Multidimensional partitions of unity and Gaussian terrains

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ABSTRACT

Partitions of unity play an important rôle as amplitude-preserving windows for nonstationary type filters. Multidimensional partitions of unity can be constructed by combining one-dimensional partitions as factors. By grouping neighbouring windows, arbitrary nonuniform partitions of unity can be formed, which can greatly reduce computation times.

The Gaussian function displays a host of desirable properties, making it particularly well suited for use as an "atom" in multidimensional partitions of unity.

INTRODUCTION

Windowing functions, or "atoms", form the basis for new Gabor transform (Gabor, 1946) processing techniques such as Gabor phase-shift migration (Grossman, et al., 2002b, this volume) and nonstationary deconvolution (Margrave and Lamoureux, 2001, and Grossman, et al., 2001). They also arise in other seismic methods such as Gaussian beam migration, and, more generally, in time- or space-variant filtering.

A required property of such windows is that they may be used to form partitions of unity by summing together shifted versions of the window. A partition of unity is a collection of functions that sum to one everywhere on the domain of interest. Thus, when used in conjunction with analysis and synthesis windowing, they ensure an overall amplitude-preserving transformation (see Grossman, et al., 2002a, this volume, for more details). Another expected quality of a windowing function is a high order of differentiability, or smoothness. This gives good spectral domain behaviour and avoids undesirable effects such as Gibb's phenomenon, or spectral ringing. In addition, for Gabor analysis we wish to retain, as much as possible, resolution in both the spatial and Fourier domains. Ideally, the atoms should also be compact (i.e. supported on a finite connected domain) or at least pseudo-compact (i.e. negligible outside such a domain).

Gabor phase-shift extrapolators (Grossman et al., this volume) have employed partitions of unity along one spatial dimension in the context of 2D migration. In order to extend this approach to 3D migration, it is necessary to consider partitions of unity in two spatial dimensions. This motivates the generalization of existing partition of unity theory to the multidimensional setting. In this case, it is also desirable that the resulting multidimensional atoms display isotropic behaviour. This equates to having radial symmetry so that no direction is artificially preferred.

In this paper, we show how a multidimensional partition of unity may be constructed from any one-dimensional partition of unity. We also demonstrate that the Gaussian window almost completely satisfies the above requirements.

THEORY AND EXAMPLES

Constructing a 2D partition of unity

Assume that we have an exact partition of unity along one spatial dimension, *x*, given by the atom, $f(x)$, and its translations $f_m(x) \equiv f(x - m\Delta x)$, where $m \in \mathbb{Z}$, the integers. This is expressed mathematically as follows:

$$\sum_{m=-\infty}^{\infty} f_m(x) \equiv 1, \forall x \,. \tag{1}$$

Assume also that we have an exact partition of unity along another orthogonal dimension, *y*, given by the atom $g(x)$ and its translations:

$$\sum_{n=-\infty}^{\infty} g_n(y) \equiv 1, \forall y \,. \tag{2}$$

It may be that *f* and *g* are the same function or they may be different. We now multiply equation (1) by equation (2) to obtain

$$\sum_{m=-\infty}^{\infty} f_m(x) \sum_{n=-\infty}^{\infty} g_n(y) \equiv 1, \forall x, y \,,$$

which may be rewritten as

$$\sum_{m,n \in \mathbb{Z}} h_{mn}(x, y) \equiv 1, \forall x, y \,, \tag{3}$$

where $h_{mn}(x, y) \equiv f_m(x) g_n(y)$.

Clearly, equation (3) describes a covering of the plane by the 2D atom $h(x, y) \equiv f(x)g(y)$ and its translations to grid points $h_{mn}(x, y) \equiv h(x - m\Delta x, y - n\Delta y)$. Since the right-hand side of equation (3) is 1, we have shown how to construct a 2D partition of unity from any pair of 1D partitions of unity.

This can be extended to the general N-dimensional case in the obvious way.

The Gaussian atom

A Gaussian function that can be used as an atom has the form,

$$g(x) = \frac{\Delta x}{\sigma \sqrt{\pi}} \mathrm{e}^{-x^2/\sigma^2} \,, \tag{4}$$

where σ is the "halfwidth" of the Gaussian (standard deviation) and Δx is the separation between atoms. The Gaussian has the following useful properties:

Smoothness

$g(x)$ is infinitely differentiable, as may be readily verified. In fact, the Fourier transform of a Gaussian function is another Gaussian, (of halfwidth π / σ) which has no ringing in the frequency domain.

Resolution

The Gaussian atom has optimal resolution properties in the spatial and Fourier domains: it is the only function that exactly minimizes Heisenberg's uncertainty relationship, the fundamental limit on the product of resolutions in both domains. The proof that the Gaussian satisfies this limit is given in Appendix A.

Compactness

The Gaussian is *not* compact, since it has support on all of $\mathbb{R}$. However, for a given machine precision *eps*, we have the following:

$$\left|\exp(-\frac{x^2}{\sigma^2})\right| < eps \quad \Leftrightarrow \quad |x| > \sigma\sqrt{\left|\ln(eps)\right|} \tag{5}$$

for a normalized Gaussian (i.e., $g(x)/g(0)$). This provides a limit on the support of any Gaussian *g*, beyond which its numerical representation on the computer is effectively zero. Thus, we have *effective* compactness. In MATLAB®, for example, we find using (5) that the Gaussian decays quickly enough to allow truncation after about six halfwidths, σ.

Partition of Unity

The Gaussian does *not* yield an exact partition of unity. However, it is a very good approximation to one for good choices of Δx and σ. Margrave and Lamoureux (2001) show that

$$\sum_{n\in\mathbb{Z}} g(x-n\Delta x) = 1 + 2\cos(2\pi x/\Delta x)\,\mathrm{e}^{-(\pi\sigma/\Delta x)^2} + \cdots, \tag{6}$$

with exponentially smaller higher order terms. The deviation from 1 is approximately 10^{-4} when $\sigma = \Delta x$, i.e. when the spacing of atoms is equal to the width of the Gaussian. Arbitrarily small errors may be achieved by further reduction in the spacing Δx.

An illustration of a one-dimensional partition of unity using the Gaussian atom is shown in Figure 1. Note that the roll-off at the edges can be removed by including atoms outside of the domain, or by appropriate normalization.

The 2D Gaussian atom

If Gaussian atoms are substituted for both *f* and *g* in equation (3), we obtain

$$\begin{aligned} h(x,y) = f(x)g(y) &= \frac{\Delta x}{\sigma_x\sqrt{\pi}}\mathrm{e}^{-x^2/\sigma_x^2}\,\frac{\Delta y}{\sigma_y\sqrt{\pi}}\mathrm{e}^{-y^2/\sigma_y^2} \\ &= \frac{\Delta x\Delta y}{\sigma_x\sigma_y\pi}\mathrm{e}^{-\left(x^2/\sigma_x^2+y^2/\sigma_y^2\right)} \end{aligned}, \tag{7}$$

where σ_x, σ_y are the widths and $\Delta x, \Delta y$ are the spacings of the atoms in the *x* and *y* directions. If we set $\sigma_x = \sigma_y = \sigma$ and $\Delta x = \Delta y = \Delta$, then we have the 2D Gaussian function:

$$h(x,y) = \frac{\Delta^2}{\sigma^2 \pi} \mathrm{e}^{-r^2/\sigma^2}, \tag{8}$$

where $r^2 = x^2 + y^2$. Equation (8) is radially symmetric, so the 2D Gaussian has the desired property of isotropic behaviour, and is unique in doing so (see Appendix B). The 2D Gaussian atom is shown in Figure 2. We have set $\sigma = \Delta = 1$, for simplicity.

As mentioned in the 1D case, the 2D Gaussian atom *h* does not construct an exact partition of unity. Applying equation (6) along both *x* and *y* axes, we obtain:

$$\sum_{m,n \in \mathbb{Z}} h_{mn}(x,y) = 1 + 2\left[\cos(2\pi x/\Delta) + \cos(2\pi y/\Delta)\right] \mathrm{e}^{-(\pi\sigma/\Delta)^2} + \cdots, \tag{9}$$

showing that the maximum error is at most double that for the 1D case.

To help visualize the Gaussian (approximate) partition of unity concept, we redisplay the atom in 3D, as a "Gaussian hill" (Figure 3a). We now sum all translations of this hill for a given value of *x*, to form a "Gaussian ridge":

$$\sum_{n \in \mathbb{Z}} h_{mn}(x,y) = f_m(x) \sum_{n \in \mathbb{Z}} g_n(y) \approx f_m(x), \tag{10}$$

which is (approximately) invariant with *y* (Figure 3b). Such a ridge can be constructed for each value of *m*, and then a summation over *m* gives the "Gaussian plateau" (Figure 3c), which is close to 1 everywhere. As for the 1D case, the roll-off at the edge can be removed by extending the summation to atoms outside of the domain.

Nonuniform multidimensional partitions of unity

A nonuniform partition of unity may be constructed by starting with a maximally redundant partition of unity, where the translation shift Δx is equal to the grid sampling of the domain. In areas where larger windows are desired, for example where variations of the property being sampled are mild, several adjacent atoms can be summed to form a "molecule".

This principle can be applied to multidimensional cases in two alternative ways. The nonuniform partitioning may be applied along each axis, and then the resulting partitions multiplied to obtain the multidimensional partition. This results in rectangular partitions. Alternatively, the full maximally redundant set of multidimensional atoms may be constructed. These may then be used to form multidimensional molecules by summing arbitrary subsets. Such a molecule is constructed in Figure 4. Figure 4(a) shows the mask of grid points used to construct the molecule, whilst a colour display of the atom is shown in Figure 4(b). Three-dimensional displays of the molecule, nicknamed "Gaussian Butte" and its complement are shown in Figure 5.

DISCUSSION

Although the Gaussian is not an exact partition of unity, it can be normalized to make it so (as can any chosen atom, for that matter). This is achieved simply by dividing the individual atoms by the sum of all atoms. Doing so achieves exact partition at the

expense of losing exact isotropy for the multidimensional cases. Whether this trade-off is worthwhile no doubt depends on the application.

The ability to generate molecules of a general shape in the multidimensional case may prove to be useful in representing velocity models for migration. For example, one might use a single molecule to characterize a salt body, whilst using smaller molecules or the underlying atoms for the surrounding sediments.

Though we have focused on radially symmetric 2D Gaussian atoms, equation (7) may be used to construct atoms that are elliptical in shape. These might be useful in cases, such as layered media, where material changes are significantly faster in one direction than in another. An alternative way to deal with this is to start with radially symmetric atoms and combine them to form elongated atoms.

CONCLUSIONS

Any one-dimensional partition of unity can be used to construct multidimensional partitions of unity. Such a partition forms a set of amplitude-preserving windowing functions.

The multidimensional Gaussian is almost ideal as an atom, since it behaves isotropically, has infinite differentiability, and optimizes resolution in spatial and Fourier domains simultaneously. Whilst not strictly compact, it is effectively so. It forms a near partition of unity, with errors that can be reduced to any desired level by sufficiently close spacing of the atoms.

The concept of a nonuniform partition of unity has been extended in a general way to the multidimensional case.

ACKNOWLEDGEMENTS

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APPENDIX A

Heisenberg's Uncertainty Principle and the Gaussian

The dispersion (about a) of a square differentiable function, g, is given by:

$$\Delta_a g \equiv \frac{\int_{-\infty}^{\infty} (x-a)^2 \left|g(x)\right|^2 dx}{\int_{-\infty}^{\infty} \left|g(x)\right|^2 dx}. \tag{A1}$$

Let the Fourier transform of g be denoted $\hat{g}$:

$$\hat{g}(\xi) \equiv \int_{-\infty}^{\infty} e^{-2\pi i x\xi} g(x) dx \quad . \tag{A2}$$

The dispersion in the Fourier domain (about α) is likewise given by:

$$\Delta_\alpha \hat{g} \equiv \frac{\int_{-\infty}^{\infty} (\xi-\alpha)^2 \left|\hat{g}(\xi)\right|^2 d\xi}{\int_{-\infty}^{\infty} \left|\hat{g}(\xi)\right|^2 d\xi}. \tag{A3}$$

Heisenberg's Uncertainty Principle places a lower limit on the product of dispersions as follows:

$$\Delta_a g \Delta_\alpha \hat{g} \geq \frac{1}{16\pi^2}. \tag{A4}$$

This describes a trade-off between resolution in one domain and achievable resolution in its Fourier dual domain. In quantum physics, this translates to uncertainty between position and momentum, due to the relationship between momentum and wavenumber (e.g. Landshoff and Metherell, 1979).

We now compute equations (A1) and (A3) in the case of a Gaussian. Substituting equation (4) into (A2) gives another Gaussian:

$$\begin{aligned} \hat{g}(\xi) &= \frac{\Delta x}{\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-2\pi i x\xi} e^{-x^2/\sigma^2} dx \\ &= \Delta x e^{-\pi^2\sigma^2\xi^2} \end{aligned}. \tag{A5}$$

The minimal dispersion for these Gaussians occurs for $a = 0$ and $\alpha = 0$. In this case, we have:

$$\Delta_{a=0} g = \frac{\int_{-\infty}^{\infty} x^2 e^{-2x^2/\sigma^2} dx}{\int_{-\infty}^{\infty} e^{-2x^2/\sigma^2} dx}. \tag{A6}$$

Integrating by parts:

$$\int_{-\infty}^{\infty} x^2 e^{-2x^2/\sigma^2} dx = -\frac{\sigma^2}{4} \int_{-\infty}^{\infty} x \frac{d}{dx}\left(e^{-2x^2/\sigma^2}\right) dx$$
$$= \frac{\sigma^2}{4} \int_{-\infty}^{\infty} e^{-2x^2/\sigma^2} dx$$

Thus:

$$\Delta_{a=0} g = \frac{\sigma^2}{4}. \tag{A7}$$

Similarly, it can be shown that:

$$\Delta_{\alpha=0} \hat{g} = \frac{1}{4\pi^2\sigma^2}. \tag{A8}$$

Hence, combining (A7) and (A8), equality holds in the Heisenberg relationship (A4).

This shows that the Gaussian function attains the best possible simultaneous resolution in both spatial and Fourier domains. It can also be demonstrated that only the Gaussian has this property (pers. comm., P. Gibson).

APPENDIX B

Suppose that $f(x)$ is a symmetric, bounded, non-negative, real-valued function, which is at least twice continuously differentiable everywhere, and with $f(0) > 0$. These are reasonable conditions to impose on a potential window candidate. We want to determine the class of functions, f, that satisfy these requirements, plus the following condition: if h factors as

$$h(x,y) = f(x)f(y), \tag{B1}$$

then h is radially symmetric, i.e., for any $\theta \in [0, 2\pi]$, we require

$$h(r,\theta) = h(r,0), \text{ where}$$
$$r(x,y) = \sqrt{x^2 + y^2}, \quad x = r\cos(\theta), \text{ and } \quad y = r\sin(\theta). \tag{B2}$$

We claim that any such f can be represented as a Gaussian. We first show that f cannot have compact support (so $f(x) > 0$ everywhere). Indeed, if $h(r_0) = 0$ at some $r_0 > 0$, (note $h(0) > 0$) then for any $\theta \in [0, 2\pi]$, we would have:

$$0 = h(r_0, 0) = h(r_0, \theta) = f(r_0 \cos\theta) f(r_0 \sin\theta). \tag{B4}$$

However, this would imply that f vanishes at $r_0/\sqrt{2}$. Iterating by way of expression (B4), we conclude that f vanishes at a sequence of points, namely $r_n = r_0/2^{n/2}$. But $r_n \to 0$ as $n \to \infty$, so $f(0) = 0$ by continuity, a contradiction. In particular, f cannot have compact support.

Next, observe that we can represent f as follows:

$$f(x) = f(0) e^{x^2 g(x)}, \tag{B5}$$

provided
$$g(x) = \begin{cases} \dfrac{\ln\left(f(x)/f(0)\right)}{x^2}, & \text{if } x > 0 \\ f''(0)/[2f(0)], & \text{if } x = 0 \end{cases} \tag{B6}$$

Here $g(0)$ is computed as the limit of $g(x)$ as $x \to 0$ (apply l'Hopitals rule twice, using the fact that f is twice continuously differentiable). This ensures continuity of g, and validates the representation (B5).

We now argue that $g(x)$ is in fact a negative constant, and hence deduce from (B5) that f must be a Gaussian. First, since h is radially symmetric,

$$h(r, 0) = h(r, \theta) = f(r\cos\theta) f(r\sin\theta), \tag{B7}$$

we are free to choose any value of θ, say $\theta = 0$ and $\theta = \pi/4$ without changing the product (B7). Substitution of these values into (B7) thus yields

$$\left[f\left(\frac{r}{\sqrt{2}} \right) \right]^2 = f(0) f(r). \tag{B8}$$

If we then represent both sides of (B8) as in (B5), we easily conclude that

$$g\left(\frac{r}{\sqrt{2}} \right) = g(r). \tag{B9}$$

This implies that g is constant for any given $r > 0$, and by continuity of g, we see that

$$g \equiv \lim_{x \to 0} g(x) = f''(0)/[2f(0)]. \tag{B10}$$

Now g is either positive, negative or 0. The case $g = 0$ gives a constant f, which is not of interest, though might be regarded as a degenerate Gaussian. If $g > 0$, f is no longer bounded. Hence $g < 0$ and we have:

$$\begin{aligned} f(x) &= f(0) e^{-x^2/\sigma^2}, \text{ where} \\ \sigma^2 &= -2f(0)/f''(0). \end{aligned} \tag{B11}$$

Equation (B11) completes the proof that f can be represented as a Gaussian.

FIGURES

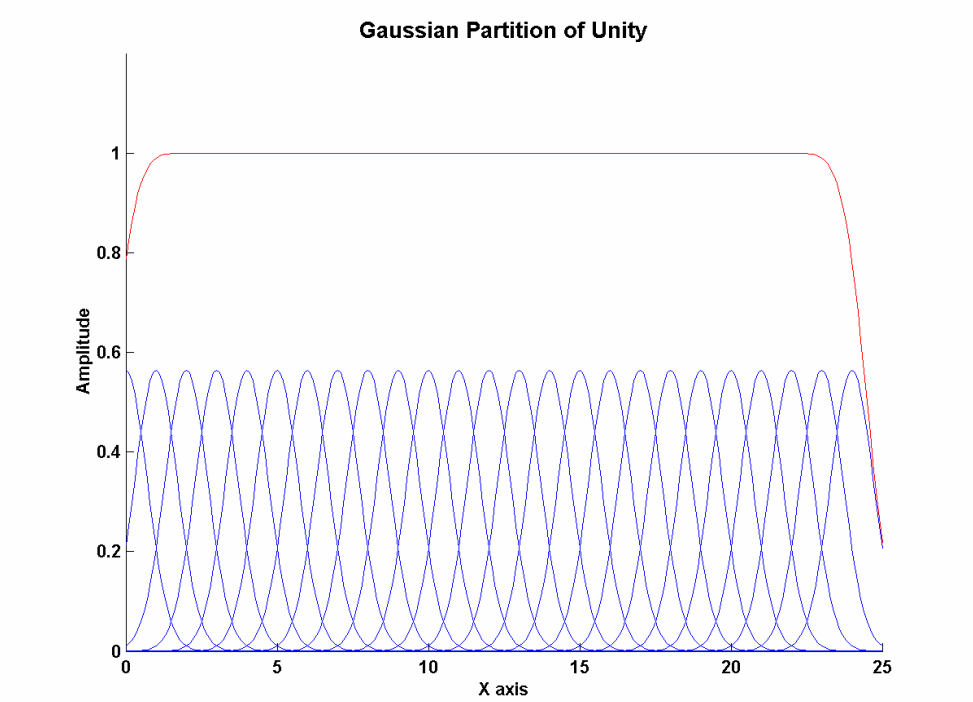


FIG 1. One-dimensional partition of unity using a Gaussian atom. The numbering along the x-axis refers to the Gaussian atom number. Roll-off effects occur because a finite number of Gaussians are used.

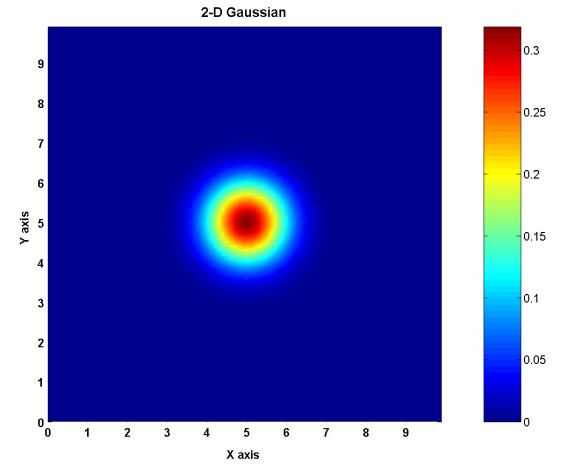
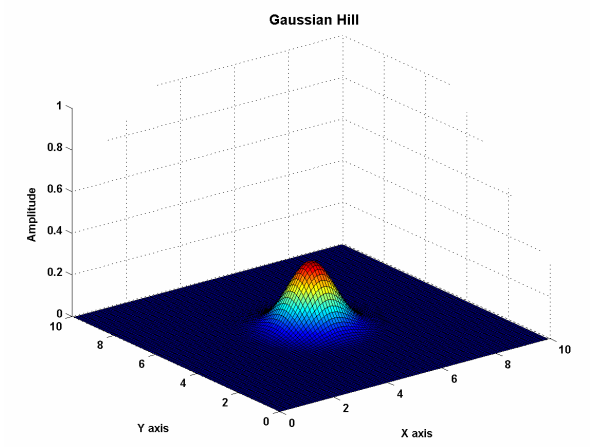
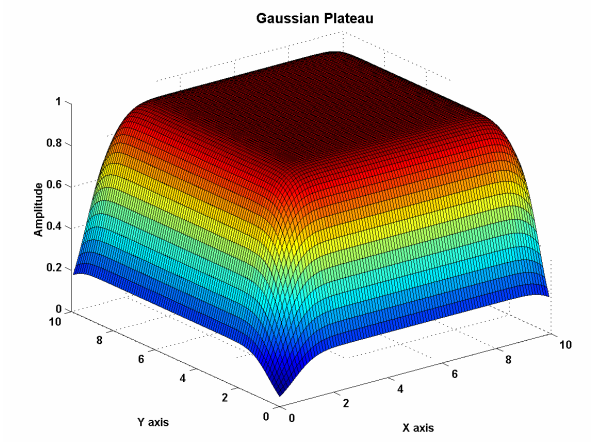


FIG 2. Two-dimensional Gaussian atom displayed as contour plot.



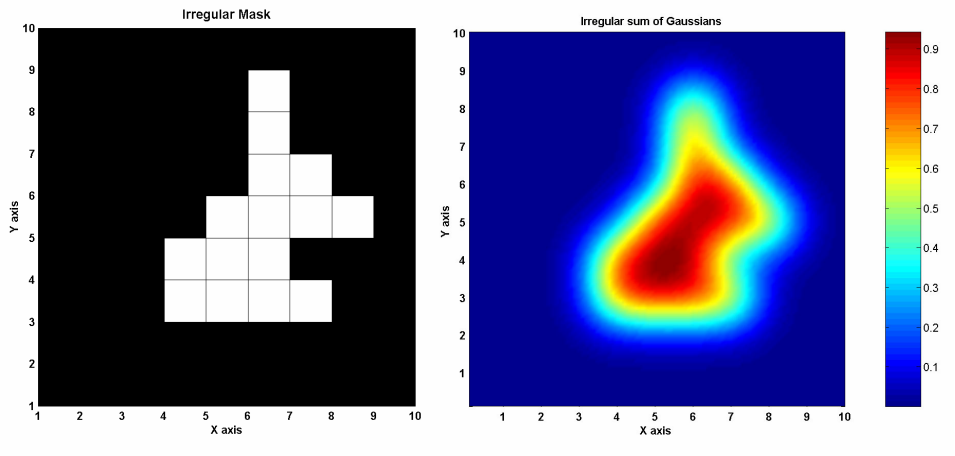
(a)

(b)



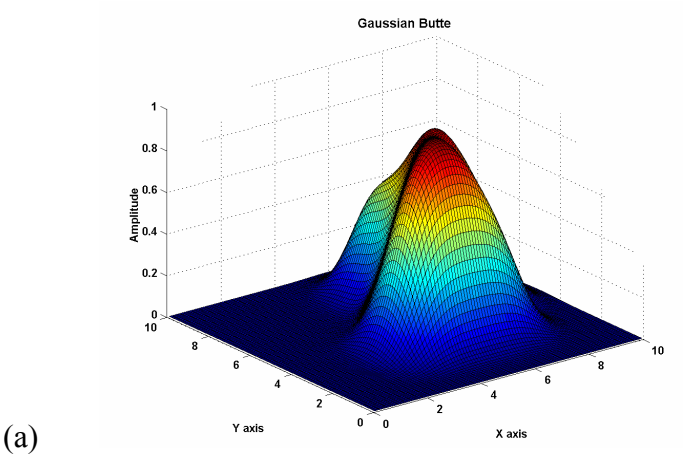
(c)

FIG 3. Almost exact partition of unity using 2D Gaussian (“Gaussian hill”) atoms (a). Summation of 2D Gaussians along a constant x coordinate of grid gives the “Gaussian ridge” (b). Summing Gaussian Ridges for all x coordinates gives the Gaussian plateau (c). Edge effects can be removed by extending summation beyond domain of interest.

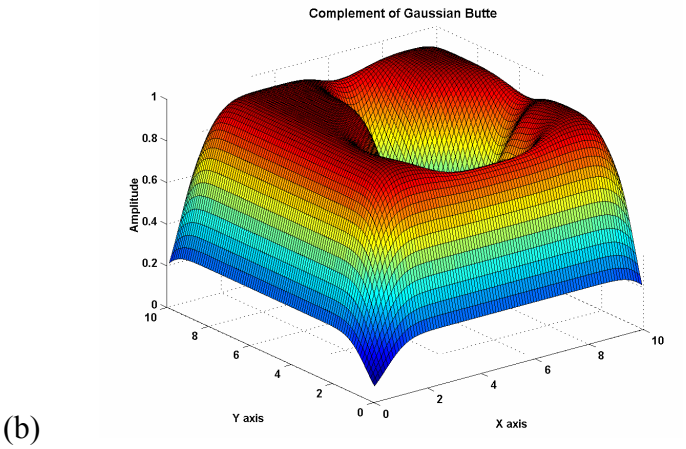


(a) (b)

FIG 4. A non-uniform partition of unity. (a) Mask of grid-points used in molecule; (b) resulting molecule.



(a)



(b)

FIG 5. Non-uniform partition of unity, based upon “Gaussian butte” (a), and its complement (b).