More on implementing the derivative filter

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ABSTRACT

An update is presented on finding faster and or more accurate implementations of differentiator filters in the sampled domain (space or time). A previous work (Bancroft and Geiger, 1997) is reviewed, a description of the recursive filter is provided, and a theoretical derivation of the ideal filter is presented.

INTRODUCTION AND REVIEW

The differentiation of a long sequence (say $a(i)$, where $-N \le i \le N$) is accurately applied in the frequency domain ($D(f) = \cdot j \lrcorner$) by applying a $\pi / 2$ or ninety–degree phase–shift with an amplitude scaling that is proportional to the frequency. In the sampled domain the derivative of $a(i)$ is $d(i)$, where $-N \le i \le N$. The derivative at location n, (dn), may be approximated by a forward difference

$$d_{n-forward} = \left(a_{n+1} - a_n \right), \tag{1}$$

or a backward difference

$$d_{n-backward} = \left(a_n - a_{n-1} \right). \tag{2}$$

The backward difference may also be defined by shifting the location of the derivative to use the same inputs as the forward method. We then have

$$d_{n+1-backward} = \left(a_{n+1} - a_n \right), \tag{3}$$

These methods are fast and usually their performance is good enough for most applications. The forward and backward methods are crude in that the derivative should be located at the midpoint between the two selected samples. Since this is not possible with a defined sequence, it is located at either a forward or backward location. Averaging these two estimates of equations (1) and (3) we start with

$$d_n = \frac{\{(a_{n+1} - a_n) + (a_n - a_{n-1})\}}{2}, \tag{4}$$

to give a better result that is referred to as the central-difference differentiator, i.e.,

$$d_n = \frac{(a_{n+1} - a_{n-1})}{2}. \tag{5}$$

This result is now centred at n, but its wider aperture requires the input sequence to be sampled at a rate that is at least twice that defined by the Nyquist criteria. Examples of these differentiators are taken from Bancroft and Geiger (1997) and show in Figure 1 the spectrums with the amplitudes represented by a solid line, and the phase by a dashed line. The phase is in radians, and the frequency normalised to the sampling frequency. Figure 1 displays in a) the ideal spectrum, b) the backwards-difference spectrum, and c) the central-difference spectrum.

Analysis of these figures indicate that a useful frequency range be limited to less than half the Nyquist frequency. A more accurate 5 point differentiator, (-0.15, 1, 0, -1, 0.15), that is not shown in the figures would slightly increase this range (Bancroft and Geiger, 1997).

Using the *z*-transform to design a recursive filter

The forward-difference differentiator filter could also be defined using the “z” transform as

$$d(z) = (1 - z), \tag{6}$$

and the backward definition

$$z \times d(z) = (1 - z). \tag{7}$$

Summing these two “z” equations gives

$$d(z)(1 + z) = 2(1 - z), \tag{8}$$

or the rational filter

$$d(z) = 2\frac{1-z}{1+z}. \tag{9}$$

This form can be expressed in the time domain as

$$d_n + d_{n-1} = 2(a_n - a_{n-1}), \tag{10}$$

or in the recursive form of

$$d_n = 2(a_n - a_{n-1}) - d_{n-1}, \tag{11}$$

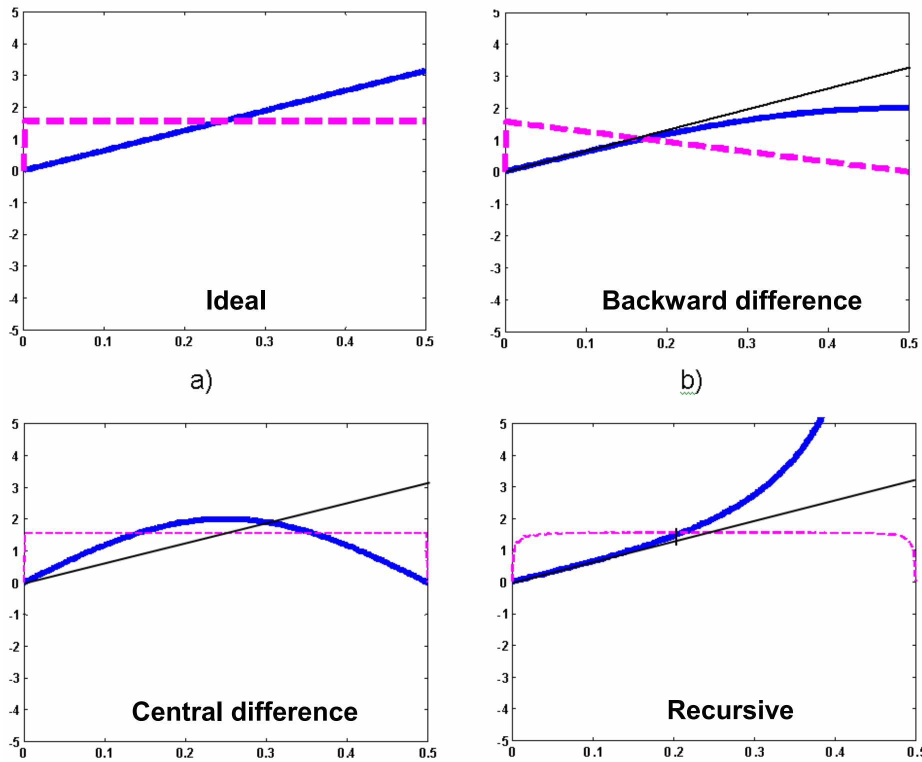
where d_{n-1} is a feed-back term from the previous output. While this is an exact expression, is has implementation problems due to the pole at z=-1. This is resolved by including a damping factor r giving

$$d_n = 2(a_n - ra_{n-1}) - rd_{n-1}. \tag{12}$$

A damping factor r = 0.985 provides a reasonably constant phase shift over the entire frequency band and an amplitude scaling factor that is reasonably accurate to half the Nyquist frequency, as illustrated in Figure 1d

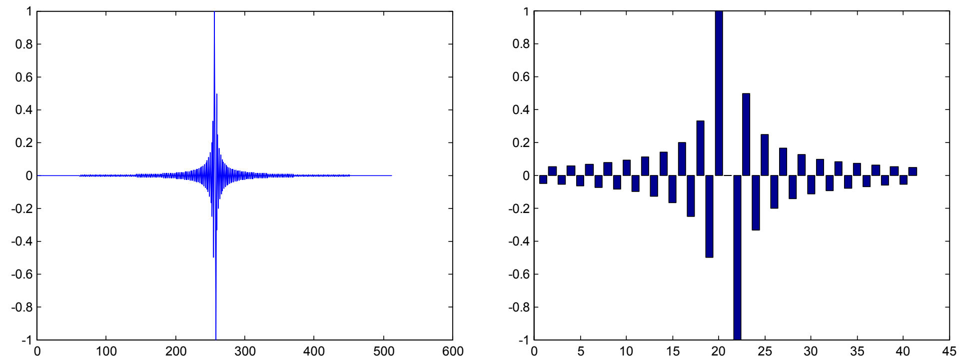
The ideal differentiator in the sampled domain

An ideal time domain form of the differential filter can be found by inverse transforming the ideal form that is defined in the frequency domain. This result is shown in Figure 2 where part a) shows a 512 point representation, and b) a close up view showing forty two points. In both these plots, the time zero (with zero amplitude) has been shifted towards the center of the trace for easier viewing.



c) d)

FIG. 1. The spectrum (amplitude and phase) of time domain filters that approximate the differential filter with a) the exact frequency domain filter, b) the two-point forward difference spectrum, c) the two-point centred difference spectrum, and d) the damped recursive spectrum (r = 0.985). The phase is in radians with the same scale as the amplitude, and the frequency range is normalised to the sampling frequency.



a) b)

FIG. 2. The ideal time domain response of a differentiator filter that is defined in the Fourier transform domain with a) the complete filter, b) a close up showing the amplitudes at discrete locations. These displays have been shifted spatially with the zeroth sample at the centre and with zero amplitude.

ESTIMATING THE TIME DOMAIN RESPONSE

Interpolation

A method for estimating the time domain coefficients of a differentiator was obtained using an interpolator. The interpolation of samples may be dependent on assumptions such as over-sampling the data at three or five times the rate defined by the Nyquist criterion. Now six to ten samples can reasonably define the shape of one period of the maximum frequency, and linear interpolation is usually sufficient.

An accurate interpolator is defined by the bandwidth at the Nyquist frequency, i.e. half the sampling frequency. If we assume a boxcar shape in the frequency domain, where the amplitude is unity to the Nyquist frequency, then time domain interpolator becomes a sinx/x or sinc function. The sinc function has unity at zero time, and is then zero at all other sample locations where the sample interval is defined as π. A continuous interpolation of the discrete input samples is obtained by convolving with the continuous sinc function.

The derivative in the time domain

When obtaining the impulse response of a differentiator, we start with an input spike at zero time. The continuous function of this spike is then found by convolving it with the sinx/x operator, i.e. defining the spike in the continuous domain as the sinx/x operator. The differential operator can now be defined by taking the derivative of the sinx/x function. This is a straight forward derivative by parts (duv = udv + vdu), that gives

$$\frac{d\frac{\sin(x)}{x}}{dx}=\frac{\cos x}{x}-\frac{\sin x}{x^2} \qquad (13)$$

The sin$x$/x function and its derivative are plotted in Figure 3. The derivative is plotted using the analytic function of equation and also by using the central difference operator. These plots overlie each other, but differences occur at the end points where the central difference is zeroed.

The value of the differential operator is defined at the sample locations. In Figure 3 the sample location at n = 1 is identified and the value of the derivative operator identified by the dot. Dots also identify the amplitudes of other samples on the differential operator. A comparison of these amplitudes with those in Figure 2b, show the same amplitude shape.

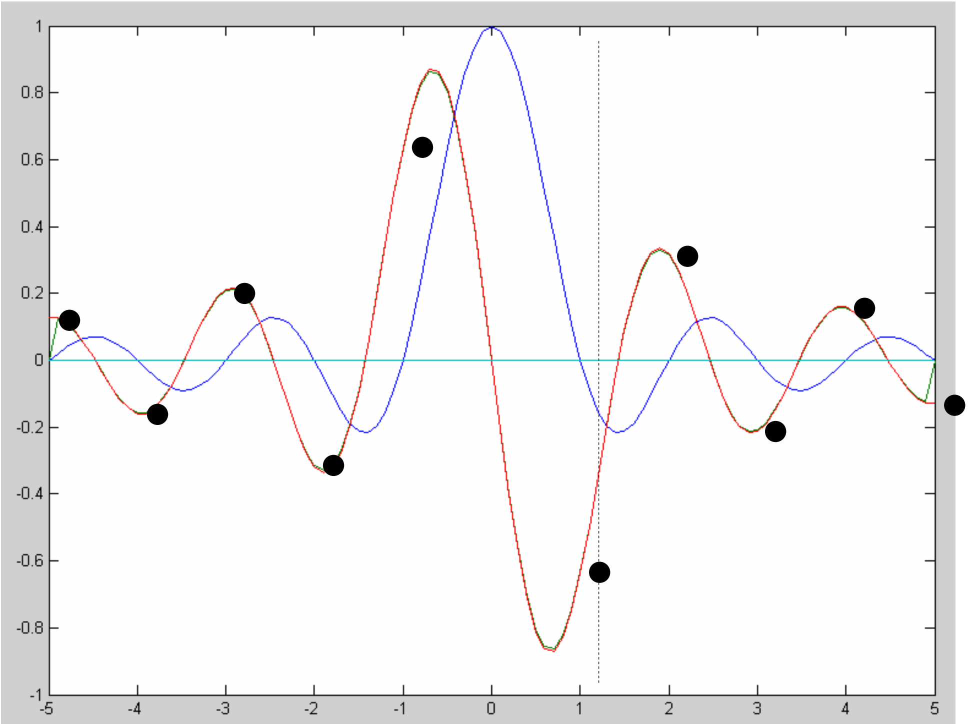


FIG. 3. Plot of the sin$x$/x operator and its derivative. The horizontal axis defines the sample number of the input array (not in radians). [*SincFn.m*]

Using a quadratic interpolator

I will now simplify the interpolation to a quadratic equation that fits three consecutive points, and use the same concept as above by deriving the derivative of this quadratic equation.

Assume three consecutive points with amplitudes *P*, *Q*, and *R*, and with the *x* origin defined at the central point, and unity increment between samples. We now fit a quadratic equation *y(x)* through these points, i.e.

$$y(x) = ax^2 + bx + c \quad . \tag{14}$$

When $x = 0$, it should be more than obvious that

$$c = Q \quad . \tag{15}$$

When $x = 1$, $R = a + b + Q$, giving

$$a + b = R - Q \quad , \tag{16}$$

and when $x = -1$, $P = a - b + Q$, we have

$$a - b = P - Q \quad . \tag{17}$$

From addition and subtraction, these last two equations give

$$b = \frac{R - P}{2} \tag{18}$$

$$a = \frac{P - 2Q + R}{2} \tag{19}$$

Our intuition should be hinting that we may have done extra work, but we now take the derivative of the quadratic equation (14) giving

$$\frac{dy}{dx} = 2ax + b \quad . \tag{20}$$

However, we only want the derivative at $x = 0$, giving,

$$\left.\frac{dy}{dx}\right|_{x=0} = b = \frac{R-P}{2}, \tag{21}$$

the same result as the central difference of equation (5). Higher order approximations are in the works that may give better solutions.

CONCLUSIONS

Conventional methods for estimating the derivative of a sequence were presented. A simple algebraic derivation of the exact time function of a differentiator filter was presented using the concepts of interpolation with a sin$x$/x operator. Another method using a quadratic equation to fit three samples was shown to be identical to the simple central difference method.

COMMENTS

This is basic material and I am sure it must be documented elsewhere; however, to date, I have not been able to identify any references.

I use the term "perfect" for a filter if it is bound or there is a finite limit to its size, and "ideal" if a filter has an infinite response, but is designed with a large aperture. We note that the differentiator filter has infinite response and that the best we can do in the space domain is build an "ideal" filter.

Differentiator filters with half–cycle delays may also be defined using software packages that define spectral shapes. These may be found in Rabiner and Gold (1975).

ACKNOWLEDGEMENTS

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REFERENCES

Bancroft, J.C. and Geiger, H.G., 1997, Analysis and design of filters for differentiation: CREWES Research Report, **9**, Ch. 21.

Rabiner, L.R. and Gold, B., 1975, Theory and Application of Digital Signal Processing: Prentice-Hall