Research Note: Improved approximations for anisotropic reflectivities

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ABSTRACT

Linear approximations for VTI reflection coefficients are reviewed. Two methods are shown for improving on these coefficients. One is to incorporate the average angles in a manner analogous to the Aki-Richards approximation for isotropic coefficients. A second form is obtained through a reasonable guess based on optimal approximations for isotropic coefficients. Both of these methods yield more realistic coefficients than the purely linear approach, the latter having no critical behaviour.

THEORY

Thomsen (1993) presented a method for calculating linear approximations to reflection coefficients in VTI (vertical transverse isotropy) media. These employ the anisotropy parameters δ and ε introduced by him earlier (Thomsen, 1986). Rüger (1996) employed this method and provided corrections to Thomsen's results to obtain the following expression:

$$\begin{aligned}R_{PP}^{VTI}(\theta_i) &= R_{PP}^{iso}(\theta_i) + R_{PP}^{aniso}(\theta_i)\\ &= \left[\frac{1}{2\cos^2\theta_i}\frac{\Delta\alpha_0}{\alpha_0} - 2\left(\frac{\beta_0}{\alpha_0}\right)^2 \sin^2\theta_i \frac{\Delta G}{G} + \frac{1}{2}\frac{\Delta\rho}{\rho}\right] + \left[\frac{1}{2}\Delta\delta\sin^2\theta_i + \frac{1}{2}\Delta\varepsilon\sin^2\theta_i\tan^2\theta_i\right]\\ &= \frac{1}{2}\frac{\Delta Z}{Z} + \frac{1}{2}\left\{\frac{\Delta\alpha_0}{\alpha_0} - \left(\frac{2\beta_0}{\alpha_0}\right)^2\frac{\Delta G}{G} + \Delta\delta\right\}\sin^2\theta_i + \frac{1}{2}\left\{\frac{\Delta\alpha_0}{\alpha_0} + \Delta\varepsilon\right\}\sin^2\theta_i\tan^2\theta_i\end{aligned} \quad (1)$$

The above expression employs the following definitions:

α_0 = average vertical compressional velocity across an interface

$\Delta\alpha_0$ = difference in vertical compressional velocity across an interface

β_0 , $\Delta\beta_0$ = average and difference in vertical shear velocity across an interface

ρ , $\Delta\rho$ = average and difference in density across an interface

Z , ΔZ = average and difference in vertical P-impedance ($\rho\alpha_0$) across an interface

G , ΔG = average and difference in vertical shear modulus ($\rho\beta_0^2$) across an interface

$\Delta\delta$, $\Delta\varepsilon$ = difference in δ and ε parameters across an interface

θ_i = the angle of incidence at the interface

We explore two approaches to improving this approximation. The first is simple, and consists simply of replacing the incidence angle, θ, with the average angles used in the

Aki-Richards approximation. The second is to extend the pseudo-linear approach applied earlier in work on isotropic reflection coefficients (Ursenbach 2002, this volume).

Effect of Average Angles

Not all researchers realize the significance of Aki and Richards (1980) choice of angle in their famous "linearization" of the Zoeppritz coefficients. The significance is that their approximation is not truly a linearization. For instance, their expression for R_{PP} is as follows:

$$R_{PP}^{A-R}(\theta) = \frac{1}{2\cos^2\theta}\frac{\Delta\alpha}{\alpha} - 2\left(\frac{\beta}{\alpha}\right)^2 \sin^2\theta \frac{\Delta G}{G} + \frac{1}{2}\frac{\Delta\rho}{\rho}. \qquad (2)$$

This is the same as the R_{PP}^{iso} of Rüger on the previous page, except that the incident angle has been replaced by the average angle, and α and β are of course isotropic. We will define the upper layer as "1" and the lower layer as "2", so that $\theta_1 (=\theta_i)$ is the incidence angle and θ_2 is the angle of transmission. Then, using standard trigonometric identities, we can manipulate $\sin\theta$ as follows:

$$\begin{aligned}\sin\theta &= \sin[(\theta_1+\theta_2)/2] \\ &= \sin(\theta_1/2)\cos(\theta_2/2) + \cos(\theta_1/2)\sin(\theta_2/2) \\ &= \sin(\theta_1/2)\sqrt{1+\cos\theta_2}/2 + \cos(\theta_1/2)\sqrt{1-\cos\theta_2}/2\end{aligned}$$

This shows explicitly the dependence of $\sin\theta$ on $\cos\theta_2$, and treatment of $\cos\theta$ shows a similar dependence on $\cos\theta_2$. After the P-P transmission critical point, $\cos\theta_2$ of course becomes imaginary, so that the Aki-Richards coefficient becomes complex, just as does the exact Zoeppritz coefficient. However, if one replaces the average angle with the incident angle, then this critical behaviour is lost. This can be observed clearly by using the CREWES AniZoeppritz Explorer (available at www.crewes.org) where one can plot Aki-Richards and Rüger approximations simultaneously. If the anisotropy parameters are all set to zero in the Explorer, then the only difference between the two approximations is that Rüger uses angle of incidence and Aki-Richards uses the average angle, yet one still observes a strong qualitative difference between the two at critical points. This then suggests that Rüger's theory can be improved at larger offsets in a very simple way by formally replacing θ_i with θ.

Pseudo-linear approach

If we further write

$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-\left(\frac{\alpha_2}{\alpha_1}\right)^2 \sin^2\theta_1} = \sqrt{1-\left(\frac{1+\frac{1}{2}\frac{\Delta\alpha}{\alpha}}{1-\frac{1}{2}\frac{\Delta\alpha}{\alpha}}\right)^2 \sin^2\theta_1},$$

then we make explicit the implicit $(\Delta\alpha/\alpha)$-dependence of the coefficients in the Aki-Richards approximation. Hence it is not truly linear, but only "pseudo-linear"; i.e., in principle the solutions of inversion by Aki-Richards should be iterated to self-consistency. In another paper (Ursenbach 2002) we complete the logic of this approach by fully including the exact (Δα/α)-dependence to create a far more accurate "pseudo-linear" alternative.

We could then go further and try extending the isotropic "pseudo-linear" concept (Ursenbach, 2002) to anisotropic coefficients. To do this rigorously would be a highly non-trivial exercise. However, as an exploratory effort, one might try guessing an appropriate form based on the isotropic result. This results in the following expression:

$$\begin{aligned} R_{PP}^{P-L} = &\frac{4\cos\theta_1\cos\theta_2}{\left[(1+\Delta\alpha/[2\alpha])\cos\theta_1+(1-\Delta\alpha/[2\alpha])\cos\theta_2\right]^2}\left\{\frac{1}{2\cos\theta_1\cos\theta_2}\frac{\Delta\alpha}{\alpha}\right. \\ &\left.-2\left(\frac{\beta}{\alpha}\right)^2\sin\theta_1\sin\theta_2\frac{\Delta G}{G}+\frac{1}{2}\left[1-\left(\frac{\Delta\alpha}{2\alpha}\right)^2\right]\frac{\Delta\rho}{\rho}+\frac{1}{2}\left[\Delta\delta+\Delta\varepsilon\tan\theta_1\tan\theta_2\right]\sin\theta_1\sin\theta_2\right\} \end{aligned} \quad . (3)$$

This is just the isotropic pseudo-linear expression with a modified version of Rüger's R_{PP}^{aniso} added on. The latter has been modified to include both θ_1 and θ_2, but is multiplied by $\cos\theta_2$ to prevent a singularity in $\tan\theta_2$ at the critical point.

We will now compare these theoretical results numerically, both to each other and to the exact VTI coefficients due to Daley and Hron (1977).

CALCULATIONS

Figure 1, below, compares R_{PP} for the various approximations.

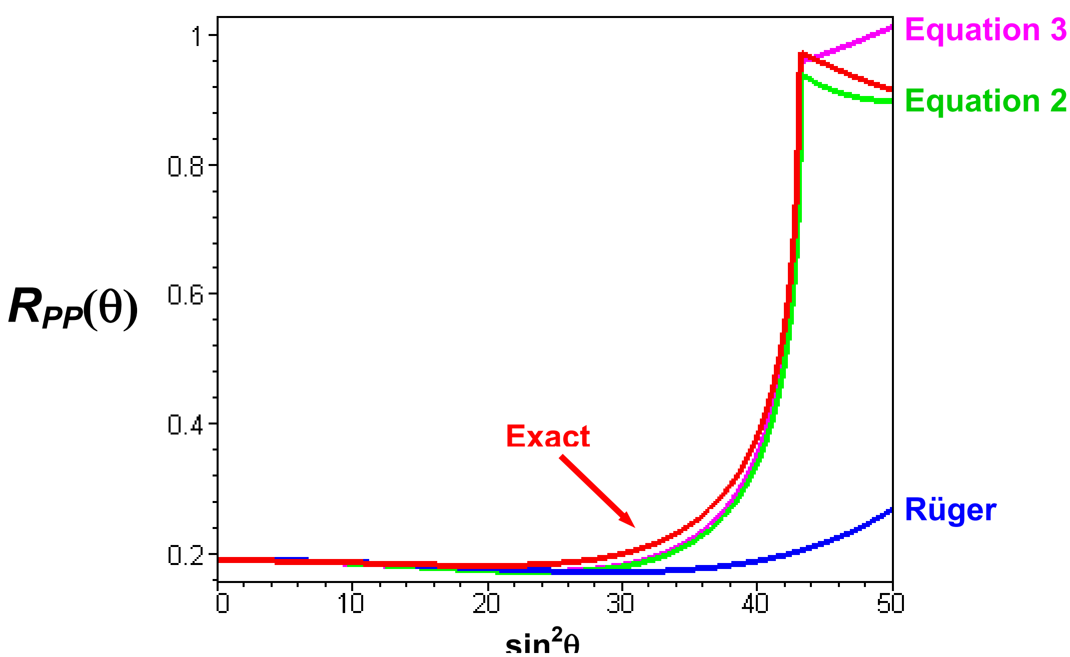

FIG. 1: A comparison of R_{PP} approximations. The earth parameters chosen are given in Table 1. The red line represents the exact solution due to Daley and Hron (1977). Rüger's approximation (blue) lacks the critical point at ~40°. Equation 2 (green) and Equation 3 (magenta) are of similar accuracy, but Equation 3 is slightly more correct at the critical point.

In Figures 2 and 3, we present the data of Figure 1 in the form of differences and relative differences.

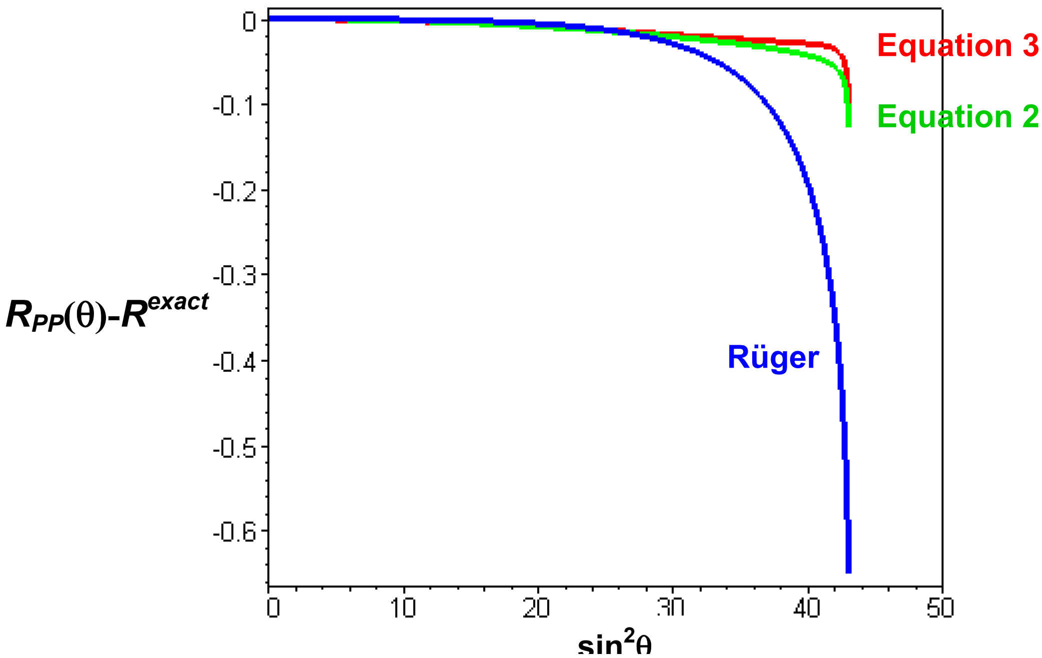

FIG. 2: The absolute difference between various R_{PP} approximations and the exact R_{PP}, for data in Figure 1.

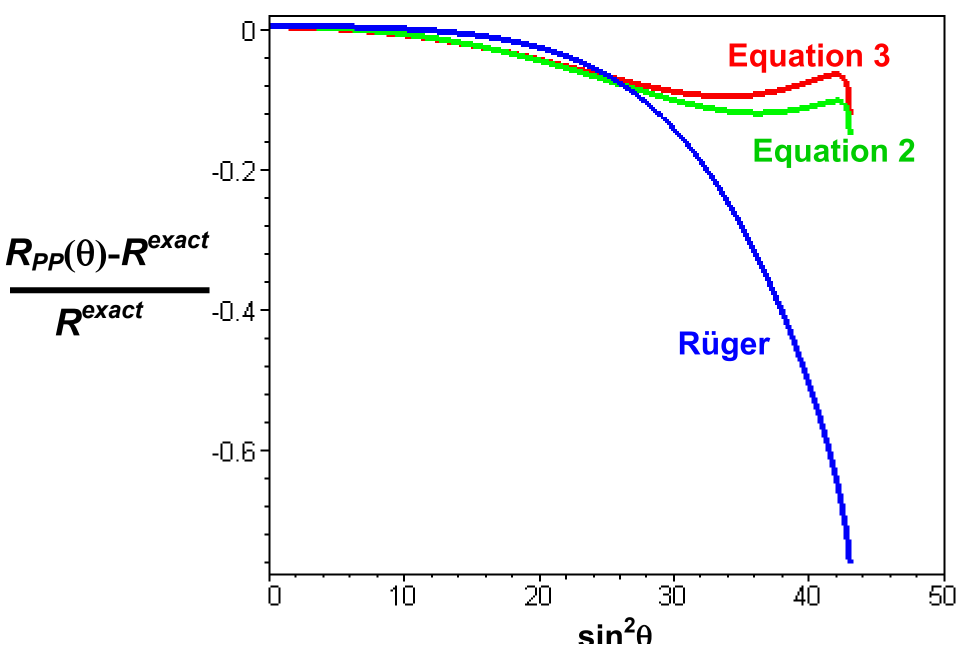

FIG. 3: The relative difference between various R_{PP} approximations and the exact R_{PP}, for data in Figure 1.

It is clear that the average-angle and pseudo-linear results are generally superior to the Rüger expression, although the latter can fortuitously be slightly better in some regions for certain earth parameter combinations. It is also clear that, unlike the isotropic case, the pseudo-linear expression given here does not provide a significant improvement over the simple angle average version. One would guess that this is due to linearization of the anisotropy coefficients, an assumption supported by some of our preliminary investigations.

Table 1 below presents the particular earth parameter values employed for the above calculations.

	Layer 1	**Layer 2**
α (m/s)	3000	4000
β (m/s)	1500	2000
ρ (kg/m^3)	2000	2200
δ	0.0	0.1
ε	0.0	0.1

Table 1: Earth parameters employed in the calculations for functions plotted in Figures 1-3 above.

These parameters are typical for geological systems, so the behaviour exhibited from the calculations is expected to be typical for real earth data.

CONCLUSIONS

We have demonstrated that including average angles into Rüger's VTI reflectivity expressions can significantly increase their accuracy, particularly at large offsets. A simple, ad hoc pseudo-linear expression is of similar accuracy to the average angle expression. A rigorous treatment of the anisotropic elements would be necessary to provide any further significant improvement.

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