. Equation (10) states that nonstationary convolution can be achieved by a forward Fourier transform of h(t), a nonstationary combination in frequency, and an inverse Fourier transform to yield g(t). The nonstationary combination function in the frequency domain, A(f,f-F), is a frequency shifted version of the 2-D Fourier transform of the nonstationary impulse response function a(u,v). The meaning of equation (14) is similar.

Figure 5 shows the nonstationary Q filter of figure 2 being applied in the Fourier domain via equation (10). Using the same mathematical formalism to move the stationary operation of figure 1 into the Fourier domain results in figure 4. (In both figures 4 and 5, only the amplitude spectrum of the complex frequency connection function is depicted.) The matrix in figure 4 is purely diagonal and, if the diagonal were displayed in profile, it would show the amplitude spectrum of the stationary filter. The matrix in figure 5 is non-zero everywhere but contains significant power only near the main diagonal. Any linear measure of the width of the off diagonal energy will be inversely proportional to the time scale over which significant nonstationarity occurs. That is, it will be directly proportional to the degree of nonstationarity.

The stationary limit

In the stationary limit, $a_{stat}(u,v) = a(u)$, and (13) reduces to

$$A_{stat}(p,q) = \int\int_{-\infty}^{\infty} a(u) e^{-2\pi i p u} e^{-2\pi i q v} du\, dv = A(p) \int_{-\infty}^{\infty} e^{-2\pi i q v} dv = A(p)\, \delta(q) \qquad (15)$$

where $\delta(q)$ is the Dirac delta function and $A(p)$ is the Fourier spectrum of $a(u)$. Insertion of this result in (10) or (14) collapses the F integration to yield

$$G(f) = A(f)\, H(f) \qquad (16)$$

which is the expected stationary result.

Similarly, if $A(p,q) = A(q)$ then equation (10) expresses stationary convolution in the Fourier domain. In this case $a(u,v) = \delta(u)\, a(v)$ (see equation (25)) and equations (8) or (9) collapse to simple multiplications.

Thus nonstationary convolution and combination both have the same stationary limit which is the stationary convolution theorem (equation (16)). In other words, if $a(u,v)$ shows no variation with v, then the Fourier transform over v yields only a dc (i.e. 0 Hz.) term in q. When this is substituted into (10), the matrix corresponding to $A(f,f\text{-}F)$ is diagonal with the dc term along the diagonal (figure 4). If $a(u,v)$ becomes nonstationary (i.e. varies with v), then the matrix form of $A(f,f\text{-}F)$ generates off diagonal terms which describe the nonstationarity (figure 5).

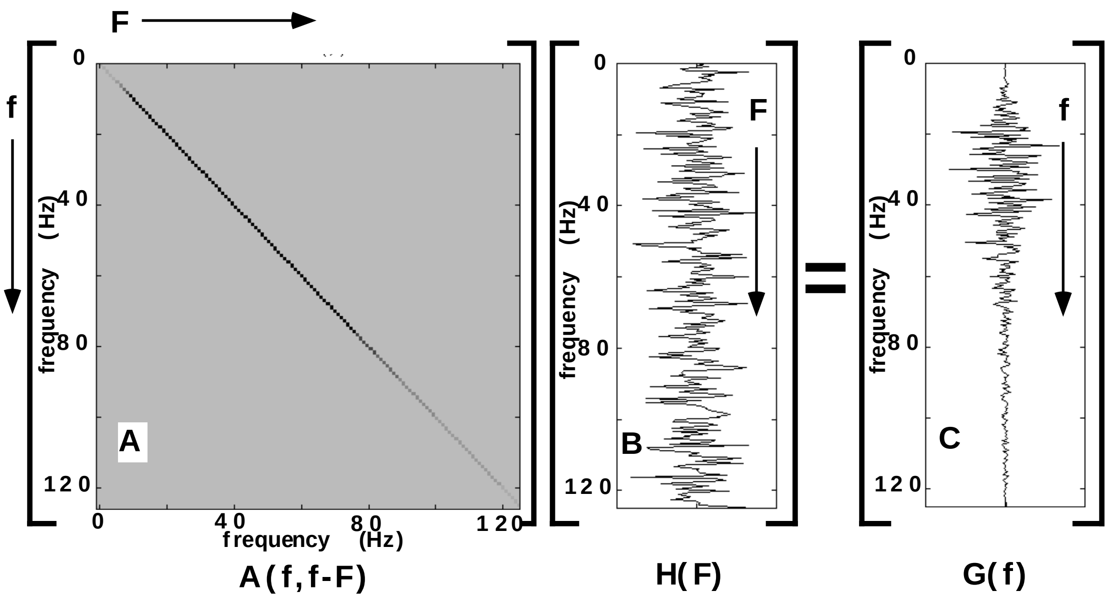

Fig. 4. The application of the stationary filter of figure 1 in the Fourier domain using the formalism of equations (10) and (13). (A) is the frequency connection matrix (only the amplitude spectrum is shown), (B) is the complex spectrum of the input trace, and (C) is the complex spectrum of the output trace. The matrix (A) is purely diagonal which is a direct consequence of the fact that the impulse response matrix (figure 3 (A)) is stationary. The spectra (B) and (C) are each related to their time domain equivalents in figure (1) by an ordinary inverse Fourier transform.

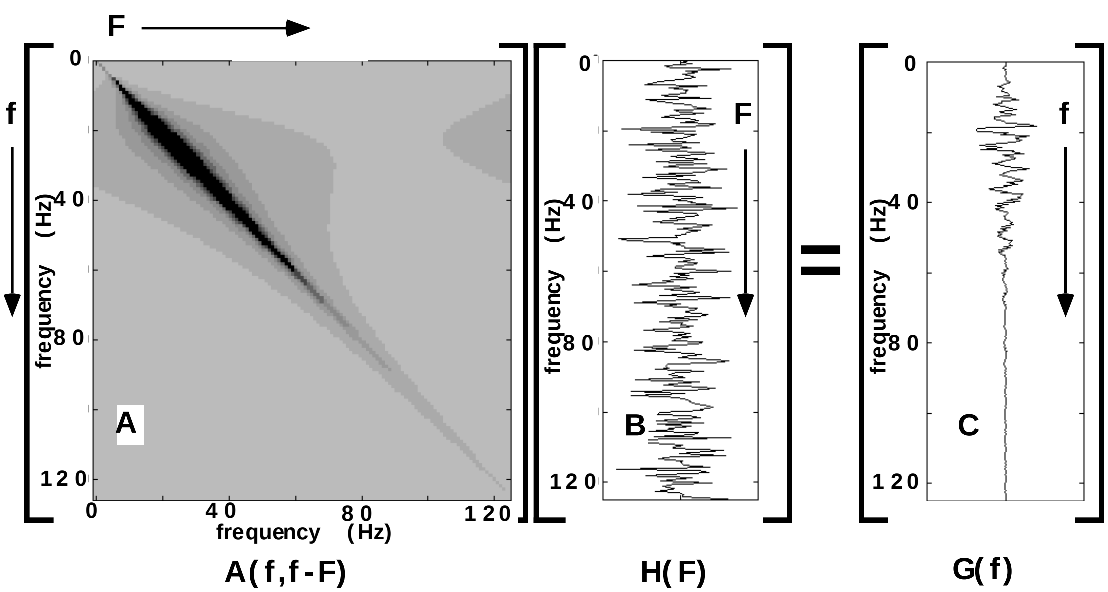

Fig. 5. The application of the nonstationary forward Q filter of figure 2 in the Fourier domain using the formalism of equations (10) and (13). (A) is the frequency connection matrix (only the amplitude spectrum is shown), (B) is the complex spectrum of the input trace, and (C) is the complex spectrum of the output trace. (A) is computed from the 2-D Fourier transform of the impulse response matrix (figure 3 (B)) and the off diagonal energy arises because the later matrix is nonstationary. The spectra (B) and (C) are each related to their time domain equivalents in figure (2) by an ordinary inverse Fourier transform. Compare with figure 4.

Mixed domain formulations

In addition to the pure time and pure frequency domain formulations, two mixed domain representations of nonstationary filtering are possible. Nonstationary convolution is most naturally expressed in the (f,τ) domain while nonstationary combination has asimple appearance in the (t,F) domain. As shown in the appendix, nonstationary convolution (equations (8) or (10)) can be expressed in (f,τ) as

$$G(f) = \int_{-\infty}^{\infty} \alpha(f,\tau)\, h(\tau)\, e^{-2\pi i f \tau} d\tau \tag{17}$$

while nonstationary combination (equations (9) or (14)) in (t,F) becomes

$$\bar{g}(t) = \int_{-\infty}^{\infty} \alpha(F,t)\, H(F)\, e^{2\pi i F t} dF \tag{18}$$

where $\alpha(p,v)$ is the *nonstationary transfer function* given by

$$\alpha(p,v) = \int_{-\infty}^{\infty} a(u,v)\, e^{-2\pi i p u} du \tag{19}$$

The frequency dependence of $\alpha(p,v)$ is simply the Fourier transform of each column of a(u,v). Thus it gives the filter spectrum directly as a function of the time. The stationary limit is found by $\alpha(p,v) = A(p)$, which inserted into (17) leads to the simple multiplication of spectra, and when inserted into (18) leads to the inverse Fourier transform of a spectral multiplication. Thus equations (17) and (18) are generalized Fourier integrals which achieve nonstationary filtering.

Figure 6 depicts the application of the stationary bandpass example of figures 1 and 4 using equation (17). Figure 7 shows the corresponding result for the nonstationary Q filter example of figures 2 and 5. Neither of these figures is a true equation since they both omit a graphical representation of the Fourier exponential in equation (17).

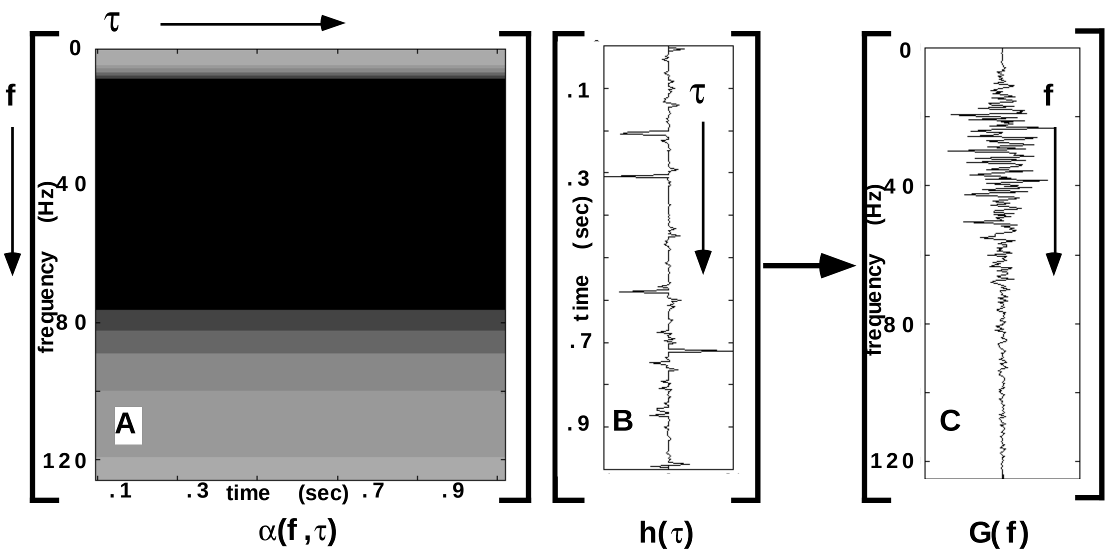

Fig. 6. Nonstationary convolution is represented in the (f,t) domain for the stationary case of figures 1 and 4. The computation of equation (17) is depicted except for a Fourier matrix representing exp(-2πift). (A) is the "nonstationary" transfer function (amplitude spectrum only) which is computed by Fourier transforming the columns of the matrix in figure 3 (A), (B) is the input reflectivity series in the time domain, and (C) is the Fourier spectrum of the desired result.

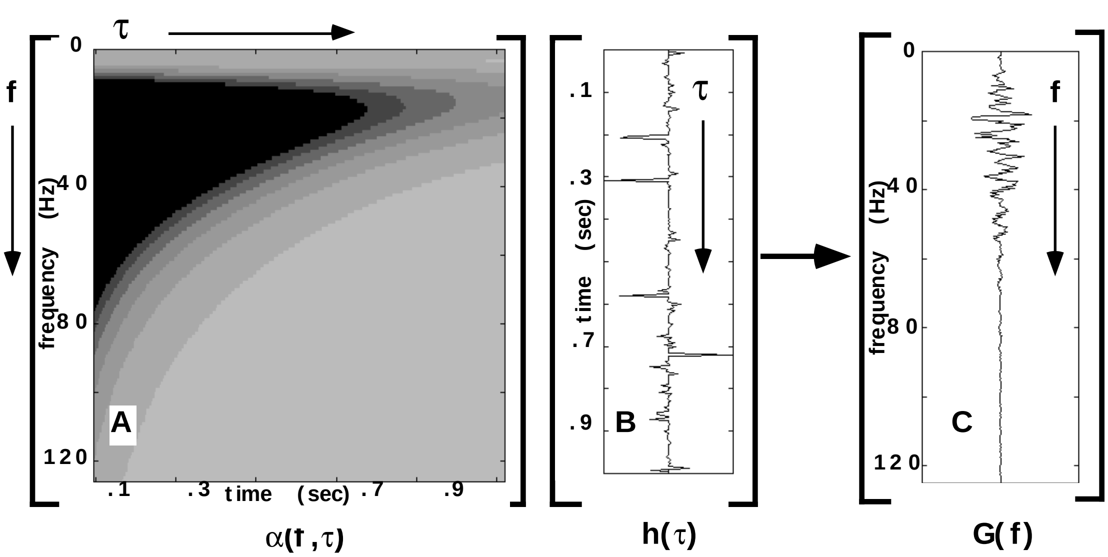

Fig. 7. Nonstationary convolution is represented in the (f,t) domain for the nonstationary case of figures 2 and 5. The computation of equation (17) is depicted except for a Fourier matrix representing exp(-2πift). (A) is the nonstationary transfer function (amplitude spectrum only) which is computed by Fourier transforming the columns of the matrix in figure 3 (B), (B) is the input reflectivity series in the time domain, and (C) is the Fourier spectrum of the desired result.

Equation (18) can be used to show that nonstationary combination can achieve an arbitrarily abrupt temporal change in the spectral content of the output filtered trace, a property not found with nonstationary convolution. Consider the computation of (18) when the nonstationary transfer function is set to its value at a particular time.

$$\bar{g}_j(t) = \int_{-\infty}^{\infty} \alpha(F,t = t_j)\, H(F)\, e^{2\pi i F t} dF \qquad (20)$$

Since $\alpha(t,\tau = \tau_j)$ is independent of time, equation (20) is an ordinary inverse Fourier transform and represents the application of an ordinary stationary filter in the frequency domain. Obviously

$$\bar{g}(t = t_j) = \bar{g}_j(t = t_j) \qquad (21)$$

Which simply says that the computation of the stationary filter (20) gives the same result as nonstationary combination when both are evaluated at precisely the time $t = t_j$. Therefore, given the set of functions $\{ g_j(t) \}$ where the subscript j is assumed to run over all possible times as given by (20), the nonstationary combination can be regarded as a "slice" through them evaluating each at $t = t_j$ (figure 8). An abrupt temporal change of $\alpha(F,t)$ is simply handled because the $\{ g_j(t) \}$ are all computed with ordinary stationary filter theory and then "sliced". Thus any discontinuities in the temporal variation of $\alpha(F,t)$ are manifest as discontinuities in the spectral content of the output trace.

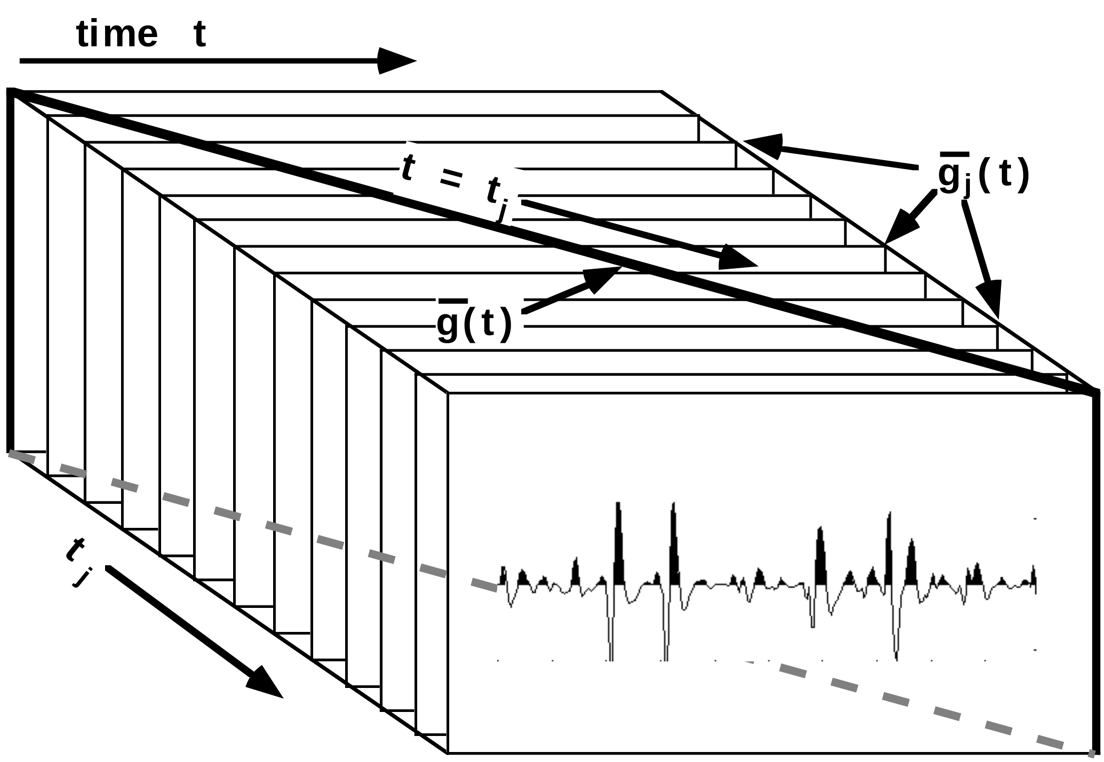

Figure 8 illustrates the construction of g(t) by nonstationary combination as a slice through the family of stationary filtered results gj(t). Each gj(t) may be considered as a conventionally filtered trace while the index j runs across all possible filters in the nonstationary transfer function.

As a direct consequence, nonstationary combination may be closely approximated by a suitable temporal interpolation between a few stationary filtered results. The stationary filtered signals may be regarded as a sparse sampling of the $\{ g_j(t) \}$ and, provided that the desired filter is only slightly nonstationary, an interpolation scheme may be devised to approximate it. Thus the limiting form of the method of time variant filtering by interpolating between stationary filter panels (Yilmaz, 1986, p25-26) is nonstationary combination and not nonstationary convolution.

This ability to change the spectral content of the output trace with arbitrary suddenness is a direct consequence of the fact that nonstationary combination changes the filter parameters as a function of output time. Nonstationary convolution cannot achieve such an abrupt change in the properties of the output trace because it varies the filter with input time. The overlap of the superimposed impulse responses softens any abrupt temporal changes in filter properties.

A similar analysis can be made of nonstationary convolution as given by equation (17) to show that any discontinuities in the frequency variation of $\alpha(t,\tau)$ are preserved in the spectrum, G(f), of the output trace. This is done by considering the set $\{G_j(t)\}$ formed by setting $\alpha(t,\tau) = \alpha(t = t_j, \tau)$ in (17) and arguing as before that G(f) is a slice through that set.

DISCUSSION AND FURTHER EXAMPLES

Relations between the different filter application domains

Equations (8), (10), and (17) are all different ways of applying a linear nonstationary filter by convolution. Similarly equations (9), (14), and (18) can apply the same filter by nonstationary combination. The functions a(u,v), A(p,q), and $\alpha(p,v)$ are all ways of specifying the nonstationary filter in the different domains. Given any one of these functions, the other two may be computed by ordinary Fourier transform operations. For filter design, it is usually preferable to specify $\alpha(p,v)$ in the mixed frequency-time domain and then convert to the domain most advantageous for numerical application. As such, in addition to equations (13) and (19), the following formulae, all ordinary Fourier transforms, are of use

$$a(u,v) = \int_{-\infty}^{\infty} \alpha(p,v)\, e^{2\pi i p u} df \tag{22}$$

$$A(p,q) = \int_{-\infty}^{\infty} \alpha(p,v) e^{-2\pi i q v} dv \tag{23}$$

$$\alpha(p,v) = \int_{-\infty}^{\infty} A(p,q)\, e^{2\pi i q v} dq \tag{24}$$

$$a(u,v) = \int\int_{-\infty}^{\infty} A(p,q)\, e^{2\pi i p u} e^{2\pi i q v} dp\, dq \tag{25}$$

Figure 9 illustrates the relationships between a(u,v), A(p,q), and $\alpha(p,v)$.

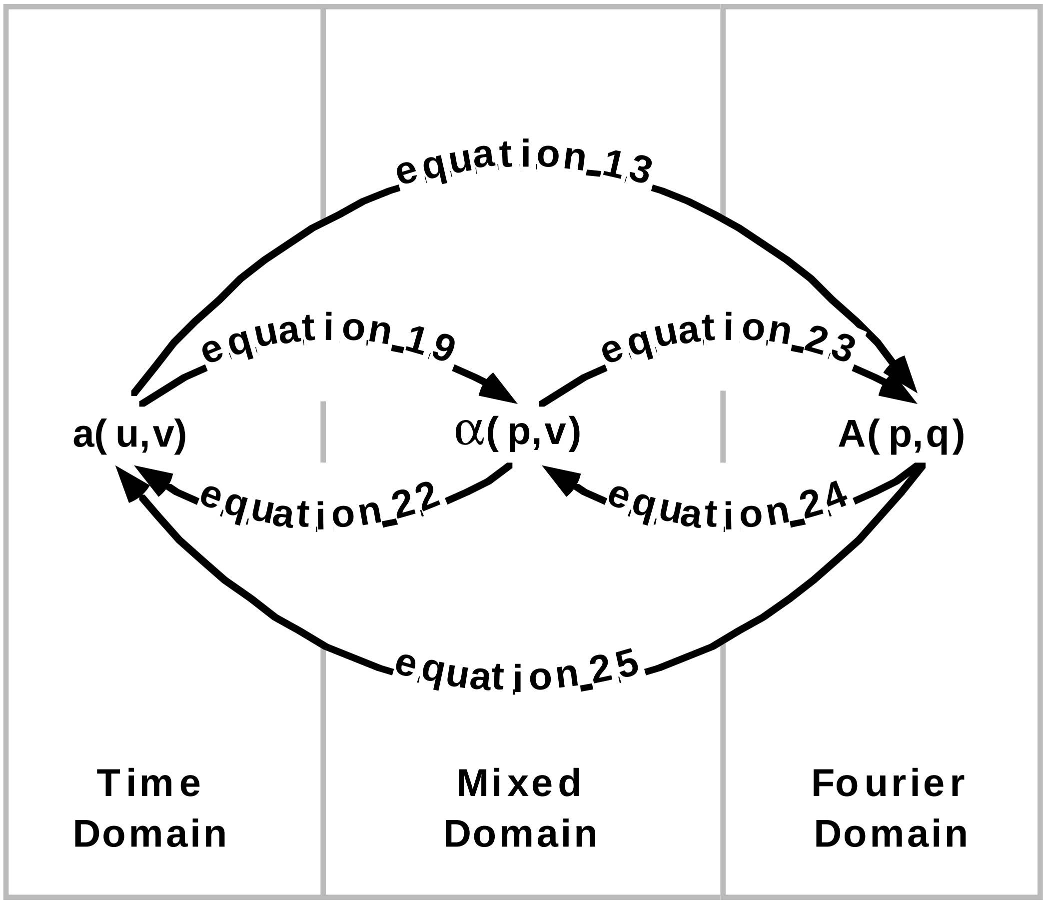

Figure 9 Illustrates the relationship between the various forms of nonstationary filter specification. Arrows to the right represent forward Fourier transforms while those to the left are inverse Fourier transforms. The four shorter arrows are 1-D transforms while the two longer arrows are 2-D transforms.

From an analytic perspective, the domain of the filter application makes no difference in the eventual result. However, in a numerical application it can have a dramatic effect. For example, it is well known that stationary time domain filters often need to be unrealizably long to achieve the same spectral performance that frequency domain filters produce. This is true for nonstationary filters as well. Also, many "almost stationary" filters may be implemented with great efficiency in the Fourier domain since, like the filter in figure 5, their description is dominated by a narrow band around 0 Hz of the frequency connection function (eqns (13) and (23)).

As in stationary theory, there will also be cases when the time domain implementation is preferred. Unlike stationary theory, there is now a third possibility, the mixed domain, which also has its advantages. If a filter is designed in the mixed domain, then there can be considerable computational effort required to transform it to either other domain. If the filter is to be applied to many traces before it must be redesigned, then the cost of moving it to another domain may be justified. On the other hand, if the filter must be redesigned for each trace, then it may make more sense to simply apply it directly in the mixed domain.

A bandpass filter example

A common use of a nonstationary filter is the time variant bandpass filter. Successful techniques for such filters have been presented by others (e.g. Pann and Shin 1976, Scheuer and Oldenburg, 1988, and Park and Black 1995); however, unlike previous methods, the technique developed here places no practical limits on the shape of the amplitude and phase spectra or their temporal variation. Indeed, all of stationary filter theory can be applied to the columns of the nonstationary connection function (or it's Fourier transforms).

Figure 10 shows a design for a minimum phase nonstationary bandpass filter. The filter bandwidth, as displayed in the nonstationary amplitude spectrum of figure 10 (A), is 10-80 Hz at time 0 and ramps linearly down to 10-40 Hz at 1 second where it becomes stationary. Filter slopes are gaussian shapes of width 5 Hz on the low end and 20 Hz on the high end. (The large width of the gaussian taper on the high end has broadened the effective bandwidth by nearly 10 Hz.) The corresponding nonstationary minimum phase spectrum (figure 10 (B)) was computed as the Hilbert transform of the log of the amplitude spectrum for each column of the matrix.

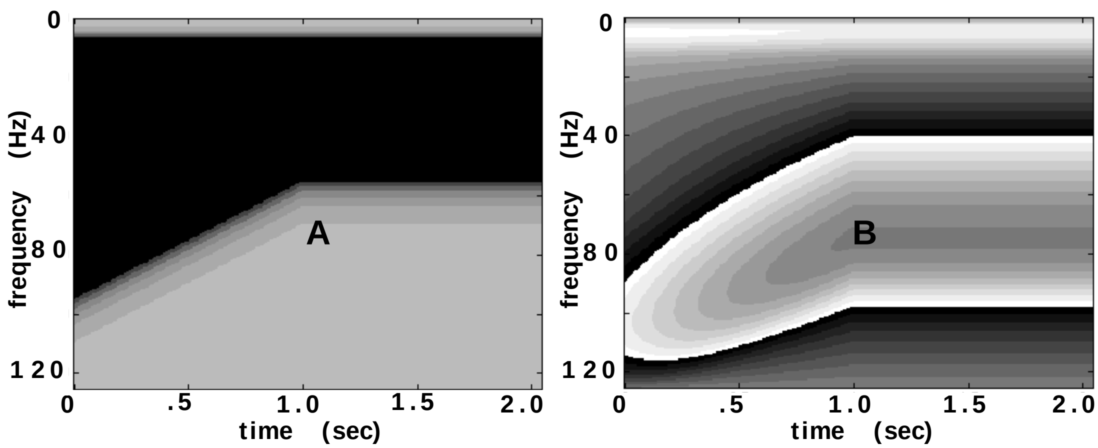

Fig. 10. A design for a time variant minimum phase bandpass filter. (A) is the amplitude spectrum which varies from about 10-80 Hz at time 0 to 10-40 Hz at time 1.0 seconds. It is stationary from 1.0 to 2.0 seconds. (B) is the minimum phase spectrum. A column of (B) is computed as the Hilbert transform of the logarithm of a column of (A).

The filter design requires that the temporal axis and frequency axis of the nonstationary transfer function be sampled compatibly with each other and with the data to be filtered. Thus, for 4 mil data with a 2.044 second record length, there are 512 temporal samples and at least 257 frequency samples. For long traces, the design matrix can consume significant computer memory and, for minimum phase, can require a large number of Hilbert transforms. It is often the case that the filter may be designed on a more sparse grid and interpolated (carefully!) to the desired dimensions. (Strictly speaking, the interpolated columns are unlikely to be truly minimum phase; however, the discrepancy is controllable.)

Figure 11(A) shows the frequency combination matrix, A(f,f-F), as appropriate to apply the filter of figure 10 as a nonstationary convolution in the Fourier domain with equation (10). Figure 11(B) is similar except that a zero phase spectrum was used. Considering 11(B) first, note that the matrix is essentially diagonal below 40 Hz because the amplitude spectrum (figure 10(A)) is stationary in this range. Above 40 Hz, there is significant off-diagonal energy with a half width of about 5 Hz. In contrast, the minimum phase matrix is nonstationary throughout because the phase is nonstationary at all frequencies as is evident in figure 10(B).

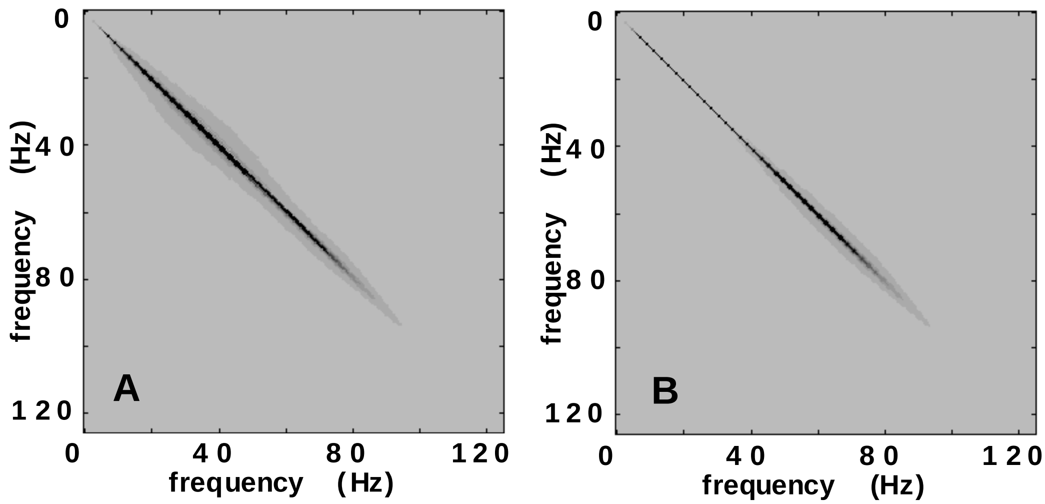

Fig. 11. (A) is the frequency connection matrix, A(f,f-F), (amplitude spectrum only) for the minimum phase filter of figure 10. (B) is similar except that a zero phase spectrum was used. When either matrix is used to multiply the spectrum of a seismic trace, the result is the spectrum of the time variant filtered trace. (Compare to figure 5).

Figure 12(A) and 12(B) show the nonstationary convolution matrix for the minimum phase and the zero phase cases respectively. The broadening of the convolutional wavelet with increasing time is clearly evident in both.

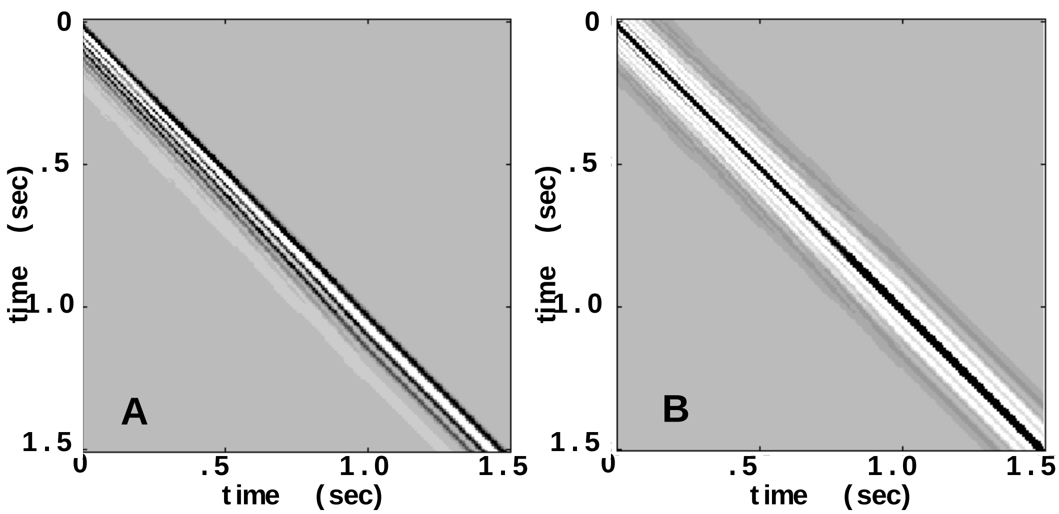

Fig. 12. (A) is the time domain nonstationary convolution matrix for the minimum phase filter design of figure 10. (B) is similar except that a zero phase spectrum was used. When either matrix multiplies a seismic trace, the result is a time variant filtered trace. (Compare to figure 3.)

Figure 13 presents the results of applying the filters to a sparse comb of samples of alternating sign. In addition to the minimum phase and zero phase cases, a result with linear phase variation, from 0 degrees at time 0 to 90 degrees at 1 second, is also shown. Fourier wrap around of the non-causal filters is also evident.

Though the filters in figure 13 obviously are nonstationary, it is not immediately evident that they have the desired spectral shape. Figure 14 shows a time variant amplitude spectrum computed from one of the results (they all have the same amplitude spectrum) and, when compared with figure 10(A), and provides the desired confirmation. Figure 14 was computed using a "short time Fourier transform" (or "windowed Fourier transform", see Kaiser (1994) for a discussion) with a window length of .3 seconds and a window overlap of 95%. The notches in the spectrum are caused by the spikes of the input comb function.

Comparison of nonstationary convolution and combination

The separation of convolution into two distinct forms when extended to nonstationary signals may be regarded as arising from the dual nature of the Toeplitz symmetry of the stationary convolution matrix (equation 3). As remarked previously the impulse response of the stationary filter can be found in each column of the matrix or, in time reverse, in each row. Nonstationary convolution and combination amount to choosing to preserve the impulse response in the columns or (its time reverse) in the rows. It follows that, given a nonstationary convolution matrix such as those shown in figure 12, the matrix which achieves nonstationary combination can be formed by transposing and then, in each row, flipping the order of samples about the diagonal. Since the multiplication of a time series by either matrix can be considered to be a scaled superposition of the columns of the matrix, nonstationary combination cannot correspond to the scaled superposition of filter impulse responses. However, since both processes have stationary convolution as their limiting form, it is expected that the differences will be slight for quasi-stationary filters.

Phrases such as "minimum phase combination" do not have quite the same meaning as "minimum phase convolution". In the latter case, it is true that each sample of the input trace was replaced by a waveform which was minimum phase. In the former case, this cannot be precisely true and all that can be said is that the filter design function was formed with minimum phase wavelets.

Following on the previous example, figure 15 shows a comparison between these two forms. That is, the filter of figure 10 was applied to the comb function of figure 13(A) using both convolution and combination.

Comparing figure 15(A) with 15(B) and then 15(D) with 15(E) it can be seen that combination and convolution are visually similar for this quasi-stationary filter. The zero phase results show a smaller difference than the minimum phase results since the minimum phase filter is less stationary. (See figure 11). For times greater than 1.0 seconds, both filters were stationary and the difference traces are zero.

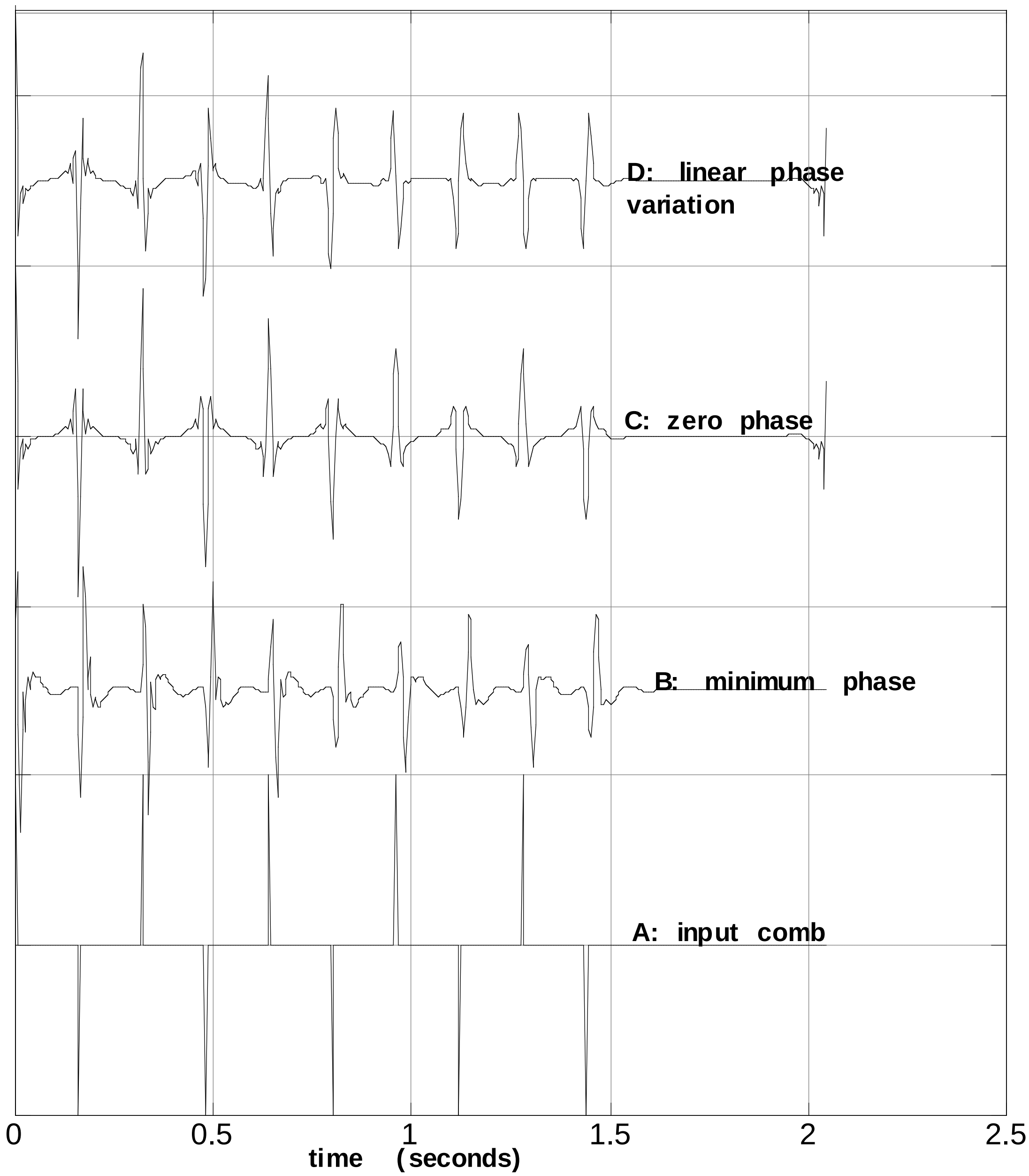

Fig. 13. Several results of the application of time variant filters to an input "comb function" (A) are shown. The minimum phase result (B) used the filter design of figure 10. The zero phase result (C) used the amplitude spectrum from figure 10 with a zero phase spectrum. The linear phase result (D) used the same amplitude spectrum with a phase shift that varied linearly from 0 degrees at time zero to 90 degrees at 1 second.

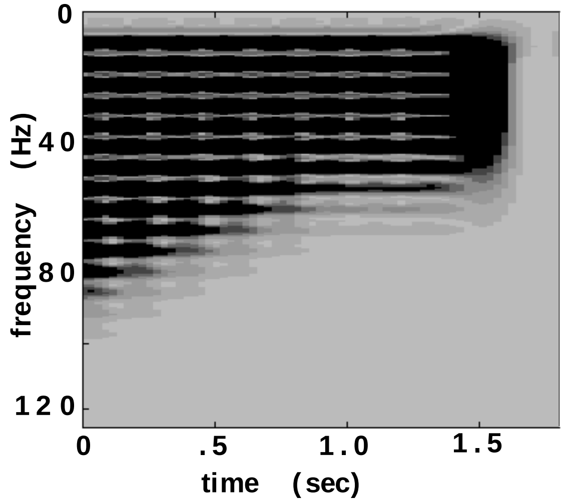

Fig. 14. Is a time variant amplitude spectrum computed from one of the filtered traces in figure 13. It was computed using a "short time Fourier transform" with a .3 second window and 95% overlap between windows. Comparison with figure 10 shows that the design spectrum was achieved.

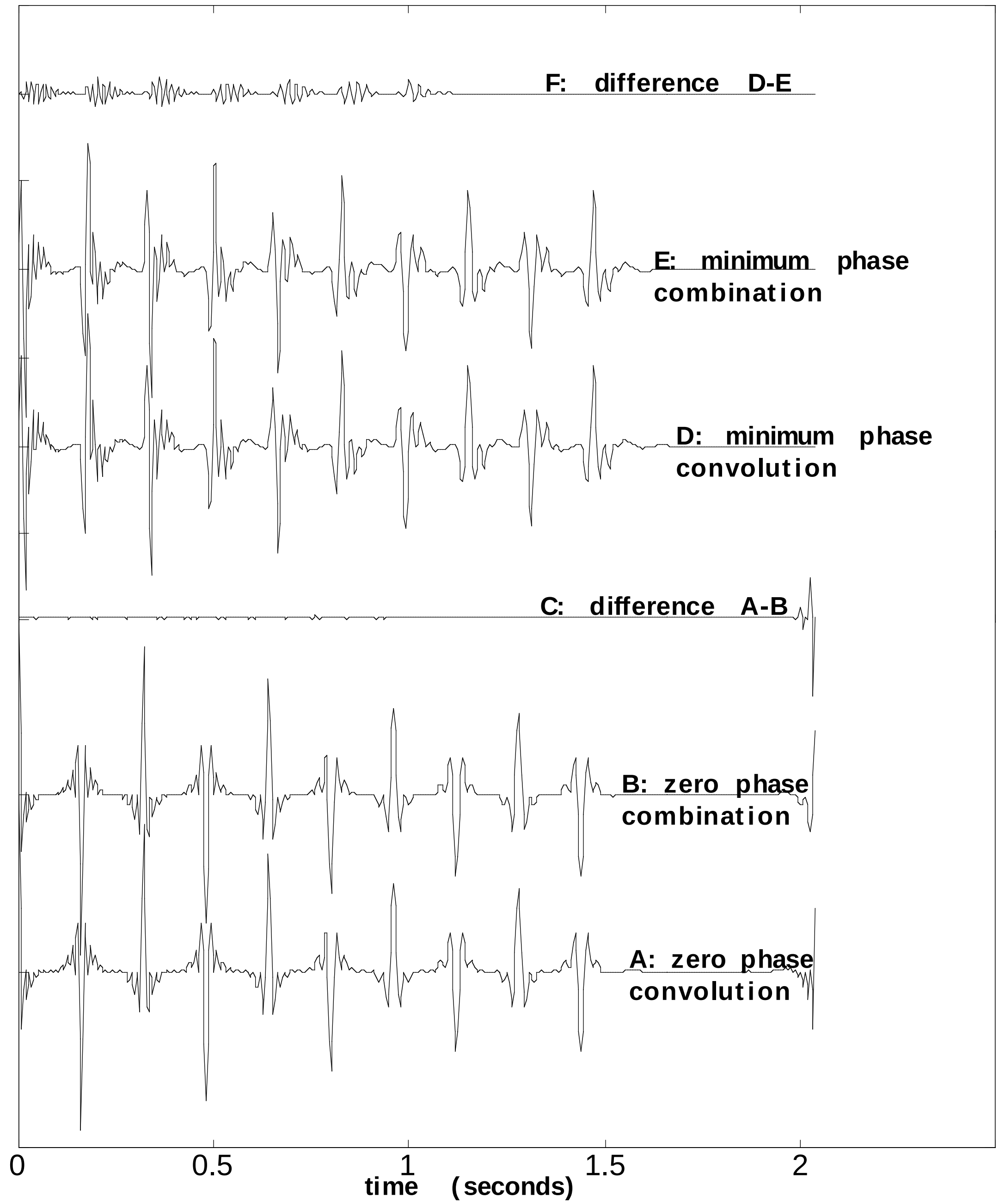

Fig. 15. A comparison of nonstationary convolution (equation 8) and nonstationary combination (equation 9) as applied to the comb function of figure 13. (A) and (B) applied the amplitude spectrum of figure 10(A) with a zero phase spectrum while (D) and (E) applied the full minimum phase design in figure 10. Both processes have ordinary convolution as their stationary limit. Though similar, the processes differ more for the minimum phase filter because it is less stationary.

Figure 16 is an example specifically chosen to contrast convolution and combination. The filter design contains an abrupt discontinuity in bandwidth at .5 seconds as shown in (A). In (B) is a synthetic reflectivity with a sequence of large spikes placed around the time of the filter discontinuity. 15(C) is a minimum phase convolution of (A) with (B) and 15(D) is a minimum phase combination. Their difference is shown in 15(E) and is quite dramatic from .5 to .6 seconds. Also a close inspection of (C) and (D) near .5 seconds shows that the latter changes apparent frequency content abruptly at .5 seconds while the former does so gradually.

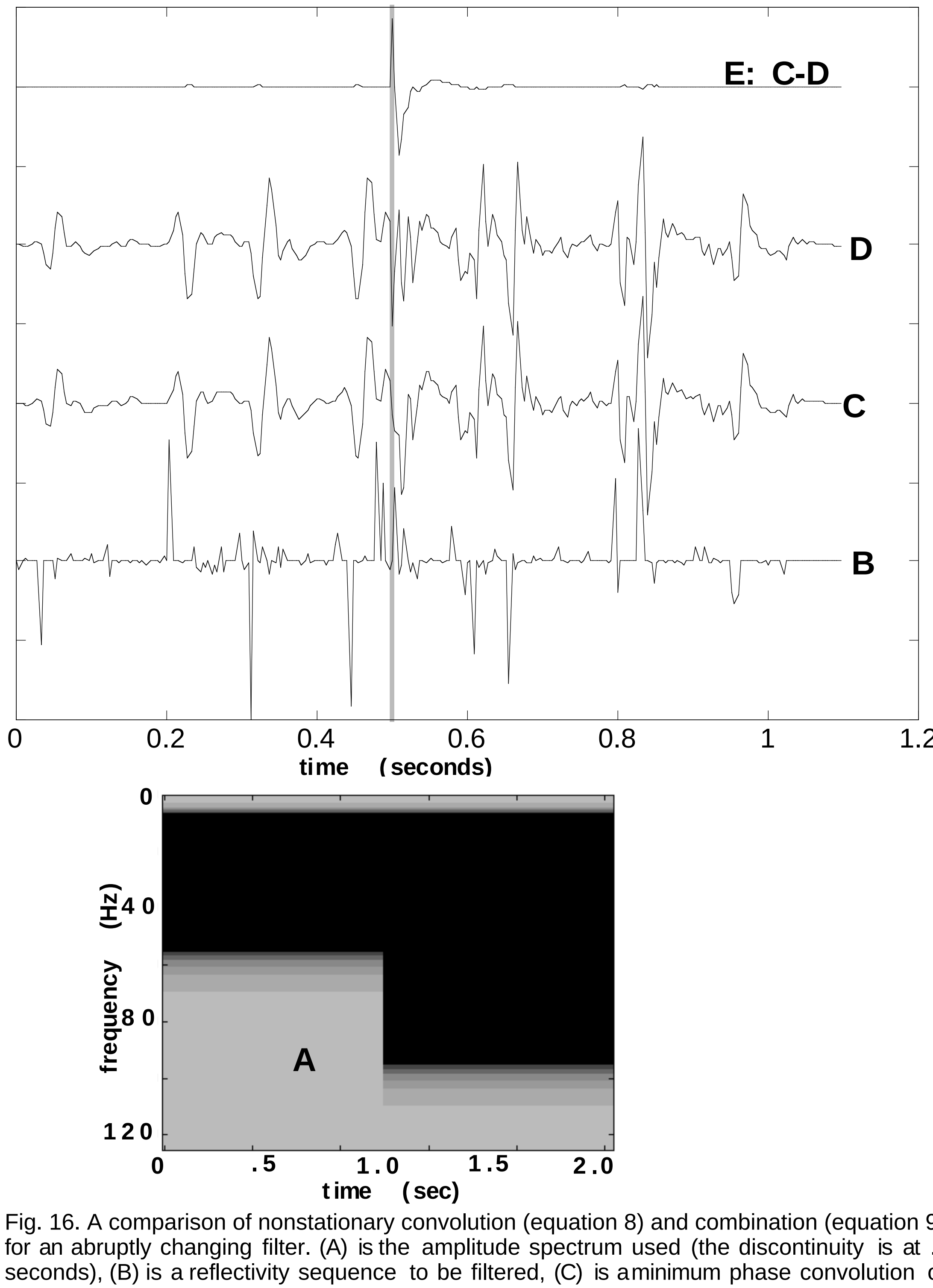

Fig. 16. A comparison of nonstationary convolution (equation 8) and combination (equation 9) for an abruptly changing filter. (A) is the amplitude spectrum used (the discontinuity is at .5 seconds), (B) is a reflectivity sequence to be filtered, (C) is a minimum phase convolution of (A) with (B), (D) is a minimum phase combination, and (E) is the difference of (D) from (C).

Though nonstationary combination is not the direct scaled superposition of the filter impulse response, its ability to change the spectrum discontinuously with time makes it a potentially useful data processing tool. This does not violate the uncertainty principle in any way since that principle is an inequality governing the widths of a temporal function and its Fourier dual. In this context, the uncertainty principle merely places limits upon our ability to measure the spectral discontinuity after-the-fact with local Fourier spectra.

CONCLUSIONS

Nonstationary filtering can be formulated as a natural extension of stationary convolution. The concept of a time-invariant impulse response is generalized to that of a two dimensional impulse response function where one dimension is the time of the impulse response and the other is the time of the impulse. The nonstationary filter can be applied digitally by forming the nonstationary convolution matrix which has in each column the impulse response for the column time delayed to start at the main matrix diagonal.

This generalization of convolution also gives rise to an alternative nonstationary filter which is called here nonstationary combination. The major difference between convolution and combination is that the former prescribes the nonstationarity as a function of input time while the latter uses output time. Nonstationary combination does not correspond to a scaled superposition of filter impulse responses but is capable of achieving arbitrarily abrupt spectral changes in the output signal. Both processes have stationary convolution as their limiting form and so are quite similar for quasi-stationary filters.

Both nonstationary convolution and combination may be moved into the Fourier domain where they are also nonstationary matrix operations. The nonstationary spectral matrix, which achieves nonstationary convolution in the Fourier domain, is formed from the 2-D Fourier transform of the generalized impulse response function. When the impulse response function is stationary, its 2-D Fourier transform yields a frequency connection function which is a Dirac delta function times the filter spectrum. This results in a diagonal nonstationary spectral matrix with the stationary filter spectrum along the diagonal. Multiplication of a signal spectrum by such a diagonal matrix achieves stationary convolution, as expected from the convolution theorem. A general nonstationary spectral matrix contains off-diagonal power which describes the temporal variation of the filter. The stronger the nonstationarity, the broader the band of significant power about the diagonal.

Both processes may also be applied in a mixed time-frequency domain. Nonstationary convolution becomes a generalized forward Fourier integral of the product of the input time domain signal and the nonstationary filter which yields the spectrum of the filtered signal. Nonstationary combination is recast as a generalized inverse Fourier integral of the spectrum of the input signal times the nonstationary filter to yield the time domain filtered signal.

ACKNOWLEDGMENTS

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REFERENCES

Brigham, E.O., 1974, The Fast Fourier Transform, Prentice-Hall
Berkhout, A.J., 1985, Seismic Migration, Developments in Solid Earth Geophysics, 14A, Elsevier
Chakraborty, A., and Okaya, D., 1994, Application of wavelet transforms to seismic data: Expanded abstracts, 64th Annual Mtg. Soc. Expl. Geophy., p 725-728
Chakraborty, A., and Okaya, D., 1995, Frequency-Time decomposition of seismic data using wavelet based methods: Geophysics, 60, 1906-1916
Claerbout, J.F., 1985, Imaging the earth's interior, Blackwell Scientific Publications
Courant, R, and John, F., 1974, Introduction to Calculus and Analysis (volumes I and II), John Wiley
Futterman, W.I., 1962, Dispersive Body Waves, JGR, 73, 3917-3935
Gabor, D., 1946, Theory of Communication: J. Inst. Electr, Eng., 93, 429-457
Gazdag, J., and Squazzero, P., Migration of seismic data by phase shift plus interpolation: Geophysics, 49, 124-131
Kabir, M.M.N., Faqi, L., Verschuur, D., 1995, Cascaded application of the linear radon and wavelet transforms in preprocessing: Expanded abstracts, 65th Annual Mtg. Soc. Expl. Geophy., p 1373-1376
Kaiser, G., 1994, A Friendly Guide to Wavelets, Birkhauser
Karl, J.H., 1989, An Introduction to Digital Signal Processing, Academic Press
Kjartansson, E., 1979, Constant Q Wave propagation and attenuation: JGR, 84, 4737-4748
Korner, T.W., 1988, Fourier Analysis, Cambridge University Press
Morse, P.M., and Feshbach, H., 1953, Methods of Theoretical Physics, McGraw-Hill
Papoulis, A., 1984, Signal Analysis,
Pann, K, and Shin, Y, 1976, A class of convolutional time-varying filters: Geophysics, 41, 28-43
Park, C, and Black, R.A., 1995, Simple time-variant, band-pass filtering by operator scaling: Geophysics, 60, 1527-1535
Scheuer, T, and Oldenburg, D.W., 1988, Aspects of time-variant filtering: Geophysics, 53, 1399-1409
Strang, G., 1986, Introduction to Applied Mathematics, Wellesley-Cambridge Press
Taner, M.T., Koehler, F., and Sheriff, R.E., 1979, Complex seismic trace analysis (amplitude, phase, instantaneous-frequency color sections): Geophysics, 44, 1041-1063
Yilmaz, O, 1987, Seismic Data Processing, Society of Exploration Geophysicists

APPENDIX: MATHEMATICAL DETAILS

The Fourier domain formulation for nonstationary convolution

Equation (8) is repeated here as

$$g(t) = \int_{-\infty}^{\infty} a(t-\tau,\tau)\, h(\tau)\, d\tau \qquad \text{(A-1)}$$

The spectrum, G(f), of g(t) is computed by the forward Fourier transform of (A-1)

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-2\pi i f t} dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} a(t-\tau,\tau) h(\tau) d\tau \right] e^{-2\pi i f t} dt \qquad \text{(A-2)}$$

The next step is to reverse the order of integration in (A-2). Strictly speaking, this requires some justification. For integrals with non-infinite limits, a double integral over a rectangular region can have its integration order interchanged provided only that the integrand is continuous (see any calculus text, for example Courant and John, 1974, (Vol. II page 398)). In improper integrals such as the 2-D Fourier transform, more consideration is required. Korner (1988) gives the general conditions under which the order reversal may be justified (his theorem 48.8) and these amount to requiring that the integrand is continuous, that both one dimensional transforms of the absolute value of the integrand are continuous, and that the 2-D transform itself converges absolutely. Of course, if the integrand has only compact support, then the infinite limits may be replaced by finite ones and the conditions are relaxed. All of the theory in this paper assumes that these conditions have been met in one way or the other. (Note that similar integration reversals are required to prove the Fourier transform theorem and the convolution theorem. For example see Korner (1988) or Morse and Feshbach (1953).) Therefore, reversing the order of integration in (A-2) gives

$$G(f) = \int_{-\infty}^{\infty} h(\tau) \left[\int_{-\infty}^{\infty} a(t-\tau,\tau) e^{-2\pi i f t} dt \right] d\tau \qquad \text{(A-3)}$$

The term in brackets is the Fourier transform (over the columns) of the delayed nonstationary impulse response matrix. The nonstationary filter can be moved entirely into the Fourier domain by substituting for $h(\tau)$ in (A-3) its expression as an inverse Fourier transform of its spectrum H(F)

$$h(\tau) = \int_{-\infty}^{\infty} H(F) e^{2\pi i F \tau} dF \qquad \text{(A-4)}$$

When this result is substituted into (A-3) and the order of integrations is again changed, we have

$$G(f) = \int_{-\infty}^{\infty} H(F)\, \lambda(f,F)\, dF \qquad \text{(A-5)}$$

where

$$\lambda(f,F) = \iint_{-\infty}^{\infty} a(t-\tau,\tau) e^{-2\pi i f t} e^{2\pi i F \tau} dt\, d\tau \qquad \text{(A-6)}$$

Equation (A-6) can be rewritten by letting $u = t - \tau$, $du = dt$, $t = u + \tau$ to give

$$\lambda(f,F) = \iint_{-\infty}^{\infty} a(u,\tau) e^{-2\pi i f u} e^{-2\pi i (f-F)\tau} du\, d\tau \qquad \text{(A-7)}$$

Define the 2-D Fourier transform of the nonstationary impulse response function, a(u,v), as in equation (13) which is repeated here

$$A(p,q) = \iint_{-\infty}^{\infty} a(u,v)\, e^{-2\pi i p u} e^{-2\pi i q v} du\, dv \qquad \text{(A-8)}$$

Comparing (A-7) and (A-8) yields

$$\lambda(f,F) = A(f,f-F) \qquad \text{(A-9)}$$

So (A-5) becomes

$$G(f) = \int_{-\infty}^{\infty} H(F)\, A(f,f-F)\, dF \qquad \text{(A-10)}$$

This is equation (10).

The Fourier domain formulation for nonstationary combination

The derivation of equation (14) proceeds along similar lines except we begin with equation (9) instead of (8). Corresponding to equations (A-5), (A-6), and (A-7) are the following

$$\bar{G}(f) = \int_{-\infty}^{\infty} H(F)\, \bar{\lambda}(f,F)\, dF \tag{A-11}$$

$$\bar{\lambda}(f,F) = \iint_{-\infty}^{\infty} a(t-\tau,t) e^{-2\pi i f t} e^{2\pi i F \tau} dt\, d\tau \tag{A-12}$$

and, letting $u = t - \tau$, $du = d\tau$, $\tau = t - u$

$$\bar{\lambda}(f,F) = \iint_{-\infty}^{\infty} a(u,t) e^{-2\pi i (f-F) t} e^{-2\pi i F u} dt\, du \tag{A-13}$$

Comparing (A-13) and (A-8) results in

$$\bar{\lambda}(f,F) = A(F, f-F) \tag{A-14}$$

so we finally get

$$\bar{G}(f) = \int_{-\infty}^{\infty} H(F)\, A(F, f-F)\, dF \tag{A-15}$$

which is equation (14).

The mixed domain formulation of nonstationary convolution

This result proceeds directly from (A-3) which can be rewritten as

$$G(f) = \int_{-\infty}^{\infty} \gamma(f,\tau)\, h(\tau)\, d\tau \tag{A-16}$$

where

$$\gamma(f,\tau) = \int_{-\infty}^{\infty} a(t-\tau,\tau)\, e^{-2\pi i f t} dt \tag{A-17}$$

letting $u = t - \tau$, $du = dt$, $t = u + \tau$ gives

$$\gamma(f,\tau) = e^{-2\pi i f \tau} \int_{-\infty}^{\infty} a(u,\tau)\, e^{-2\pi i f u} du \tag{A-18}$$

The nonstationary transfer function is defined as the Fourier transform over the first temporal coordinate (i.e. u) of the nonstationary impulse response function a(u,v) as in equation (19) which is repeated here

$$\alpha(p,v) = \int_{-\infty}^{\infty} a(u,v)\, e^{-2\pi i p u} du \tag{A-19}$$

Comparing (A-18) and (A-19) gives

$$\gamma(f,\tau) = e^{-2\pi i f \tau}\alpha(f,\tau) \tag{A-20}$$

Substitution of this into (A-16) gives

$$G(f) = \int_{-\infty}^{\infty} \alpha(f,\tau)\, h(\tau)\, e^{-2\pi i f \tau} d\tau \tag{A-21}$$

This is equation (17).

The mixed domain formulation of nonstationary combination

We begin with equation (9) which is repeated here

$$\bar{g}(t) = \int_{-\infty}^{\infty} a(t-\tau,t)\, h(\tau)\, d\tau \tag{A-22}$$

Now substitute equation (12), which expresses $h(\tau)$ in terms of its spectrum $H(F)$, into (A-22) to get

$$\bar{g}(t) = \int_{-\infty}^{\infty} a(t-\tau,t) \left[\int_{-\infty}^{\infty} H(F)\, e^{2\pi i F \tau} dF \right] d\tau \tag{A-23}$$

changing the order of integration results in

$$\bar{g}(t) = \int_{-\infty}^{\infty} \bar{\gamma}(F,t)\, H(F)\, dF \tag{A-24}$$

where

$$\bar{\gamma}(F,t) = \int_{-\infty}^{\infty} a(t-\tau,t)\, e^{2\pi i F \tau} d\tau \tag{A-25}$$

let $u = t - \tau$, $du = -d\tau$, $\tau = t - u$ in (A-25)

$$\bar{\gamma}(F,t) = e^{2\pi i F t} \int_{-\infty}^{\infty} a(u,t)\, e^{-2\pi i F u} du \tag{A-26}$$

Comparison of (A-26) and (A-19) gives

$$\bar{\gamma}(F,t) = e^{2\pi i F t}\, \alpha(F,t) \tag{A-27}$$

When this result is substituted into (A-24), it becomes

$$\bar{g}(t) = \int_{-\infty}^{\infty} \alpha(F,t)\, H(F)\, e^{2\pi i F t}\, dF \tag{A-28}$$

This is equation (18).