A review of the finite-element method in seismic wave modelling

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ABSTRACT

Numerical solutions of the scalar and elastic wave equations have greatly aided geophysicists in both the forward modelling and migration of seismic wavefields in complicated geologic media. In P- and S-wave propagation, the finite-element method is a powerful tool for determining the effect of structural irregularities on wave propagation. Dependence of the wave equation on both spatial and temporal differentials requires solving both spatial and temporal discretization. In the spatial discretization step in 1D and 2D, piecewise linear basis functions and the Galerkin method are the most commonly used tools. After solving spatial discretization with the finite-element method, the wave equation reduces to an ODE (ordinary differential equation). In this regard, different authors used different ODE solver including Runge-Kutta method and finite-difference method. This paper will familiarize the reader with the diverse approaches of solving temporal discretization. An application of finite-element methods to solve seismic wave motion in linear viscoelastic media (where inelastic strains development depends not only on the current state of the stress and strain but on the full history of their development), using memory variable formalism in spatial discretization step, is one of the reviewed sections.

INTRODUCTION

Numerical solutions of the scalar and elastic wave equations have greatly aided geophysicists in both forward modelling and migration of seismic wavefields in complicated geologic media, and they promise to be invaluable in solving the full inverse problem. One of the numerical methods that can be applied to the problem of wave propagation is the finite-element method. The finite-element method has become the most widely accepted general-purpose technique for numerical simulations in engineering and applied mathematics. Principal applications arise in continuum mechanics, fluid flow, thermodynamics, and field theory. In these areas, computational methods are essential and benefit strongly from the enormous advances in computer technology. The finite-element method is a powerful tool for the numerical modelling of seismic body-wave propagation in a heterogeneous elastic media. The finite-element results agree with finite-difference results (Smith, 1975) but in spite of this, the method has never become popular in the geophysical literature. Perhaps this is because implementation is more difficult that other methods (Kay, 1996).

The finite-element method is a general technique for constructing approximate solutions to boundary-value problems to solve physical problems. In the finite-element analysis, a body is considered to be an assemblage of discrete finite elements interconnected at nodal points on element boundaries. One-, two-, and three-dimensional elements have a variable number of nodes. Figure 1 shows some typical continuum elements. Bathe (1996) considers the following steps to solve a problem with the finite-element method:

1) The idealization of the physical problem to a mathematical model requires certain assumptions that together lead to differential equations governing the mathematical model.

2) The identification of the unknown displacements that completely define the displacement response of the structural idealization.

3) The formulation force-balance equations corresponding to the unknown displacement and the solution of these equations.

4) The interpretation of the displacement predicted by the solution of the structural idealization (Based on the assumption used).

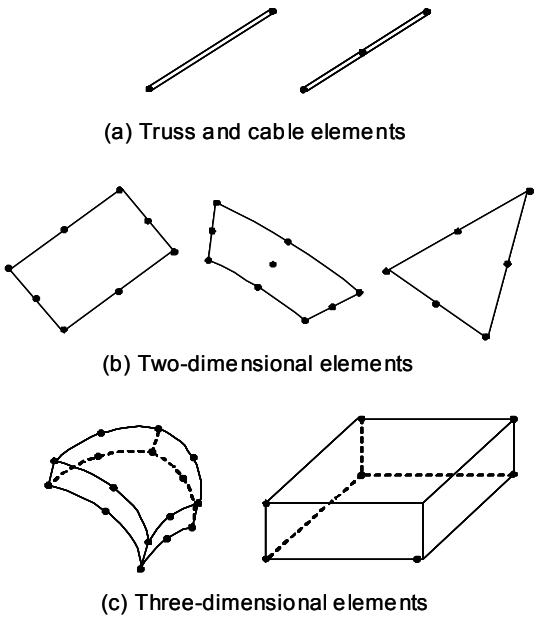


FIG. 1. Typical elements for finite-element method (after Bathe 1998).

Since the finite-element solution technique is a numerical procedure, it is necessary to assess the solution accuracy. If the criteria are not met, the finite-element solution has to be repeated with refined solution parameters (such as finer mesh) until sufficient accuracy is reached. Appendix A shows the finite-element process briefly.

Dependence of wave equation on both spatial and temporal derivatives requires both spatial and temporal discretization. In spatial discretization in 1D and 2D piecewise linear basis functions, the Galerkin method in variational form is one of the finite-element methods that is used by many authors.

WAVE PROPAGATION IN ELASTIC MEDIA

1D Spatial Discretization

Consider a one dimensional (1D) domain, *X*, in an elastic medium. The equation of motion for one degree of freedom elastic wave is

$$\rho\ddot{u}(x,t) = \rho\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(M_u u'(x,t)\right), \tag{1}$$

with simpler form as

$$\rho\ddot{u}(x,t) = (M_u u'(x,t))', \tag{2}$$

where M_u is the appropriate modulus related to the density ρ through the phase velocity. Here the finite-element solution of simple wave equation with the boundary conditions $u(0,t) = u(l,t) = 0$ and initial condition $u(x,0) = 0$ is reviewed. The PDE (2) is our finite-element model problem.

The difficulty is the requirement that a solution $u(x,t)$ to equation (2) satisfies the differential equation at every point x, $0 < x < l$, is too large. To overcome this difficulty, we shall reformulate the boundary-value problem in a way that will admit weaker conditions in the solution and its derivatives. Such reformulations are called weak or variational formulations of the problems. One weak statement of the model problem 2 is given as follow: find the function $u(x,t)$ such that the differential equation, together with boundary conditions, are satisfied in the sense of weighted averages (Becker, 1981). By the satisfaction of all "weighted averages" of the differential equation, as Becker (1981) defines, we require

$$\int_0^l \left(\rho\ddot{u}(x,t) - \left(M_u u'(x,t)\right)'\right) v(x)dx = 0, \text{ for all } v(x) \in H\,. \tag{3}$$

$v(x)$ belongs to class H (Linear Hilbert space) and has zero values at x=0 and x=l (the boundaries). The first derivative of such Hilbert functions $v(x)$ is square-integrable $\left|\int (v'(x)^2\,dx\right| < \infty$. Many authors called this $v(x)$ the weighting function. Taking integration by parts for equation (3) yields

$$\int_0^l v(x)\rho(x)\ddot{u}(x,t)dx = \int_0^l v(x)M_u u'(x,t)dx - \int_0^l v(x)'M_u u'(x,t)dx\,. \tag{4}$$

It is clear that if equation (3) is true, there can be no portion of finite element length of the interval $0 < x < l$ within which the differential equation (2) fails to be satisfied. If function $u(x,t)$ satisfies equation (3), it will satisfy equation (2) as well. Becker (1981) states that for having a symmetric weak formulation we assume that the solution functions $u(x,t)$ also belong to class H. Now we choose a set of basis orthogonal functions for H such that every function in H can be expressed as a linear combination of such basis functions. Becker (1981) represents $v(x)$ in the form

$$v(x) = \sum_{i=1}^{\infty} a_i \psi_i(x)\,, \tag{5}$$

where the coefficients a_i are given by

$$a_i = \int_0^l v(x)\psi_i(x)dx\,. \tag{6}$$

If we take only a finite number N of terms in the series 5, then we will obtain an approximation of $v_N(x)$ of $v(x)$:

$$v_N(x) \cong \sum_{i=1}^{N} a_i \psi_i(x)\,. \tag{7}$$

The N basis functions $\{\psi_1(x), \psi_2(x), \ldots, \psi_N(x)\}$ define an N-dimensional subspace H^N of H. After setting the weighting functions as a linear combination of basis functions; we are now ready to consider the Galerkin method for constructing an approximated solution to the variational boundary-value problem (equation (3)). Becker (1981) says that the Galerkin method consists of seeking an approximate solution $u_N(x,t)$ to equation (3) in subspace H^N rather than in a whole space H of the form

$$u_N(x,t) \cong \sum_{i=1}^{N} d_i(t)\psi_i(x)\,. \tag{8}$$

The coefficients $d_i(t)$ take on the value of the displacement u at a discrete number of nodal points (points $i = 1, \cdots, N$).

For a better understanding of $v_N(x)$, suppose the value of $v_N(x)$ at nodes 1, 2, and 3 are 0.9, 0.7, and 0.2, respectively, with basis functions as in Figure 2b. Substituting these values into equation (7) gives $v_N(x) = 0.9\psi_1(x) + 0.7\psi_2(x) + 0.2\psi_3(x)$, so that these three components combine to give the continuous piecewise-linear function shown in Figure 3a and the piecewise-linear function $v_N(x)$ has the form of Figure 3b. In the Galerkin method the solution function $u(x,t)$ is approximated by piecewise-linear functions $v_N(x)$, with values coinciding with those of $u(x,t)$ at the nodes, the result is a polygonal function which closely resembles $u(x,t)$. This is the piecewise-linear interpolation of the exact solution $u(x,t)$. As the mesh is refined (i.e., as the number of elements is increased), the finite-element interpolant becomes progressively closer to $u(x,t)$ (see Figure 3c).

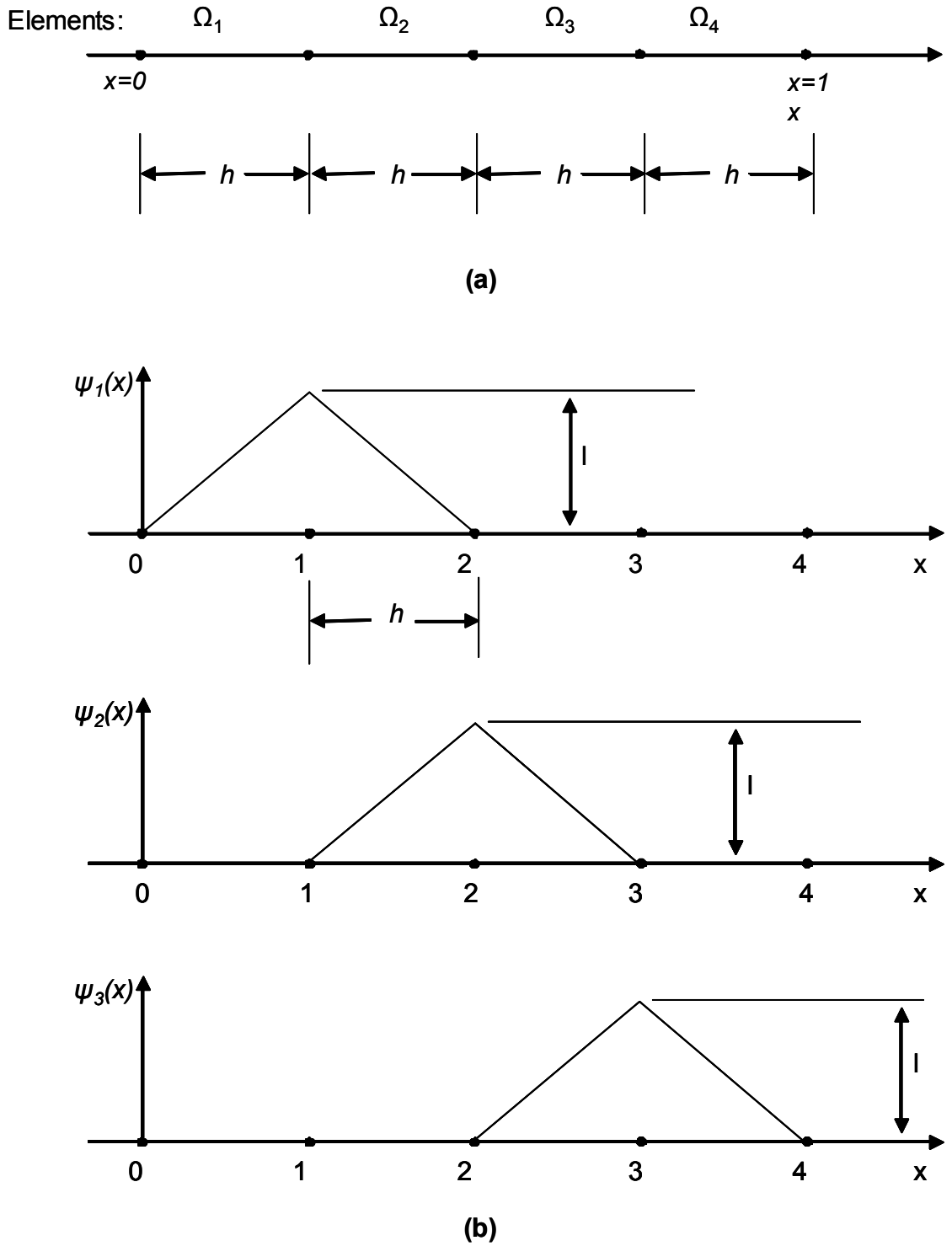


FIG. 2. (a) A simple 1D domain mesh (b) Example of finite-element basis functions (both, after Becker, 1981).

There is no need for weighted function $v(x)$ to be expressed by N-dimensional basis functions $\{\psi_1(x),\psi_2(x),\ldots,\psi_N(x)\}$. Many authors consider them as separate sets (Bathe, 1996; Hughes, 1987; and Marfurt, 1984). Marfurt (1984) states that choosing weighting functions as linear combination of basis functions such that the weighting and basis are identical, generates the Galerkin formulation of the finite-element method. Marfurt (1984) also mentions while the Galerkin method of setting the weighting function $v(x)$ equal to that of the basis functions ψ_i has been among the most popular finite-element techniques, there is no compelling reason to do so. Furthermore, there is no compelling reason to use the same ψ_i in evaluating the mass, damping, stiffness, and load matrices.

Substitution of the expansion of $u_N(x,t)$ and $v_N(x)$, equations (7) and (8), into equation (4) results in

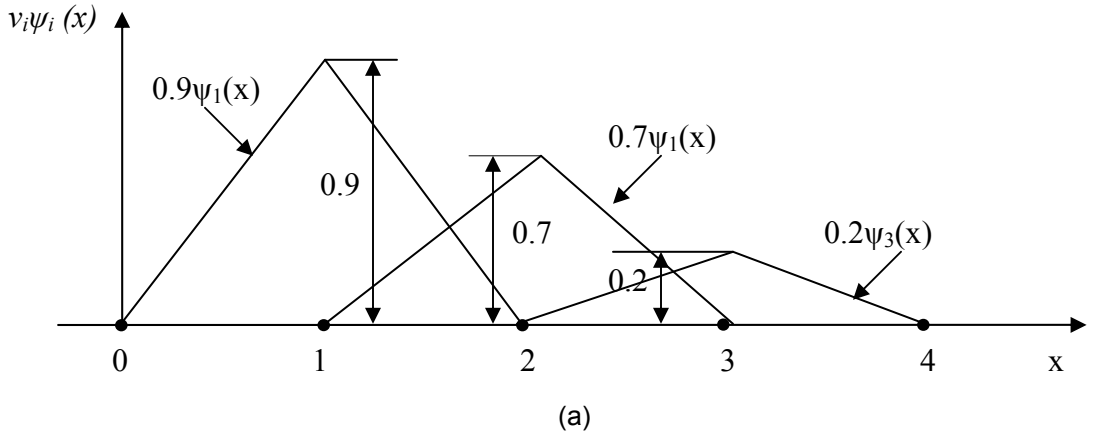
$$\sum_i a_i \left[\sum_j \left[\int \psi_i(x) \rho \psi_j(x) dx \right] \ddot{d}_j(t) + \sum_j \left[\int \psi_i(x)' m \psi_j(x)' \right] d_j \right] = 0 . \qquad (9)$$

This is the same result as (Kay and Krebes, 1999). Note that in getting this result the boundary values are already applied. Since the functions $v(x)$, and thus the coefficients a_i, are arbitrary, the expression in the brackets of equations (9) must equal zero. The structure of equation (9) can be written in more compact form as

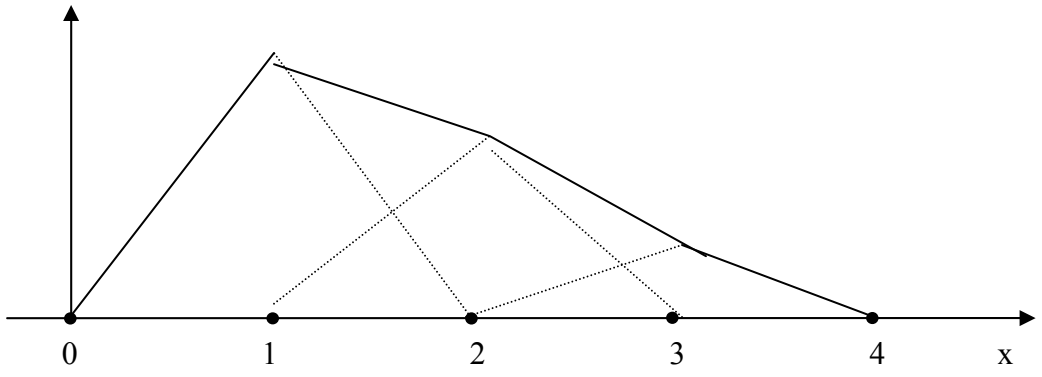
$$\sum_{i=1}^{N} a_i \left(\sum_{j=1}^{N} (M_{ij} \ddot{d} + K_{ij} d_j) \right) = 0 , \qquad (10)$$

whose M and K matrices are

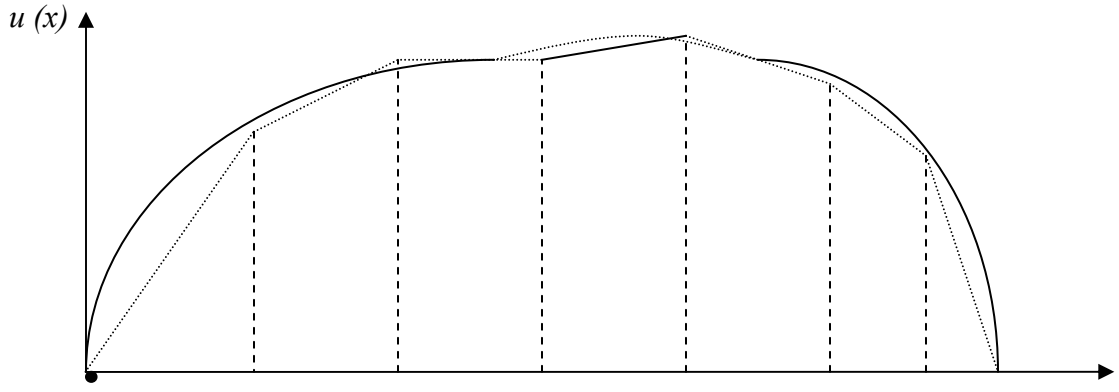
$$\begin{aligned} M_{ij} &= \int_0^l \psi_i(x) \rho \psi_j(x) dx \\ K_{ij} &= \int_0^l \psi_i'(x) M_u \psi_j'(x) dx. \end{aligned} \qquad (11)$$



(a)



(b)



(c)

FIG. 3. (a) $v_N(x) = 0.9\psi_1(x) + 0.7\psi_2(x) + 0.2\psi_3(x)$, (b) piecewise-linear finite-element representation of the function $v_N(x)$ (both, after Becker, 1981). (c) Local linear interpolation of smooth function *u*.

The resulting ordinary differential equations in time for the unknown displacements at the nodal points can be written as

$$M\ddot{d}(t) + Kd(t) = 0\,. \tag{12}$$

The matrices M and K are called mass and stiffness matrices, respectively. Equation (12) is the correspondence finite-element equation. (The terms mass and stiffness matrix are misnomers for the acoustic wave equation where the reciprocal of the density is actually included in the stiffness matrix and the reciprocal of stiffness in the mass matrix (Marfurt, 1984).)

Calculating the mass matrix M, from equation (11), leads to a sparse matrix as in Figure 4.

$$M = \begin{bmatrix} 0 & 0 & & & & & & \\ 0 & x & 0 & & & & & \\ & 0 & x & 0 & & & & \\ & & 0 & x & 0 & & & \\ & & & 0 & x & 0 & & \\ & & & & 0 & x & & \\ & & & & & & \bullet & \\ & & & & & & & \bullet \\ & & & & & & & & \bullet \end{bmatrix}$$

FIG. 4. Cartoon of mass matrix *M* assembly in 1D (After Kay, 1996).

Considering a source F and a damping term C in wave equation, the corresponding finite-element equation becomes

$$M\ddot{d} + C\dot{d} + Kd = F. \tag{13}$$

Equation (13) is used for the finite-element model of wave equation by Smith (1975), Jianlin (1994), Kay (1999), Hughes (1987), Cohen (2000), Sullivan (1983), and Marfurt (1984), among others. The initial-value problem for equation (13) consists of finding a displacement, $d = d(t)$, satisfying equation (13) and the given initial data:

$$\begin{aligned} d(0) &= d_0\,, \\ \dot{d}(0) &= v_0\,. \end{aligned} \tag{14}$$

Here we try to present a finite-element solution of equation (12). In this regard, we need to define the basis functions then construct the stiffness and mass matrices. As

mentioned above, finite–element methods assume that the solution can be represented with N basis functions enumerated as ψ_i. These functions are chosen to be zero outside some finite interval. In addition, if the ψ_i has the value of unity at one node (defined later) in the computational mesh and zero at every other node, a one-to-one correspondence between basis functions and the grid nodes results. Kay (1996) also mentions that the ψ_i are chosen to be sufficiently "smooth" such that all integrals will exist. The main idea as Becker (1981) says, is that the basis functions ψ_i can be defined piecewise over subregions of the domain called finite elements and ψ_i can be chosen to be very simple functions such as polynomials of low degree.

Constructing such a set of piecewise basis functions, as Becker (1981) states, we first partition the domain X (the interval $0 < x < l$) of our problem into a finite number of elements. Figure (2a) shows, for example, the domain of our model problem.

Within each element, certain points are identified, called nodes or nodal points, which play an important role in the finite-element constructions. The collection of elements and the nodal points making up the domain of the approximate problem is sometimes referred to as a finite-element mesh. The basis functions should be smooth enough to be members of a Hilbert space. One very simple sample set of basis functions for the domain showed in Figure (2a), is shown in Figure (2b).

If the coordinates of the nodes are denoted $x_i (i = 0, 1, 2, 3, 4),$ then the basis functions shown for $i = 1, 2, 3$ are given by

$$\psi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i} & \text{for } x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{h_{i+1}} & \text{for } x_i \le x \le x_{i+1} \\ 0 & \text{for } x < x_{i-1} \text{ and } x > x_{i+1} \end{cases}, \qquad (15)$$

where $h_i = x_i - x_{i-1}$ is the length of element Ω_i. The first derivatives of basis functions also can be calculated very easily. Considering equation (11) the Mass and Stiffness matrices can be computed.

Solving the wave equation (12) and finding mass and stiffness matrices by constructing the basis functions, equation (12) may be cast into the form

$$\ddot{d}(t) = M^{-1}(-Kd(t)). \qquad (16)$$

The development of boundary-value problems describing physical phenomena in two dimensions follows closely the one-dimensional treatment given above, differing only in aspects dictated by the higher dimensionality.

2D Spatial Discretization

In two dimensions, the domain is discretized into element domains similar to the 1D case. In two dimensions the element domains might be simply triangles and quadrilaterals (Figure 5). Nodal points may exist anywhere on the domain but most frequently appear at the element vertices and interelement boundaries and less often in the interiors (Hughes, 1987).

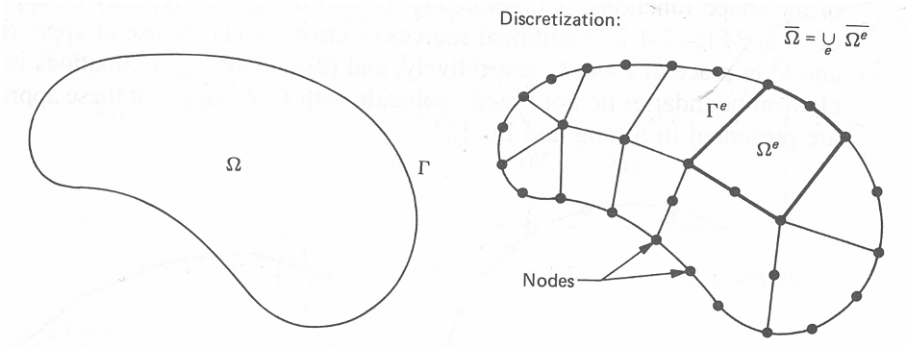


FIG. 5. 2D domain and 2D mesh for finite-element spatial discretization (after Hughes, 1987).

In the 1D domain in equation (14), we introduced a set of basis functions as linear functions $\psi(x) = a_1 + a_2 x$. For generalizing the concept to 2D, Becker (1981) considers the linear function

$$\psi(x, y) = a_1 + a_2 x + a_3 y . \tag{17}$$

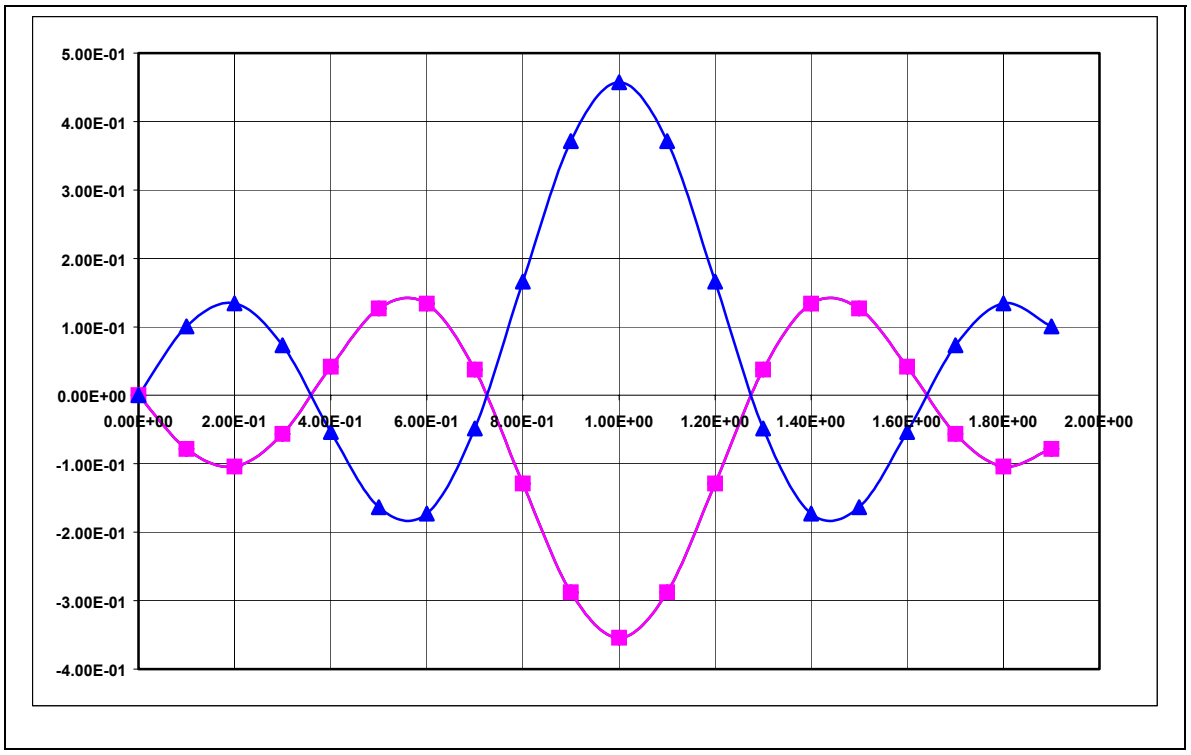
This function determines a plane surface. The use of such a basis function on a triangle element will result in the approximation of a smooth function $v(x,y)$ (that we had before as weighting functions) by a planar function,

$$\psi_i(x_i, y_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } \mathrm{i} \neq \mathrm{j} \end{cases} \tag{18}$$

for (x_1, y_1) as coordinate of each node in finite-element mesh. The basis functions $\psi_i(x, y)(i = 1,2,\ldots,N)$ are constructed in the same manner as described for 1D. The basis functions corresponding to adjacent elements in the mesh are simply patched together to produce a pyramid nodal point in the mesh (Becker, 1981).

There are many mechanical engineering software packages that solve boundary value problems by finite-element methods. Including MSC.Patran and NISA II/Display III. Figure 6 is a 1D section of a 2D finite-element solution to a sample wave propagation executed by NISA II.

The generalization of the finite-element method discussed to three dimensions is theoretically simple, though computationally more expensive (Marfurt, 1984).



a(t+ΔT/2)	time	a(t)
0.00E+00	0.00E+00	0.00E+00
-7.82E-02	1.00E-01	1.01E-01
-1.04E-01	2.00E-01	1.35E-01
-5.66E-02	3.00E-01	7.30E-02
4.16E-02	4.00E-01	-5.37E-02
1.27E-01	5.00E-01	-1.64E-01
1.34E-01	6.00E-01	-1.73E-01
3.77E-02	7.00E-01	-4.86E-02
-1.29E-01	8.00E-01	1.66E-01
-2.88E-01	9.00E-01	3.71E-01
-3.54E-01	1.00E+00	4.57E-01
-2.88E-01	1.10E+00	3.71E-01
-1.29E-01	1.20E+00	1.66E-01
3.77E-02	1.30E+00	-4.86E-02
1.34E-01	1.40E+00	-1.73E-01
1.27E-01	1.50E+00	-1.64E-01
4.16E-02	1.60E+00	-5.37E-02
-5.66E-02	1.70E+00	7.30E-02
-1.04E-01	1.80E+00	1.35E-01
-7.82E-02	1.90E+00	1.01E-01

FIG. 6. 1D finite-element solution to wave equation in an elastic (Isotropic) material. The blue plot indicates the initial wavelet and the red plot indicates the wavelet after half-period of initial wavelength in the centre point a disc defined as initial problem. Boundary and initial value both were set to zero. This is 1D section of 2D finite-element solution of a sample wave propagation, Executed by NISA II/Display III software. Displayed data table shows 1D numerical values.

Temporal Discretization

There are different ways to solving the time-step equation (15). Many authors use different numerical methods for solving this ODE (Ordinary Differential Equation). Smith (1975) uses a numerical solution using the Runge-Kutta algorithm, but there are no details of his solution in his paper. Runge-Kutta is a numerical method for solving ODE that does not use the explicit evaluation of the derivatives (Ambramowitz and Stegun, 1970). A brief summery of the Runge-Kutta method is found in Appendix B.

Becker (1981) presents a finite–difference solution for the time-dependent heat equation. The heat equation is very similar to wave equation. The heat equation and its finite-element model are

$$c\dot{u}(x,t) = c\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\big(k(x)u'(x,t)\big), \tag{19}$$

$$C\dot{u} + Ku = 0\,. \tag{20}$$

The time-step solution for the heat equation will hold for the wave equation with some slight modification. Becker (1981) states that by using the finite-element method, we have succeeded in reducing the given initial-value problems (19) to the system of ordinary differential equation (20). Since we have not yet discretized the behaviour of u_N in time, equation (20) is referred to a semi-discrete finite-element approximation. To obtain a fully discrete approximation, we must now introduce an approximation of the behaviour of $u(t)$ in time. Becker (1981) outlines one of the simplest methods, forward finite-difference approximation. The time domain $0 \le t \le T$ is divided into k equal intervals of length $\Delta t = T/k$. At t = 0, the solution is known: $u(t) = \hat{u}$ .To advance the solution in time from $t = n\Delta t$ to $(n+1)\Delta t$ the forward finite-difference operator is used

$$\frac{du(n\Delta t)}{dt} \approx \frac{u^{n+1} - u^n}{\Delta t}, \quad \mathrm{u}^{\mathrm{n}} = u(n\Delta t); \tag{21}$$

then equation (20) leads to the algorithm

$$\begin{aligned} C(u^{n+1} - u^n) &= -Ku^n\Delta t \\ u^{n+1} &= (I - \Delta t C^{-1}K)u^n. \end{aligned} \tag{22}$$

Since $u(0)$ is known, u^1 can be calculated using equation (22). In this way we integrate the solution in time from t = 0 to $t = k\Delta t$ for any desired number of time steps. It should be noted that for a given mesh size N, a limitation on the time-step size Δt is needed in order for this scheme to be numerically stable.

Taking the same steps as Becker (1981) for wave equation, and using the centered second finite-difference operator

$$\frac{d^2u}{dt^2}\Delta t^2 \approx u^{n+1} - 2u^n + u^{n-1}, \tag{23}$$

equation (16) becomes

$$u^{n+1} = (2 - \Delta t^2 M^{-1} K)u^n - u^{n-1}. \tag{24}$$

So the solution in time for any number of time steps is achieved.

Strange and Fix (1973), in solving the heat equation with finite-element methods, also uses Galerkin method; and for solving the time-dependent equation (20), they state that:

> "*It is natural to ask why finite-elements are not used also in the time direction. This has certainly been attempted, but not with great success and in fact a straightforward application of the Galerkin principle may couple all the time levels, and destroy the crucial property of propagation forward in time*".

In the time-step solution for equation (20), Strange and Fix (1973) analyze the Crank-Nicholson scheme, which is centred at $(n+\frac{1}{2})\Delta t$ and therefore achieves second-order accuracy in time:

$$C\frac{u^{n+1} - u^n}{\Delta t} + K\frac{u^{n+1} + u^n}{2} = 0. \tag{25}$$

Rewritten, the approximation u^{n+1} is determined by

$$\frac{M + K\Delta t}{2}u^{n+1} = \frac{M - K\Delta t}{2}u^n. \tag{26}$$

Strange and Fix (1973) say that in an actual computation, the matrix on the left can be factored by Gauss elimination into LL^T, where L is Cholesky's lower triangular matrix, and then u^{n+1} would be computed at each step by back substitutions,

$$Lu^{n+1/2} = \frac{M - K\Delta t}{2}u^n, \qquad L^{\mathrm{T}}u^{n+1} = u^{n+1/2}. \tag{27}$$

Strange and Fix (1973) state that because the coefficient of equation (20) is time-dependent, then in the strict Galerkin theory the mass and stiffness matrices must be computed at each time step. Now it becomes clear that why the implementation of finite-element method in wave equation is so hard.

Thornton (1982) takes the same finite-element procedure for solving the heat equation (19) and gets the finite-element equation (20). He also uses the finite-difference operator for solving the time-dependent part of the heat equation.

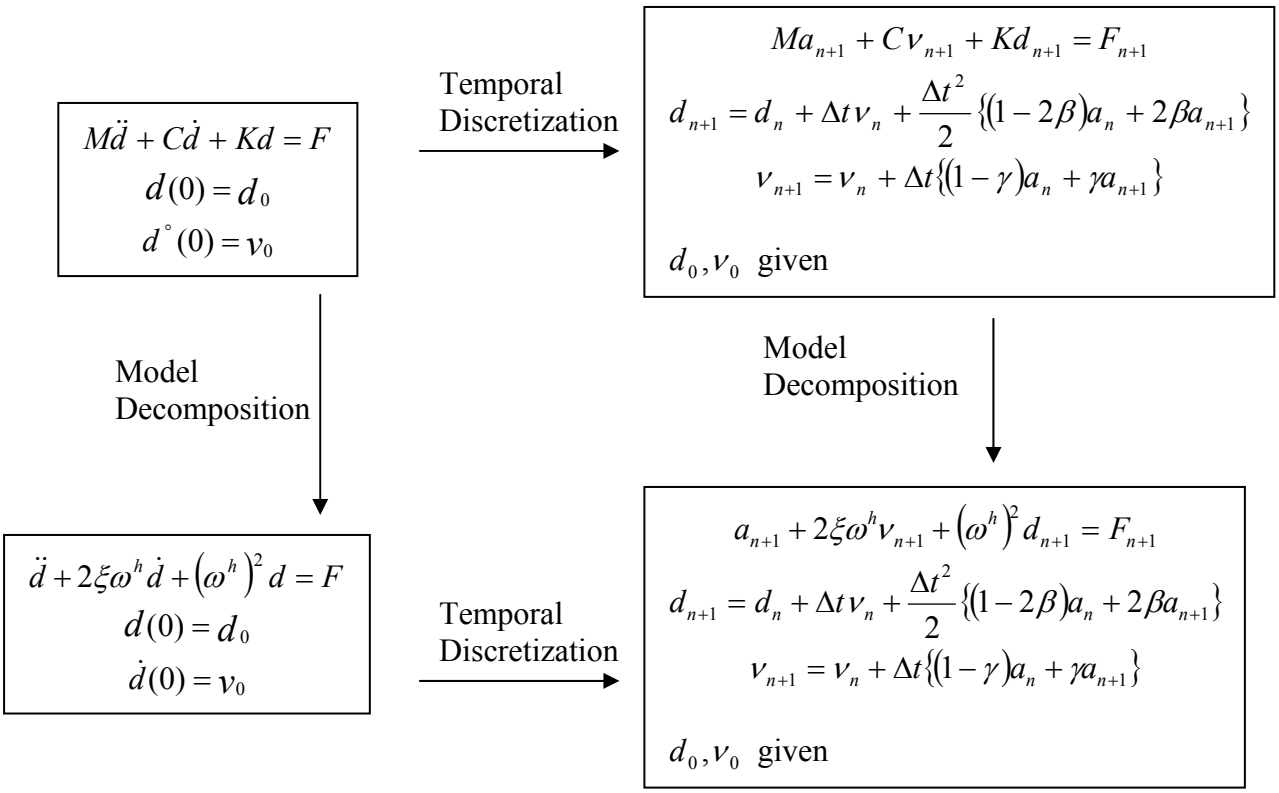


FIG. 7. The explicit finite-element method for wave equation (after Hughes, 1987).

Hughes (1987) in solving the wave equation by finite-element also drives equation (13) and states that the most widely used family of direct methods for solving equation (13) with initial conditions (14), is the Newmark family, which consists of the following equations:

$$M\ddot{d} + C\ddot{d} + Kd = F$$

$$Ma_{n+1} + Cv_{n+1} + Kd_{n+1} = F_{n+1} \tag{28}$$

$$d_{n+1} = d_n + \Delta t v_n + \frac{\Delta t^2}{2}\left[(1-2\beta)a_n + 2\beta a_{n+1}\right] \tag{29}$$

$$v_{n+1} = v_n + \Delta t\left[(1-\gamma))a_n + \gamma a_{n+1}\right], \tag{30}$$

where $d_n, v_n,$ and a_n are the approximations of $d(t_n), \dot{d}(t_n),$ and $\ddot{d}(t_n)$, respectively. Equations (29) and (30) are finite-difference formulas describing the evolution of the approximate solution. The parameters β and γ determine the stability and accuracy characteristics of the algorithm under consideration. Equations (28) to (30) may be thought of as three equations for determining the three unknowns $d_{n+1}, v_{n+1},$ and a_{n+1}. It is being assumed that $d_n, v_n,$ and a_n are known from the previous step's calculations. Hughes (1987) summarizes the explicit finite-element method for the wave equation in Figure 7 where the ω^h is the weighting function (as defined in previous parts). Hughes (1987) states when the time step restriction is not too severe, as is often the case in elastic

wave-propagation problems, the central finite-difference method is generally the most economical direct integration procedure and is thus widely used. Hughes (1987) discusses alternative approaches from the Galerkin and Newmark methods, for solving the time-step of the wave equation using the finite-element method, including: the Linear Multisteps (LMS), and Houbolt methods; Collocation Schemes; the α-Method, the Wilson-θ method, the Fox-Goodwin method, and the predictor-corrector method, which all are all weighted residual time integration schemes. All mentioned weighted residual time integration schemes lie in a grey area between finite-element and finite-differences methods (Bathe, 1996; Zeinkiewicz, 1997; and Marfurt, 1984).

Marfurt (1984) uses a three-point scheme for discretizing the homogeneous time dimension and states one can apply the temporal basis functions for ODE equation (13) (Zeinkiewicz, 1977).

WAVE PROPAGATION IN LINEAR VISCOELASTIC MEDIA

Viscoelasticity

All we have shown up until now has been in elastic media, but there is also the finite-element solution for wave propagation in *viscoelastic* and *anisotropic* media. Kay and Krebes (1999) apply the finite-element method to a solution of seismic wave motion in *linear viscoelastic* media. First we'll discuss viscoelastic media. Zienkiewicz (2000) characterizes viscoelastic phenomena by the rate at which inelastic strains develop depends not only on the current state of the stress and strain but, in general, on the full history of their development. Thus, to determine the increment of inelastic strain over a given time interval (or time-step), it is necessary to know the state of stress and strain at all preceding times. Kay (1996) describes the stress-strain relation in viscoelastic media as:

$$\sigma(t) = \int_0^t \varepsilon(t - t')dR(t'), . \tag{31}$$

where $\sigma(t)$ is stress, $\varepsilon(t)$ is strain and the tensor $R(t)$ is the strain relaxation function (we will define shortly). As stated by Aki and Richards, the $R(t)$ is the characteristic relaxation time for strain under an applied step in stress. If at $t = 0$ the system goes from a state of no strain to a state with some small but non-zero strain, then the stress for any time after $t = 0$, is given by this convolution of the strain history with the relation function. Kay (1996) shows that an integral form of the stress-strain behaviour in a viscoelastic media can be written in a convolution form as:

$$\sigma(t) = \int_0^t R(t') \frac{d\varepsilon(t - t')}{dt'} dt' . \tag{32}$$

Kay (1996) states that there is a one-to-one correspondence between the stiffness matrix of elastic theory and the relaxation function here. Bland (1960) states that the Laplace transform of the viscoelastic problem will formally have the same form as the elastic problem. The terms that correspond to the elastic constants appear as functions of the Laplace transform parameter, *t*. Bland (1960) also shows that the familiar decoupling

of SH and P-SV waves for elastic media holds for linear viscoelastic media too. Kay and Krebes (1999) say the term viscoelastic can apply to problems in any number of dimensions, whether acoustic, antiplane-strain (SH waves), or with multiple degrees of freedom at each point and exhibiting coupled P-SV waves.

Kay (1996) states that a generalized Maxwell model composed of components has a discrete relaxation spectrum with

$$R(t) = \left[M_r + \delta M \sum_{i=1}^{N_m} a_j e^{-t/\tau_j} \right] H(t), \tag{33}$$

where M_r is the relaxed modulus and M_u (in equation (2)) is equal to the sum $M_r + \delta M$, and $H(t)$ is the Heaviside step function (Bland, 1960; Ch. 1). For this form of $R(t)$ (equation (33)), the stress (equation (32)) can be expressed as

$$\sigma(t) = M_u \left[u'(t) - \sum_{j=1}^{n} X_j \right], \tag{34}$$

where the X_j are the memory variables. Kay (1996) defines memory variables in terms of the strain as

$$X_j = \frac{a_j \delta M}{\tau_j M_u} \int_0^t e^{-(t-\tau)/\tau_j} u'(\tau) d\tau . \tag{35}$$

This expression is equivalent to the differential equation

$$\dot{X}_j + \frac{1}{\tau_j} X_j = \frac{a_j \delta M}{\tau M_u} u'(t). \tag{36}$$

Spatial discretization for viscoelasticity

Kay and Krebes (1999) use the following wave equation in 1D heterogeneous medium

$$\rho(x)\ddot{u}(x,t) = \sigma(x,t)', \tag{37}$$

where $\sigma(x,t)$ is stress. It is the same form as we stated in equation (2) for elastic media, because for a single degree-of-freedom, strain is equal to $u(x,t)'$. Note that for elastic media $\delta M \equiv 0$, the equation of motion has the same form as equation (2). So all the results for viscoelastic media can also be applied to elastic media.

Kay and Krebes (1999) consider the equation (34) for stress and substitute that into equation (37). Repeating the Galerkin analysis, multiplying by the weight function $v(x)$, (as shown above in elastic media) yields to similar equation as equation (4), with the form

$$\int_0^l v(x)\rho(x)\ddot{u}(x,t)dx = \int_0^l v(x)(M_u u'(x,t))'dx - \int_0^l v(x)\left(M_u \sum_j X_j(x,t)\right)' dx . \quad (38)$$

Using the memory variable X_j, Kay and Krebes (1999) presented two methods of spatial discretization using the memory variable X_j. Here we just discuss one of those methods. This discretization scheme is analogous to equation (7) approximating the memory variable X_j as

$$X_j \cong \sum_{i=1}^{N} \xi_i \psi_i(x) . \quad (39)$$

These expansions are substituted into equation (36) and a similar Galerkin analysis is applied to equation (36). The second-order ordinary differential equation in $\ddot{d}$ (in equation (13)) is coupled to two first-order differential equations in d and v if $(d/dt)d = v$. Repeating the Galerkin analysis using this expansion for X_j will result in an ODE for d (t). Then the system of equations can be summarized as

$$\frac{d}{dt}\begin{bmatrix} d(t) \\ Mv(t) \\ M_I\xi_1(t) \\ M_I\xi_2(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ -K_M & 0 & C_M^T & C_M^T & \cdots \\ C_{\delta 1} & 0 & -M_{\tau 1} & 0 & \cdots \\ C_{\delta 2} & 0 & 0 & M_{\tau 2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} d(t) \\ v(t) \\ \xi_1(t) \\ \xi_2(t) \\ \end{bmatrix}, \quad (40)$$

where

$$\begin{aligned} M &= \int \psi_i(x)\rho(x)\psi_j(x) & \mathrm{M}_\tau &= \int \psi_i(x)\frac{1}{\tau_j}\psi_j(x) \\ \mathrm{M}_I &= \int \psi_i(x)\psi_j(x) & \mathrm{K}_\mathrm{M} &= \int \psi_i'(x)M_u\psi_j'(x) \\ C_\delta &= \int \psi_i(x)\frac{a_j(x)\delta M(x)}{\tau_j(x)M_u(x)}\psi_j'(x) & & \\ \mathrm{C}_\mathrm{M}^\mathrm{T} &= \int \psi_i'(x)M_u\psi_j(x) & & \end{aligned} . \quad (41)$$

Details can be found in Kay (1996). In addition to the spatial discretization presented above, some method of advancing the solution in time is required, and the resulting scheme must be shown to approximate the solution of the continuous equations.

As mentioned before applying the spatial discretization (39) to wave equation (37) resulted in an ODE in time domain. Kay and Krebes (1999) use a centred finite-difference operator to approximate $\ddot{d}(t)$. They discretized time t in equal steps of Δt, and a superscript k denotes evaluation at time as $t = k\Delta t$. With this notation, Kay and Krebes

(1999) propose finite-element in space and finite-difference in time. Their (FE/FD) scheme is

$$M.d^{k+1} = M.(2d^{k} - d^{k-1}) + (\Delta t)^2 \left[- K.d^{k} + \frac{1}{2} \sum_{j} K_{\delta_j} .(\xi_j^{k+1/2} + \xi_j^{k-1/2}) \right]. \qquad (42)$$

Solving the ODE resulting from equation (40), Kay and Krebes (1999) also used a sixth-order Runge-Kutta scheme with adaptive step-size control (they also mention the method of Kjartansson (1997) can be used). They compared the results from equation 42 solved by the Kjartansson method to those from Runge-Kutta method base on application of the "method of lines" (Ames, 1992). Agreement between the two different solution methods increases confidence in the results. In this way, Kay and Krebes (1999) showed the accuracy of their method and also showed their solution is also independent of the memory variable formulation. Results in homogeneous medium also agree with the frequency domain solutions of Kjartansson's constant-Q method (for more detail see Kay (1996)).

CONCLUSIONS

Many of the concepts associated with finite-element methods are more intuitive than those in finite-difference or spectral methods, but the implementation is more complicated for simple one-dimensional problems. However, for higher dimensionality and complex geometry, finite-element pays for extra the work. Irregular geometries and inhomogeneous media that represent realistic geological structures can be handled by the finite-element method better that other methods like finite-difference

On the basis of the concept of the principle of Galerkin variational method of initial-boundary-value problems, the finite-element solutions for acoustic and elastodynamic transient problems have been successfully formulated by many researchers. Kuo (1982) states the finite-element method as developed provides advantages over the more conventional finite-difference methods when applied to exploration problems for: (1) simple and accurate modelling of arbitrary seismic sources and source arrays; (2) ease of applying homogeneous and inhomogeneous boundary conditions of any type; (3) great flexibility in modelling targets of any irregular shape as well as the effects of irregular topography and weathering zones; and (4), perhaps most importantly, errors are averaged over the elements throughout the domain in question.

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APPENDIX A: FINITE-ELEMENT ANALYSIS

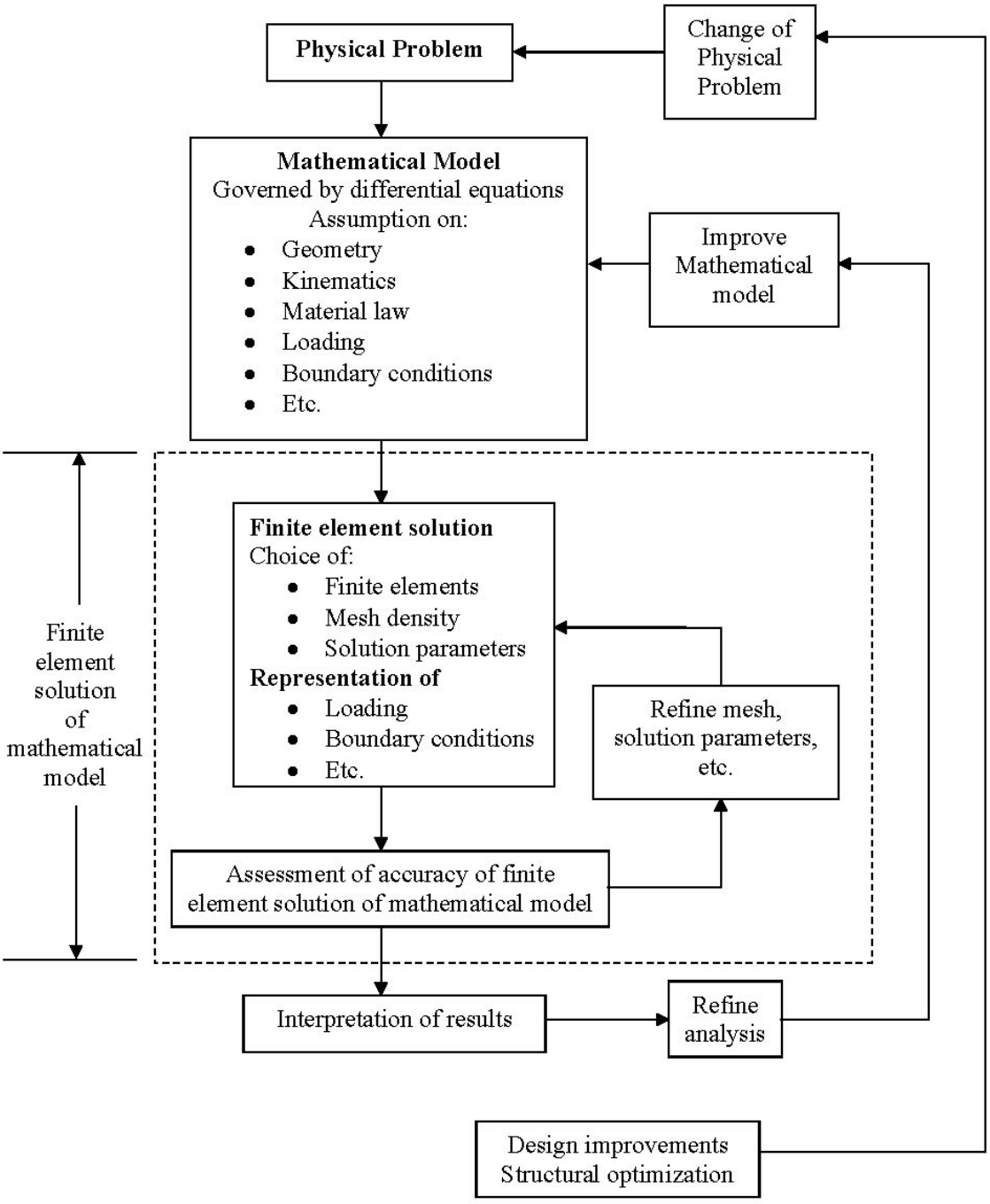

FIG. A1.The process of finite-element analysis (after Bathe, 1996).

APPENDIX B

Runge-Kutta method

The Runge-Kutta method is the most accurate numerical algorithm to solve Ordinary Differential Equations. This method fits a polynomial to the curve, which it uses to estimate the new *y* value. This is an extension on Euler's improved method. The Euler method can be considered a first-order Runge-Kutta method. The fourth-order Runge-Kutta method works by evaluating the function at four separate points and takes the average of those four points. The Runge-Kutta method provides improved accuracy with larger step-sizes and without the need of evaluate higher differentials (beyond the first derivative) of the function of interest. Higher order Runge-Kutta methods evaluate the solution at more points (i.e., sixth-order Runge-Kutta uses six solutions), and therefore become more accurate. The aim of method is generating a numerical solution to an initial value problem of the form:

$$y' = f(x, y)$$
$$y(x_o) = y_o$$

Here is a summary of the method:

$$y_{n-1} = y_n + \frac{h}{6}(k_{n1} + k_{n2} + k_{n3} + k_{n4})$$

where,

$$k_{n1} = f(t_n, y_n)$$
$$k_{n2} = f(t_n + h/2, y_n + hk_{n1}/2)$$
$$k_{n3} = f(t_n + h/2, y_n + hk_{n2}/2)$$
$$k_{n4} = f(t_n + h, y_n + hk_{n3})$$

Note that in the case where $f(t,y)$ does not depend upon y, the above reduces to

$$y_{n+1} = y_n + \frac{h}{6}[f(t_n) + 4f(t_n + h/2) + f(t_n + h)]$$

which is Simpson's rule.