Fourth-order finite-difference scheme for P and SV waves propagating in 2D transversely isotropic media

Zhengsheng Yao and Gary F. Margrave

ABSTRACT

The velocity-stress finite difference scheme formulation for wave propagation through 2D transverse isotropic media is presented. The wave equations are solved by a finite difference scheme of fourth order spatial operators and a second order temporal operator on a staggered grid. The five elastic constants for a transversely isotropic media are explicitly used in the scheme allowing it to model wave propagation in both isotropic and transversely isotropic media with an arbitrary symmetry axis.

INTRODUCTION

Finite difference methods are widely used to model seismic wave propagation in both acoustic and elastic media and to migrate seismic data (Alford, 1974; Loewenthal et al. 1991; Stephen, 1984). One major drawback of the displacement formulation is that it may becomes unstable when the velocity contrast is very sharp, e.g. a liquid/solid interface (Vireux, 1986). This disadvantage may be overcome by using velocity gradients instead of a true discontinuity (Stephen, 1984), but requires a more complicated formulation for computer code because of the mixed derivatives, especially when anisotropic media are involved. In comparison to displacement formulation, the application of higher order finite-difference to velocity-stress wave equations is a much simpler procedure. Levander (1988) developed a second order accurate time and fourth order accurate space formulation of the 2D staggered grid scheme for modeling wave propagation in elastic isotropic media. He demonstrated that this scheme is suitable for modeling a broad class of problems, such as near surface lateral heterogeneity and laterally heterogeneous acoustic layers, which are found in exploration seismology.

In this paper, Levander's scheme is extended to wave propagation in transversely isotropic media with symmetry axes of any orientation. Free surface, symmetry and absorbing boundary conditions are discussed for a practical application.

FINITE-DIFFERENCE EQUATION

In elastic media, the relationship between stresses τ_{ij} and strains u_{ij} can be written as

$$\tau_{ij} = c_{ijkl} u_{k,l} \,, \tag{1}$$

where the summation convention has been implied. In transversely isotropic media, with a coordinate system parallel to the principle axes of anisotropy, the stiffness matrix contains 12 elastic constants, five of which are independent (e.g. Thomsen, 1986). The symmetric form is

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \tag{2}$$

where $c_{55}= c_{44}$, $c_{22}=c_{11}$, $c_{23}=c_{13}$, and $c_{12}=c_{11}-2c_{66}$ when the symmetry axis is the z-axis and $c_{66}= c_{55}$, $c_{22}=c_{33}$, $c_{12}=c_{13}$, $c_{23}=c_{22}-2c_{44}$ when the symmetry axis is the x-axis. If the chosen coordinate system does not coincide with the principal axes of the anisotropy, then the stiffness matrix will contain additional non-zero dependent elastic constants. The elastic constants in one coordinate system can be transformed, generally speaking, to any other system by

$$c_{ijkl} = a_{im} a_{jn} a_{ko} a_{lp} c_{mnop} \tag{3}$$

where the summation convention is implied and a_{ij} is the direction cosines relating to the transform coordinates.

If the derivative of (1) with a time variable is written as

$$\dot{\tau}_{ij} = c_{ijkl} v_{k,l} , \tag{4}$$

where v is velocity, then, based on Newton's law

$$\rho \dot{v}_i = \tau_{ij,j} , \tag{5}$$

where the summation convention is implied and ρ is the density, leading to a set of first order coupled differential equations being formed, where the variables are the stresses and the velocities. In the case of 2D transversely isotropic media, the equations become

$$\begin{aligned} \rho \dot{v}_1 &= \tau_{11,1} + \tau_{13,3} \\ \rho \dot{v}_2 &= \tau_{13,3} + \tau_{33,3} \\ \dot{\tau}_{11} &= c_{11} v_{1,1} + c_{15} v_{1,3} + c_{15} v_{3,1} + c_{13} v_{3,3} \\ \dot{\tau}_{33} &= c_{13} v_{1,1} + c_{35} v_{1,3} + c_{35} v_{3,1} + c_{33} v_{3,3} \\ \dot{\tau}_{13} &= c_{15} v_{1,1} + c_{55} v_{1,3} + c_{55} v_{3,1} + c_{35} v_{3,3} \end{aligned} \tag{6}$$

The elastic constants and the density may vary arbitrarily with spatial position. The continuous equations of motion may be recast into discretized equivalents using a staggered-grid approach (Levander, 1988). Applying the derivative operators for forward and backward differences in the x and z directions, D_{x+}, D_{x+}, D_{z+} D_{z-}, to u (horizontal displacement), w (vertical displacement), τ_{xx} , τ_{zz} and τ_{xz} , instead of v_i and τ_{ij}, the finite-difference equations can be written as

$$
\begin{aligned}
u_{j,i}^{m+1/2} &= u_{j,i}^{m-1/2} + \tfrac{\Delta t}{\rho_{j,i}}(D_{x+}[\tau_{xxj,i}]^m + D_{z+}[\tau_{xzj,i}]^m) \\
w_{j,i}^{m+1/2} &= w_{j,i}^{m-1/2} + \tfrac{\Delta t}{\rho_{j,i}}(D_{x-}[\tau_{xzj,i}]^m + D_{z+}[\tau_{zzj,i}]^m) \\
\tau_{xxj,i}^{m+1} &= \tau_{xxj,i}^{m} + \Delta t(c_{11\,j,i}D_{x-}[u_{j,i}]^{m+1/2} + c_{13\,j,i}D_{z+}[w_{j,i}]^{m+1/2} + \\
&+ c_{15\,j,i}(D_{z-}[u_{j,i}]^{m+1/2} + D_{x+}[w_{j,i}]^{m+1/2})) \\
\tau_{zzj,i}^{m+1} &= \tau_{zzj,i}^{m} + \Delta t(c_{13\,j,i}D_{x-}[u_{j,i}]^{m+1/2} + c_{33\,j,i}D_{z+}[w_{j,i}]^{m+1/2} + \\
&+ c_{35\,j,i}(D_{z-}[u_{j,i}]^{m+1/2} + D_{x+}[w_{j,i}]^{m+1/2})) \\
\tau_{xzj,i}^{m+1} &= \tau_{xzj,i}^{m} + \Delta t(c_{15\,j,i}D_{x-}[u_{j,i}]^{m+1/2} + c_{35\,j,i}D_{z+}[w_{j,i}]^{m+1/2} + \\
&+ c_{55\,j,i}(D_{z-}[u_{j,i}]^{m+1/2} + D_{x+}[w_{j,i}]^{m+1/2}))
\end{aligned}
\tag{7}
$$

with a second order approximation to the time derivative discretised at intervals of Δt. Figure 1 shows the locations for the variables defined for each cell. The derivative operators D with plus and minus subscripts correspond to the forward and backward differences in x and z directions and their fourth order forms can be found in Levander (1988). The fourth order forward finite-difference operator, e.g. in x direction is

$$D_{x+}[w_{j,i}] = \{\tfrac{9}{8}(w_{j,i+1} - w_{j,i}) - \tfrac{1}{24}(w_{j,i+2} - w_{j,i-1})\} / \Delta x \quad . \tag{7}$$

The cells for updating the variables are shown in Figure 2. The elastic constants in Figure 2, c_{15} and c_{35} are not zero when the axis of symmetry of the anisotropy is not parallel to the coordinate axes.

Boundary conditions

a. Free surface boundary conditions. A free surface implies that there are no normal and shear stresses, i.e. for a surface normal to z-axis,

$$\tau_{zz} = 0, \qquad \tau_{xz} = 0 \quad , \tag{8}$$

Because the shear stress is exactly defined at the top boundary (Figure 1), the free surface shear stress boundary condition can be applied by simply setting τ_{xz} equal to zero and placing a fictitious set of nodes above it. The boundary condition for normal stresses can be applied by making the normal stress anti-symmetric for the two rows of fictitious nodes above the top, i.e.

$$\tau_{zz-1,i} = -\tau_{zz0,i} \qquad \tau_{zz-2,i} = \tau_{zz1,i} \quad , \tag{9}$$

which implies as normal stresses of zero at the surface. At a stress free boundary to the z-axis, the velocities are set to satisfy the equations:

$$c_{33}w_z + c_{35}u_z = -(c_{35}w_x + c_{13}u_x) \tag{10a}$$

$$c_{35}w_z + c_{55}u_z = -(c_{55}w_x + c_{15}u_x) \tag{10b}$$

b. Absorbing boundary conditions. There are a number of ways to apply absorbing boundary conditions. Radiation conditions may be satisfied explicitly (Clayton and Engquist, 1977; Stacey, 1988), or the solution may be tapered over a thin strip along the boundary (Cerjan et. al., 1985; Loewenthal et al., 1991). To meet radiation conditions, only two fictitious strips of nodes along the boundary for fourth order operators are required, whereas tapering generally requires more strips. However, for media with general elastic constants, tapering is the easiest to implement.

c. Symmetry boundary condition. A symmetry boundary condition implies that a mirror image of the model exists on the other side of the boundary. This boundary condition is easily implemented.

Source functions

Three functions are commonly used as source functions and may be expressed as follows:

Gaussian function

$$g(t) = exp(-\alpha t^2) \tag{11a}$$

the first derivative of Gaussian function

$$g(t) = -2\alpha t\, exp(-\alpha t^2) \tag{11b}$$

and the second derivative of Gaussian function

$$g(t) = (4\alpha^2 t^2 - 2\alpha)\, exp(-\alpha t^2) \tag{11c}$$

Source types

Source waveform as explosive, shear, horizontal or vertical point sources may be introduced by appropriately weighting the stresses or velocities at the source node or nodes (Aboudi, 1971). Assuming that the point source is located at grid point (iz,ix), three types of source may commonly be used for different purposes of modeling and may be implemented as follows:

Pressure source: this is used as to model the *P*-wave source and can be set by the source function acting as the stresses to the source location

$$\tau_{xx}(iz,ix) = g(t) \qquad , \qquad \tau_{zz}(iz,ix) = g(t) \tag{12a}$$

S wave source: this can be implemented by the source function acting to the velocity variables as

$$\begin{aligned} u(iz,ix) &= g(t) & w(iz,ix) &= g(t) \\ u(iz-1,ix) &= -g(t) \quad , & w(iz,ix+1) &= g(t) \end{aligned} \tag{12b}$$

Normal stress/velocity source: this can be implemented simply by acting the source function to τ_{zz} or w, respectively, i.e.

$$\tau_{zz}(iz,ix) = g(t) \qquad , \qquad w(iz,ix) = g(t) \tag{12c}$$

Stability condition

The following quantities are used for stability condition:

$$\Delta t \le 0.6\,\Delta x / v_{max} \qquad \lambda_{min} \ge 8\Delta x \quad , \tag{13}$$

where λ is the wave length and v is the propagating velocity.

EXAMPLE

Two models of elastic media are adopted for the numerical calculations. Model I is a solid plexiglass-aluminum model (White, 1982), in which $c_{11}=51.8$, $c_{33}=21.4$, $c_{13}=13.0$, $c_{55}=3.65$ and $\rho=1.95$. The maximum and the minimum velocities are *5154.04 m/s* and *1377.74 m/s*, respectively. Model II is the Gypsum-soil model (Sakai & Kawasaki 1990), in which $c_{11}=28.4$, $c_{33}=8.5$, $c_{13}=4.3$, $c_{55}=3.0$ and $\rho=2.35$. The maximum and minimum velocities are *3476.36 m/s* and *1129.87 m/s*, respectively. Both models are very highly anisotropic. The parameters, ε (Thomsen, 1986), for measuring anisotropy are *0.71* for model I and *1.17* for model II. A vertical source with the source function is chosen as the second derivative of the Gaussian function, with $\alpha=4000$. Since that the constant α controls the dominant frequency of the source function ($\alpha \approx 10$ times of the dominant frequency f_d squared), the dominant frequency is 20 Hz. The size of the grid used for finite-difference calculation is 5×5 m^2 and the time step Δt = 0.0005 second.

Figures 3 and 4 provide snapshots of both the dilation and rotation wavefields at $t=0.45$ seconds for models I and II, which display three principal effects of anisotropy: noncircular propagation of wave fronts; deviation of polarization of particle motions of qP and qSV waves from those in an isotropic medium; multi-wavefront of qSV waves.

In order to highlight these anisotropic effects, seismograms (Figures 5 and 6) recorded 500 meters away from the source location at 10^0 intervals. Both qP and qSV waves appearing at both radial and transverse components agree with the results obtained from the displacement potential method (White, 1982) and Cagniard integral method (Sakai & Kawasaki, 1990).

CONCLUSIONS

The fourth-order finite-difference scheme on staggered grids solving velocity-stress wave equations is presented. Boundary conditions, source functions and source types are discussed in relation to the practical implementations. Numerical examples indicate that the wavefront in anisotropic media, unlike wave propagation in isotropic media, is no longer circular. The polarization of particle motions of qP and qSV waves are not perpendicular and tangential to the wavefront. The multi-wavefront of qSV waves complicates the waveform. The fact that the numerical results are greatly agree with the analytical solutions proves that the finite-difference scheme works well.

ACKNOWLEDGEMENT

We thank the sponsors of the CREWES Project for their financial support.

REFERENCES

Aboudi, J., 1971, Numerical simulation of seismic sources, Geophysics, **36**, 810-821.

Alford, R. M., Kelly, K. R., and Boore, D. M., 1974, Accuracy of finite difference modeling of the acoustic wave equation: Geophysics, **39**, 834-842.

Clayton, R. and Engquist, B., 1977, Absorbing boundary conditions for acoustic and elastic wave equations, Bull. Seis. Soc. Am., **67**, 1529-1540.Levander, A.R., 1988, Fourth-order finite-difference P-SV seismograms, Geophysics, **53**, 1425-1436.

LoewenthalD., Wang, C.J. , Johnson, O.G. and Juhlin, C., 1991, High order finite difference modeling and reverse time migration. Exploration Geophysics, **22**, 533-545.

Stephen, R.A., 1984, Finite-difference seismograms for laterally varing marine model. Geophys. J. Roy. Astr. Soc. , **72**, 185-198.

Thomsen, L., 1986, Week elastic anisotropy. Geophysics, **51**, 1954-1966.

Sakai, Y. and Kawasaki, I., 1990, Analytic waveforms for a line source in a transversely isotropic medium. J. G. R., **95**, 11333-11344.

Virieux,J., 1996, P-SV wave propagation in heterogeneous media: velocity-stress finite-difference method. Geophysics, **51**, 889-901.

White, J.E., 1982, Computed waveforms in transversely isotropic media. Geophysics, **44**, 771-783.

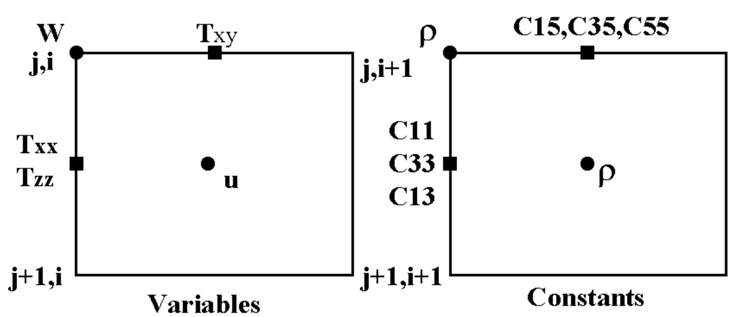

Figure 1. Location of discretized variables and constants on finite-difference grid.

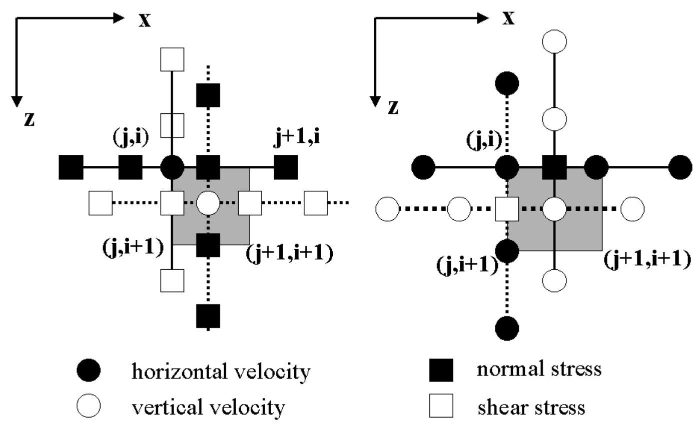

Figure 2. Staggered finite-difference grid and spatial stencils for velocity and stress update.

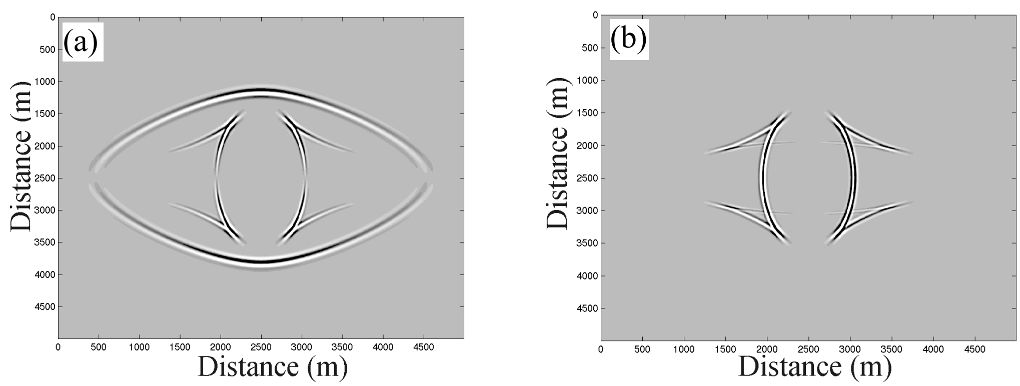

Figure 3. Snapshots of wavefield for model I (a) dilatational component (b) rotational component.

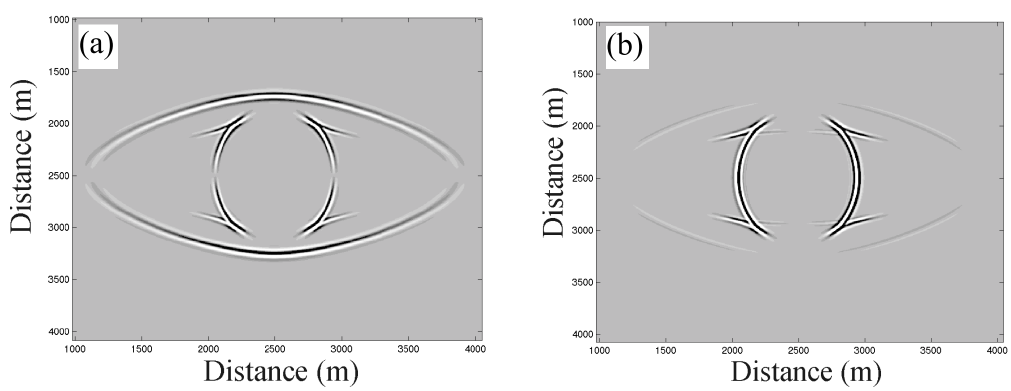

Figure 4. Snapshots of wavefield for model II (a) dilatational component (b) rotational component.

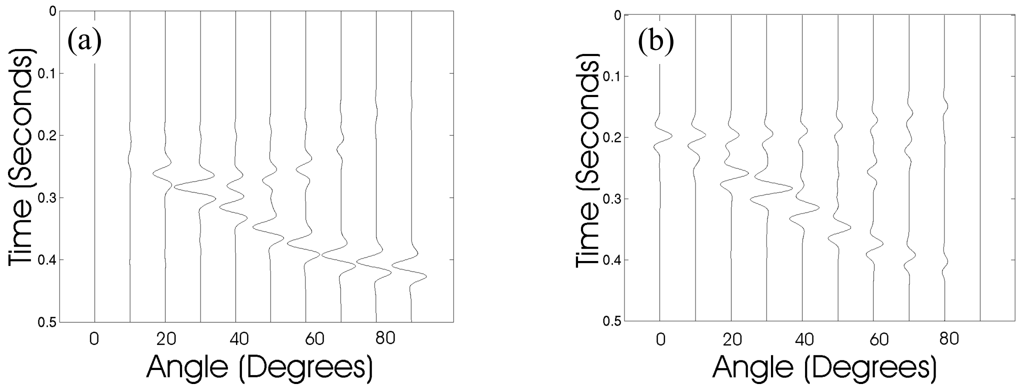

Figure 5. Seismograms recorded 500 meters away from the source location at ten degree intervals for model I. (a) Radial and (b) transverse components.

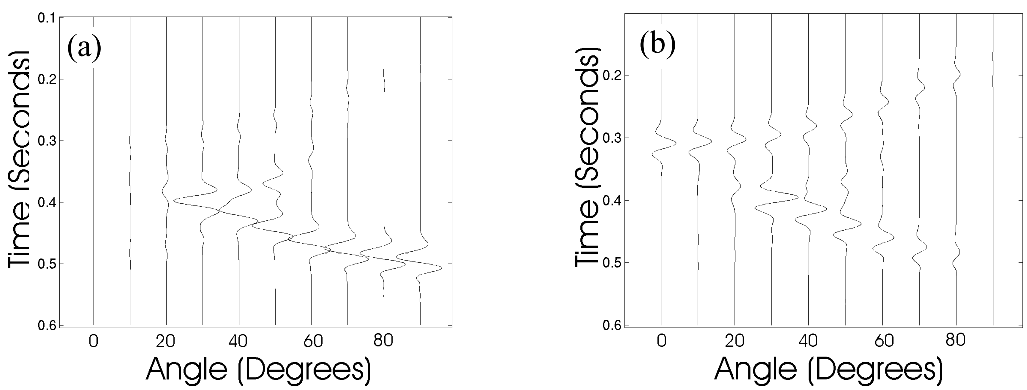

Figure 6. Seismograms recorded 500 meters away from the source location at ten degree intervals for model II. (a) Radial and (b) transverse components.

APPENDIX: DESCRIPTION OF USING TI_2D.F

Program name:

TI_2D.f

Function:

Modeling elastic wave propagation in 2D transversely isotropic inhomogeneous media.

Method:

Solving wave equation in the form of velocity-stress by Fourth-order finite-difference.

Language:

FORTRAN.

Date:

November, 1999.

Author:

Yao, Zhengsheng.

Requirements:

Input parameter file must be named as "TI_2D.in" (ASCII format).

Paremeters:

The parameter file contains only numbers, which must be given in following order:

Line 1: only one number, which time step in seconds (real).

Line 2: only one number, which is the total number of samples in time (integer).

Line 3: only one number, which is the index of the depth level where receiver is located (integer).

Line 4: only one number, which is the source wavelet dominant frequency (real).

Line 5: contain two characters, which are indications of the wave type and the source type, respectively. There are three options for wave type: "ga" for the Gaussian function, "dg" for the first derivative Gaussian function, and "d2" for the second derivative Gaussian function. There are four options are for source type: "p" for pressure sources, "s" for S-wave sources, "n" for normal stress sources, and "w" for normal velocity sources.

Line 6: contains two integer numbers, which are the horizontal and vertical distance indices, respectively, for the source location.

Line 7: contains four integer numbers, which indicate the boundary conditions at the top, the bottom, the left, and the right boundary, respectively. Each of the four number has three options: "1" for absorbing, "2" for symmetry, and "3" for free surface boundary condition, respectively.

Line 8: only one number, which is the number of files for the elastic constants of the medium, i.e., rho (the density), c11, c13, c33, c15, c35 and c55. If the symmetric axis is parallel to x or z axis, only 5 files are needed.

If there are 5 elastic constant files, the file names are give in Line 9 to line 13 with each line contains only one name (character string). If there are 7 elastic constant files, the file names are give in Line 9 to line 15 with each line contains only one name (character string).

Line 14 or Line 16: output file name (character string) of the x-component seismogram.

Line 15 or line 17: output file name (character string) of the z-component seismogram.

Line 16 or line 18: contains only one integer number, which is the time index when a snapshot is needed.

Line 17 or line 19: the file name (character string) of the snapshot indicated in line 16 or line 18.

From Line 18 or line 20, every two lines must be in a pair, where the first line contains an integer number indicating the snapshot time index and the second line gives the file name (character string) for saving the snapshot data. These pairs are in the same format as the pair of Line 16 and Line 17 or Line 19 and Line 20.

Format of files for the elastic constants:

If the number of elastic constants is 5, 5 files have to be prepared. If the number of elastic constants is 7, then 7 files are needed. Each of these files contains only the values of one fixed elastic constant. For example, if a file contains values of c13, then the values at all the nodes will be read as c13-values.

The 5 or 7 files have the save format in terms of how the numbers for the z-x grids are given. In detail format is as follows:

1. The file should be of ASCII text file.

2. The first line of the file includes 4 numbers. The first two numbers are integers, which are the total number of nodes in z-direction and the total number of the nodes in x-direction, respectively, and in this order (z first then x). The third and the fourth numbers are real, and they are the grid widths in z- and x-directions, respectively, and in this order (z first then x).

3. Start from line 2, the elastic constants should be given by sequential z-levels from the first to the last. Different z-level numbers can not be in the same line. For each z-level, the values must be given from the first to the last x-nodes.