Wavefield extrapolation: from pseudospectral method to nonstationary phase shift

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ABSTRACT

The pseudospectral method as a limit of finite differences of increasing orders has been successfully used for forward seismic modeling. In this paper, the pseudospectral method is applied to a localized wave equation for wavefield extrapolation in inhomogeneous media. As a limiting case, this method leads to the nonstationary phase shift method.

INTRODUCTION

Wavefield extrapolation has been being studied extensively in exploration seismology. Over the past two years, the nonstationary phase-shift method has been developed for wavefield extrapolation (Margrave & Ferguson 1997 & 1999a). This method approximately solves the wave equation in the frequency-wavenumber domain. It demonstrates two algorithms of wavefield extrapolation: a limiting case of phase-shift-plus-interpolation equivalent, named PSPI; and a nonstationary phase shift, called NSPS. Although both formulae (PSPI and NSPS) have a form that is similar to the constant velocity phase-shift method (Gazdag, 1978) they allow the phase shift to adapt to lateral velocity variation. In real applications, the velocity structure is often composed of a laterally inhomogeneous layer, this method makes a significant contribution to the theory of wavefield extrapolation. Moreover, when the velocity is constant, both algorithms are convergent to the exact analytic solution. As an approximate method, however, the nonstationary phase shift has its own limits. Recently, the formulae of nonstationary phase shift were derived by a Taylor series expansion technique (Margrave & Ferguson, 1999b), which suggests that the method works as a localized technique for solving the wave equation.

Generally speaking, if the medium is inhomogeneous, it may be difficult to find the exact analytic solution and numerical techniques such as the finite-difference method need to be employed (e.g. Alford et al., 1974). The accuracy of the finite-difference method usually depends upon the order of the approximation to the differential operators in the wave equation (Baysal et al., 1984). As an alternative to finite-difference, the pseudospectral is a limit of finite differences of increasing orders (Fornberg, 1987). The pseudospectral method was first proposed by Kreiss and Oliger (1972) with additional basic theory avalable in Orszag (1972), Fornberg (1975), and Gottlieb and Orszag (1977). The applications of this method can be found in many fields, e.g. in nonlinear wave dynamics and in weather forecasting. Various geophysical literatures provide specific examples of the applications of this method to 2D and 3D forward seismic modeling (wavefield extrapolation along the time coordinate) (Reshef *et. al*., 1988; Fornberg 1987; and Gazdag 1981). Nanxuen and Cheadle (1996) applied this method to wavefield extrapolation. Improved stability and accuracy with reasonable computational efficiency memory usage result of the

pseudospectral method is used instead of the finite-difference method (Reshef *et al*., 1988).

In this paper, the pseudospectral method is applied to wave equation (a discrete form of wave equation) for wavefield extrapolation by analogue to forward modeling. The results show that in the case of laterally varying velocity the limiting case of the pseudospectral method leads to the nonstationary phase shift method.

BRIEF REVIEW OF NON-STATIONARY PHASE SHIFT METHOD

Nonstationary filters (Margrave, 1998) have two distinct filter forms termed convolution (NSPS) and combination (PSPI). Applying these filters to wavefield extrapolation, the formula for NSPS can be written as

$$\varphi(k_x,z)_{NSPS} = \int \psi(x,0)\alpha(k_x,x,z)\exp(ik_x x)dx \tag{1}$$

and

$$\psi(x,z)_{NSPS} = \int \varphi(k_x,z)_{NSPS}\exp(-ik_x x)dk_x \tag{2}$$

where the nonstationary wavefield extrapolator, α, is

$$\alpha(k_x,x,z) = \exp(iz\sqrt{\frac{\omega^2}{v(x)^2}-k_x^2}) \tag{3}$$

.

The formula for PSPI can be written as

$$\varphi(k_x,0) = \int \psi(x,0)\exp(ik_x x)dx \tag{4}$$

and

$$\psi(x,z)_{PSPI} = \int \varphi(k_x,0)\alpha(k_x,x,z)\exp(-ik_x x)dx \tag{5}$$

where ψ is the wavefield representing velocity potential or pressure, t is time, and x and z are horizontal and vertical coordinates, respectively. Note that following Margrave & Ferguson (1997, 1999a) we use the term PSPI here to mean the limiting case where a reference wavefield is computed for each distinct velocity and no interpolation is required. The equations above are equivalent and exact in the stationary limit (i.e. constant velocity). Margrave & Ferguson (1997, 1999a) provided an explanation of these operators using spatial windowing when velocity is piecewise constant.

Both NSPS and PSPI are approximate methods. Therefore, the limitations of the formulae need to be clarified. As an example, Figure 1(a) shows a model composed of two blocks. The velocity on the left block is 1500 m/s and on the right it is 2500

m/s. One point source is located close to the velocity boundary on the left. After extrapolation with $\Delta z=200\ m$, the result from PSPI is shown in Figure 2(a) and from NSPS in Figure 2(b). The figures show that the results from both PSPI and NSPS are incorrect in this case. In Figure 2(a), the wavefield has discontinuities at the velocity interface. In figure 2(b) the shape of the wavefield remains constant as it spreads across velocity boundary. Furthermore, the extrapolations are not self-consistent because the wavefield cannot be extrapolated back to its original (Figure 3) when the extrapolated wavefield are used as the data. However, the operators are complementary in the sense that NSPD is invertable by the PSPI extrapolator and vice versa (Margarve & Ferguson, 1998b).

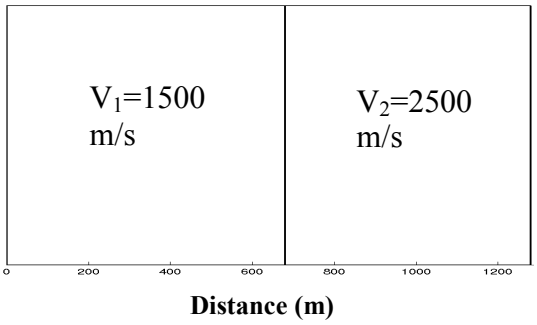

Figure 1. The velocity model.

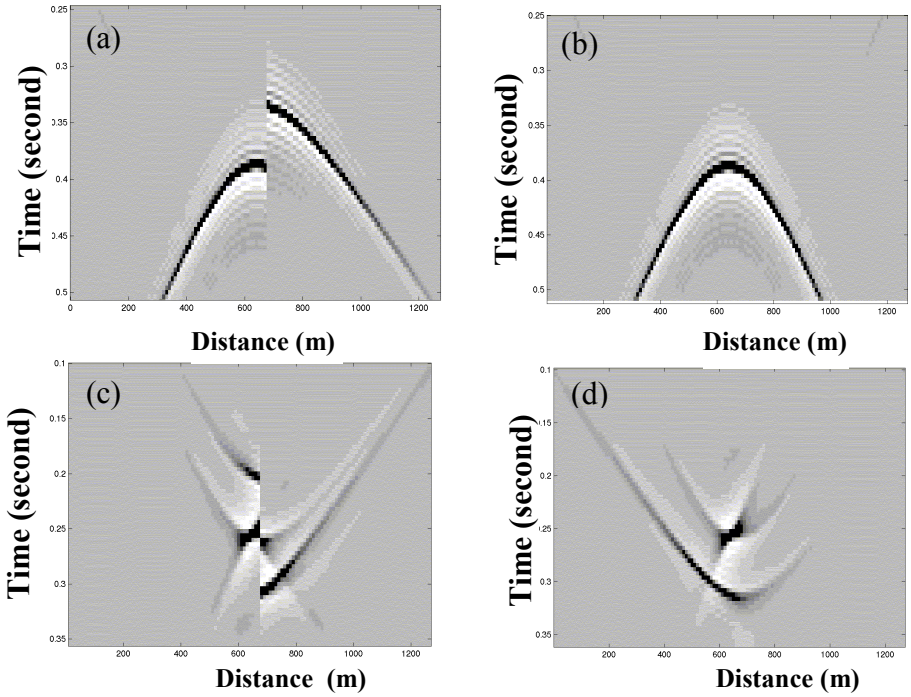

Figure 2. Results of the pulse extrapolated Δz=200m by (a) PSPI and (b) NSPS; and their extrapolated back Δz=-200m by (c) PSPI and (d) NSPS

PSEUDOSPECTRAL METHOD

The equation under consideration is

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 (\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2}). \tag{6}$$

For homogeneous media, i.e. v is constant, the solution to the initial value problem of equation (6) can be expressed by a finite Fourier series, provided that the data specifying the initial conditions are bandpass limited and periodic with respect to x and z. If these conditions can be met by ψ_0 , a solution to equation (6) may be written as

$$\psi(x,z,t) = \int\int \varphi_0 \exp\{i(k_x x + k_z z + \omega t)\} dk_x dk_z \tag{7}$$

where $\frac{\omega^2}{v^2} = k_x^2 + k_z^2$ is the dispersion relation that governs the wavenumber and the temporal frequency and

$$\varphi_0 = \varphi(k_x, k_z, t = 0) \tag{8}$$

is the finite Fourier Transform over *x* and *z* of the wavefield at *t=0*. Equation (7) is a form of wavefield extrapolation along the *t* coordinate, or wavefield modeling (Gazdag, 1981). Of course, this is not the complete solution as only $e^{+i\omega t}$ and not $e^{-i\omega t}$ is used. If $\frac{\partial \psi}{\partial t}|_{t=0}$ is known, then the general solution can be constructed.

In a similar way, when $\psi(x, z = 0, t)$ is known, the boundary value problem may also be solved. If we assume that only upward traveling waves are recorded, the solution may be expressed as

$$\psi(x,z,t) = \int\int \varphi_0 \exp\{i(k_x x + k_z z + \omega t)\} \tag{9}$$

where φ_0 is the finite Fourier transform over x and z of $\Psi(x,z=0,t)$ and $k_z = \pm\sqrt{\frac{\omega^2}{v^2} - k_x^2}$. Equation (9) is the equation for wavefield extrapolation along the spatial z coordinate, i.e. the phase-shift equation (Gazdag, 1978).

In heterogeneous media where velocity is an arbitrary function of the spatial variables, there is no simple analytic solution. In such cases, the wave equation is often solved numerically. Consider a wavefield (pressure) propagating in a *2D* velocity distribution *v(z,x)* in a grid with horizontal direction x, vertical direction z denoted by l and n, respectively. At each grid the wave equation can be written as

$$\frac{\partial^2 \psi_{l,n}}{\partial t^2} = v^2 (\frac{\partial^2 \psi_{l,n}}{\partial x^2} + \frac{\partial^2 \psi_{l,n}}{\partial z^2}). \tag{10}$$

and may be referred to as a localized wave equation. The solution to this equation for modeling can be obtained by finite-difference methods (e.g. Alford, 1974), where the differential operators are approximated by finite difference operators. Instead of using

the finite-difference operator approximation, pseudospectral methods are those that evaluate the spatial derivatives by a wavenumber multiplication in the wavenumber domain (e.g. Gottlieb & Orszag, 1977; Gazdag, 1981; Fornberg & Sloan, 1994). The derivative $\partial\Psi/\partial x$ at any fixed z, for example, is computed in a pseudospectral solution by taking the Fourier transform of

$$\frac{\partial\psi(x)}{\partial x} = F_x^{-1}[ik_x F_x(\psi(x))] \tag{11}$$

where F_x and F_x^{-1} are the forward and inverse Fourier transform to x, respectively, and k_x is the wavenumber in the x-direction. Similarly, the second derivative is obtained by multiplication of $(ik_x)^2=-k_x^2$,

$$\frac{\partial^2\psi(x)}{\partial x^2} = F_x^{-1}[-k_x^2 F_x(\psi(x))] \tag{12}$$

Differentiation in the z-direction is defined the same way, by replacing x with z in (11) and (12). The advantages of this approach over finite differencing are stability and reduction in both memory and the number of computations required for obtaining a solution of a specified accuracy. Furthermore, compared to finite differences of the same accuracy, the pseudospectral method requires fewer grid points (a factor of 25 fewer grid points in 2D and 125 in 3D, Kosloff, 1982). With pseudospectral methods, the equation to be solved becomes

$$\frac{\partial^2\psi_{l,n}}{\partial t^2} = v^2{}_{l,n} F_{xz}^{-1}[-(k_x^2 + k_z^2)F_{xz}^{-1}(\psi_{l,n})] \tag{13}$$

where F_{xz} denotes the 2D Fourier transform with variables x and z. In order to solve this equation accurately, the differential with variable t is usually approximated by a high-order finite-difference operator. Therefore, the process of the solving equation (13) has the same properties as that of finite difference, e.g. a criterion for choosing the time difference Δt is required necessarily to meet the stability conditions (e.g. Gazdag, 1981). With a suitable choice of Δt, the interference of the wavefield between the grid points at new time level can be neglected and the whole set of equations can subsequently be approximately written as

$$\frac{\partial^2\psi}{\partial t^2} = v^2(x,z)F_{xz}^{-1}[-(k_x^2 + k_z^2)F_{xz}(\psi)] \tag{14}$$

where both ψ and v are discrestized versions distributed on the grid points.

As discussed above, both extrapolation and modeling are conceptually equivalent to each other if we consider that modeling is a extrapolation in a coordinate of time. Therefore, if we exchange the Fourier transform on a spatial coordinate into the time coordinate, we can obtain the pseudospectral method for extrapolation problem. Based on this idea, equation (10) for wavefield extrapolation can be written as

$$\frac{\partial^2 \psi_{l,n}}{\partial z^2} = \frac{1}{v_{l,n}^2}\frac{\partial^2 \psi_{l,n}}{\partial t^2} + \frac{\partial^2 \psi_{l,n}}{\partial x^2} \tag{15}$$

In analogy to equations (10) and (14), let the F_{xt} to be the *2D* Fourier Transform with variables x and t, equation (6) for wavefield extrapolation can be written as

$$\frac{\partial^2 \psi(x,z,t)}{\partial z^2} = F_{xt}^{-1}(k_x^2 - \frac{\omega^2}{v^2(x,z)})F_{xt}(\psi(x,z,t)) \tag{16}$$

If we denote $\Psi(x,z,\omega)$ as the Fourier transform with variable t to $\psi(x,z,t)$, then the equation can be written as

$$\frac{\partial^2 \Psi(x,z,\omega)}{\partial z^2} = F_x^{-1}(k_x^2 - \frac{\omega^2}{v^2(x,z)})F_x(\Psi(x,z,\omega)) \tag{17}$$

and the solution to this equation together with initial condition is

$$\Psi(x,z,\omega) = F_x^{-1}[\exp(\pm i\sqrt{\frac{\omega^2}{v^2(x,z)} - k_x^2}\, z)F_x(\Psi(x,z=0,\omega))] \tag{18}$$

It should be noticed that a divergence from common procedure occurs here in that high-order finite-difference operators are not used to approximate the differential operator on the left hand of equation (17). Instead, we solve the equation directly to the following purpose: account is taken of the higher order approximation for the time derivative, when the velocity is constant, the equation can exactly extrapolate the wavefield to any depth. However, it is important to remember that this equation is derived from equation (15) and it is only valid when the interference of the local wavefield near each grid can be neglected. Just as in usual pseudospectral method, the extrapolation distance z must be carefully considered.

If the interference of the wavefield from the grid points during the extrapolation can be neglected, the term $(k_x^2 - \frac{\omega^2}{v^2})$ can be moved to inside the Fourier Transform F_x and therefore, the solution can be written as

$$\Psi(x,z,\omega) = F_x^{-1}[F_x(\exp(\pm i\sqrt{\frac{\omega^2}{v^2(x,z)} - k_x^2}\, z)\Psi(k_x,z=0,\omega))] \tag{19}$$

Equations (18) and (19) are the PSPI and NSPS of nonstationary phase shift, respectively (Margrave & Ferguson, 1997, 1999). These can be physically explained in that with the finite extrapolation distance, every point on the wavefield at the original depth level can be considered as a wavelet source. The wavelets emitted from these sources propagate with their local velocities in a circle (2D) or sphere (3D) to the new level. The superposition of all these wavelets on the new layer forms the extrapolated wavefield. In variable velocity media, when wavelet expansion from the source, the pattern of the circle will be distorted and the extrapolation distance z is required to be sufficiently small so that the distortion can be ignored. It was noticed that the differences between NSPS and PSPI become lessened when a recursive

wave-stepping scheme was applied (Margrave & Ferguson, 1997). This is because when the z is sufficiently small, the equation (18) and (19) are equivalent to each other.

STABILITY AND ACCURACY

Both equation (18) and (19) can be written in a generalized form as

$$\Psi(x, z = h, \omega) = L_h \Psi(x, z = 0, \omega) \tag{20}$$

where L_h is the extrapolation operator in NSPS or PSPI for the depth step of h. The expression of L for non-stationary phase shift is given by Margrave (1998). In laterally inhomogeneous media, the implementation of these equations needs to be done recursively, i.e. extrapolation depth h needs to divided into n small depths ($h=n\Delta z$). Therefore, equation (20) can be written as

$$\Psi(x, z = h, \omega) = L_{\Delta z}^{n} \Psi(x, z = 0, \omega) \tag{21}$$

where $L_{\Delta z}$ is the extrapolation operator for the depth equaling Δz. The process of the extrapolation is stable if the energy is conserved during the extrapolation. The energy in the wavefield is proportional to the square of the amplitudes and therefore, the stability can be expressed as

$$\| \Psi(x, z = 0, \omega) \|^2 \geq \| L_{\Delta z}^{n} \Psi(x, z = 0, \omega) \|^2 \tag{22}$$

where $\|.\|^2$ denotes the 2-norm. Based on matrix algebra theory

$$\| L_{\Delta z}^{n} \Psi \|^2 \leq \| L_{\Delta z}^{n} \|^2 \| \Psi \|^2 \tag{23}$$

and $||L^n_{\Delta z}||2 \leq (max(\sigma_i))^n$, here σ_i denotes the singular value of the matrix $L_{\Delta z}$. Therefore, the process of extrapolation is said to be unconditionally stable when $max(\sigma i) \leq 1.0$.

Instability arises because when velocity is not constant, the formulae of the stationary phase shift only satisfy the wave equation approximately. When Δz chosen improperly, the errors due to velocity gradients may interfere constructively to create unphysical effects. Figure 3 shows the unphysical effects coming from four steps of extrapolation with $\Delta z = 50$ *m* for the same previous model. The accuracy of the extrapolation in laterally inhomogeneous media is also controlled by the extrapolation distance. The more accurate the extrapolation, the more stable it is.

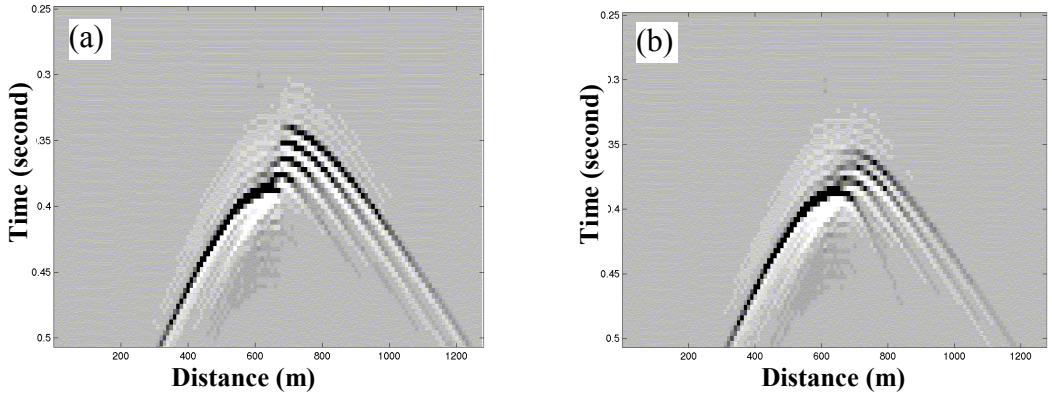

Figure 3. The results of 200m extrapolation with 4 steps of Δz=50 m by (a) PSPI and (b) NSPS.

Instead of theoretic derivation for the choice of Δz, we present a practical criterion for the choice based on Fresnel resolution. The basic assumption is that the wavelet from the secondary source expands outward with local constant velocities. Assuming there are two closed secondary sources located at both sides of a velocity boundary, when the wavelets from these sources expanding vertically to the new plane with Δz, phase difference between them is

$$\Delta\Theta = 2\pi f\left(\frac{\Delta z}{v_1} - \frac{\Delta z}{v_2}\right).$$

If $\Delta\Theta$ is less than $\pi/2$, it should be beyond the spatial resolution based on Fresenel zone concepts. Therefore, we choose

$$\Delta z \le \left| \frac{v_1 v_2}{4 \bar{f} \Delta v} \right| \tag{24}$$

where $\bar{f}$ is the dominant or average frequency and $\Delta v=|v_1-v_2|$ is the maximum velocity change of the neighbor grids in the layer, as the criteria for Δz. In numerical modeling tests, we found that the minimum extrapolation distance is limited by discrete interval Δx. The reason may because the velocity is smoothed after the Fourier Transform.

Figure 4 shows the results with the same velocity model but in this case $\Delta z = 10$ m. The first arrival of extrapolated wavefields match the travel time calculated by solving the Eiknoal equation with finite differences (Vidale, 1988). Figure 5 shows the results when extrapolating the wavefield back from the wavefield in figure 4. The original point sources are well resolved.

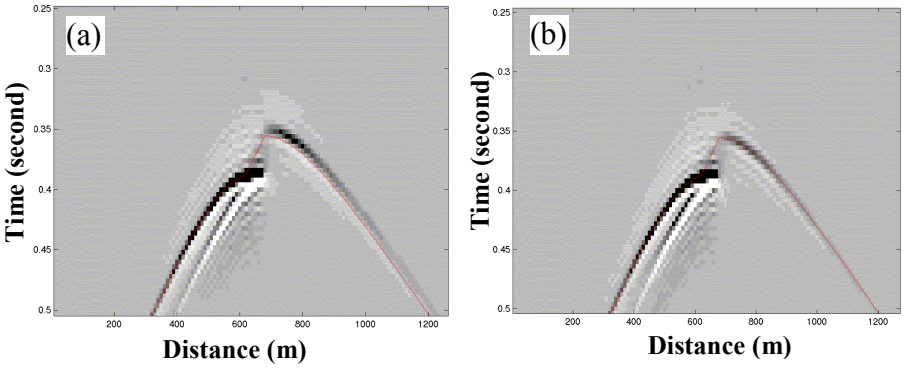

Figure 4. The results of 200m extrapolation with 20 steps of Δz=10 m by (a) PSPI and (b) NSPS. The solid lines indicate the travel time of the first arrival.

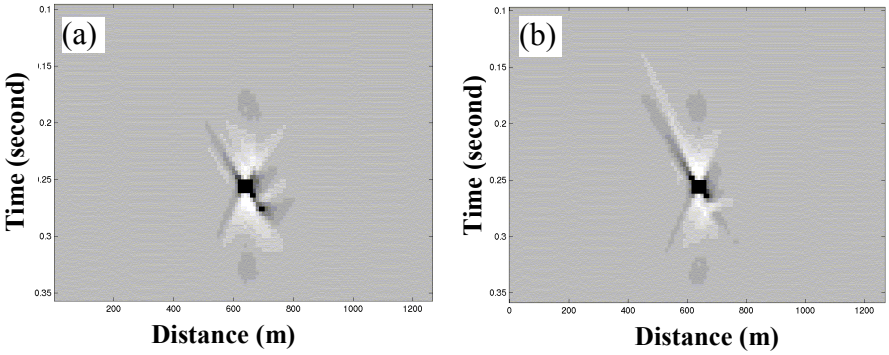

Figure 5. The result of -200m extrapolation with 20 steps of Δz=-10m from the data in figure 4 by (a) PSPI and (b) NSPS.

CONCLUSIONS

Wavefield extrapolation in laterally variable velocity structures can be calculated by the recursive nonstationary phase shift method, which can be considered as the limiting case of pseudospectral method applied to wavefield extrapolation. Like finite differences, this method may take account of any velocity variation but can give higher accuracy and can be more computationally efficient. Based on the strength of the lateral velocity gradients, the extrapolating distance can be chosen adaptively.

The derivation of the equations presented above directly applies the pseudospectral method to wave equation in discretized model and therefore, the local property of the solution is easily understood by its physical interpretation. The step length chosen based on Fresenel zone gives a practical criteria for choice of the Δz.

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