

Spatial prediction filtering in the fractional Fourier domain

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ABSTRACT

The fractional Fourier transform is a new concept in the theory of time-frequency representations. Closely linked to the Wigner distribution through the Radon transform, it introduces frequency-time hybrid domains in which the signal and the noise could be interwoven differently than in either the time or the frequency domain. The fractional transform breaks down a signal into elementary chirp functions called Hermite-Gauss functions. In contrast the Fourier transform decomposes the same signal into harmonic functions. These characteristics might be exploited in the process of separating signal from noise especially in vibroseis datasets. Algorithms for t-x and f-x spatial prediction filters are adapted to perform spatial prediction in fractional Fourier domains and tested on synthetic data. The results obtained are comparable to those obtained in the standard time or frequency domains.

INTRODUCTION

Prediction techniques, such as spatial prediction filtering, are based on the assumption that the signal to be filtered is composed of two parts: one predictable, the coherent signal and other unpredictable, the random noise. The prediction algorithm estimates future values for the predictable component based on past values of the trace and/or neighboring traces. In the spatial prediction process the error in the prediction is associated with the incoherent noise from trace to trace, for which only a unit prediction lag is necessary.

Spatial prediction filtering techniques have been developed to attenuate random noise, uncorrelated from trace to trace, that remains after stacking. The main advantage of spatial prediction methods is the preservation of the relative amplitudes and the signal character without amplitude distortion.

In the spatial prediction process, lateral coordinates are always spatial and the vertical coordinate can be either time or frequency. If the vertical coordinate is time the filtering method is known as the t-x prediction method (Hornbostel, 1991). Alternately if frequency is used as the vertical axis the prediction technique is known as the f-x prediction method (Canales, 1984). Abma, et al. (1995), suggests that the t-x method is favored when the two methods are compared. The f-x method is not favored because it requires a temporally stationary signal, it may introduce artifacts, and more random noise is passed compared to short time lengths used in the t-x method. However, the f-x method can be conducted in overlapping short windows to overcome most of these objections.

The seismic trace, especially in the case of converted-wave datasets, has non-stationary spectrum. When extended to time-frequency domains, the traditional processing methods performed in the time or the frequency domain should improve converted-wave processing techniques. Several time-frequency representations have been introduced to handle non-stationary signals. The Gabor transform (Feichtinger 1998), the wavelet transform (Chakraborty, 1994) and the Wigner distribution (Claasen, 1980) are

the most widely used. The fractional Fourier transform is closely related to the Wigner distribution and its range of applications to wave phenomena research is increasing.

The fractional Fourier transform is a generalization of the Fourier transform. Using it allows the signals to be represented in intermediate domains between time and frequency. The family of fractional Fourier transforms of a signal can be considered as interpolated representations between the signal and its Fourier transform. The spatial prediction filtering method in the fractional Fourier domain will be introduced in this paper.

THEORY

The fractional Fourier transform

The fractional Fourier transform is a generalization of the Fourier transform that has been formulated in several contexts. For example, Wolf (1979), recognized that it was just a special case of the more general set of linear canonical transforms. As well Namias (1980), determined that the fractional Fourier transform was a signal representation generated from the fractional powers of the Hermit-Gauss functions. Lohman (1993) found it studying rotation in the time-frequency plane of the Wigner distribution. Condon (1937), working on operator theory, identified it as a transformation of coordinate multiplication and differentiation operators. Finally, the fractional Fourier transform can be used to solve the quantum-mechanical harmonic oscillator equation (Cohen-Tannoudji et al., 1977).

The ordinary Fourier domain is just a special case of a continuum of fractional Fourier domains, which has a richer mathematical theory and potential applications in every area where the Fourier transform has been used. The definition of the fractional transform, equation (1), introduces a parameter a . When a is equal to zero the transform equals the function itself. When a is equal to one its Fourier transform is obtained. For any other a value between 0 and 1, a new representation of the signal is generated. These new representations may be interpreted as interpolated representations between the signal and its Fourier transform (Figure 1). For a parameter value of 2 or -2 the result is again the original function. When $a=-1$, the inverse Fourier transform is obtained. The definition of the fractional Fourier transform is periodic in a with period 4, so it is enough to consider the interval $[-2,2]$ as the parameter domain.

The a 'th order of the fractional Fourier transform of the function $f(t)$ is

$$f_a(u) = A_\alpha \int_{-\infty}^{\infty} f(t) e^{i\pi(\cot \alpha u^2 - 2 \csc \alpha ut + \cot \alpha t^2)} dt, \quad (1)$$

where the parameter a is a real number in the interval $[-2,2]$. $A_\alpha = \sqrt{1 - i \cot \alpha}$ is a phase factor, in which the square root is defined such that the argument of the result lies in interval $[-\pi/2, \pi/2]$. The kernel of the integral is not strictly defined when a is an even integer. However, it is possible to show that as a approaches an even integer the kernel behaves as a delta function under the integral sign. Generally speaking, the fractional Fourier transform of $f(t)$ exists under the same conditions required for the existence of the

ordinary Fourier transform (Ozaktas, 2001). The fractional variable u , is a hybrid variable composed of interwoven time and frequency, is exactly identical to time when a is zero, identical to frequency when a is one and a hybrid time-frequency new variable for any other value of a between zero and one. Time is a more significant component when the fractional parameter approaches zero whereas frequency dominates when the fractional parameter approaches one.

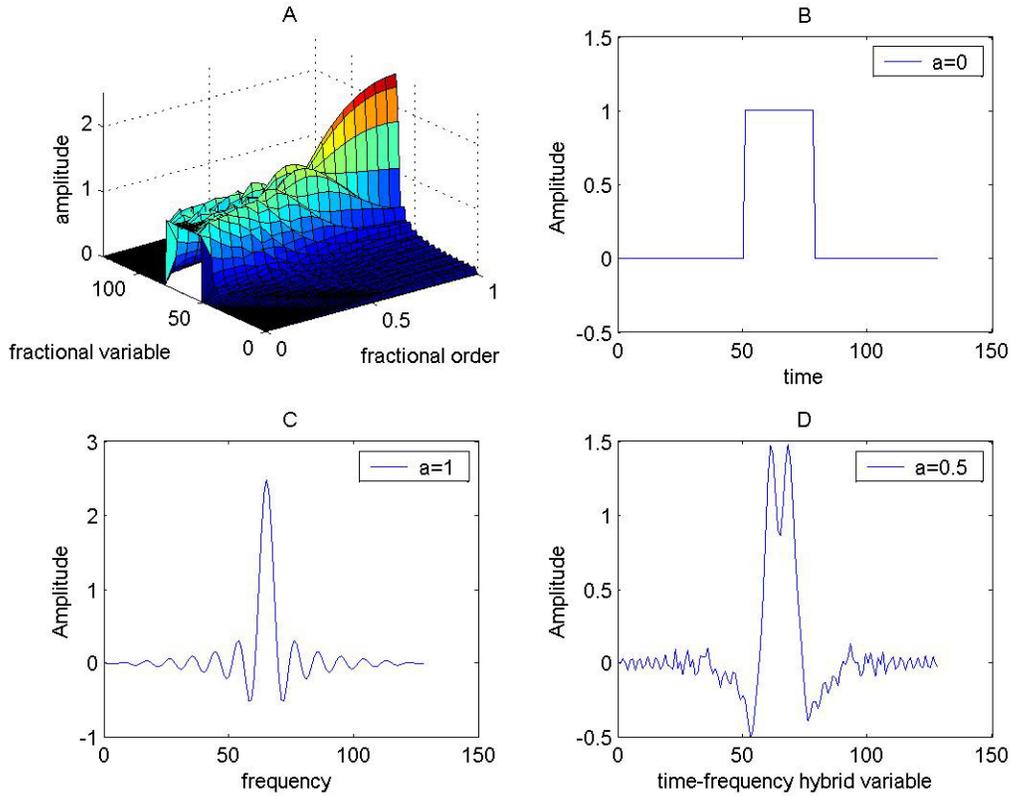


FIG 1. Fractional Fourier transform for a boxcar function. A shows the continuum of fractional transforms, interpolating from the function to its Fourier transform. B, the boxcar function: the fractional variable is time. C, its Fourier transform: the fractional variable is frequency. D, the fractional Fourier transform of order 0.5: the fractional variable is a hybrid of time and frequency. An algorithm by Ozaktas (2001) was used.

The Wigner distribution of $f(t)$ (Claasen, 1980) is defined as

$$W_f(t, \omega) = \int_{-\infty}^{\infty} f\left(t + \frac{t'}{2}\right) f^*\left(u - \frac{t'}{2}\right) e^{-i2\pi\omega t'} dt' \quad (2)$$

$W_f(u, \mu)$ can be interpreted as a function that indicates the distribution of the signal energy over space and frequency. Let \mathfrak{R} denote the Radon transform operator, which maps a two dimensional function of (t, ω) to its integral projection onto a plane making

angle α with the t axis. The fractional Fourier transform and the Wigner distribution are linked through the Radon transform. The Radon transform of the Wigner distribution of a function onto the u_α axis is equal to the squared magnitude of the α th order fractional Fourier transform of the function as expressed in equation (3),

$$\mathfrak{R}_\alpha W_f(u, \mu) = |f_\alpha(u)|^2. \quad (3)$$

$\alpha = a \frac{\pi}{2}$ is an alternate form for the fractional parameter. α can be geometrically interpreted in the Wigner distribution as the direction of the plane on which the integral projection of the Radon transform is done to obtain the squared absolute value of the fractional transform, Figure 2.

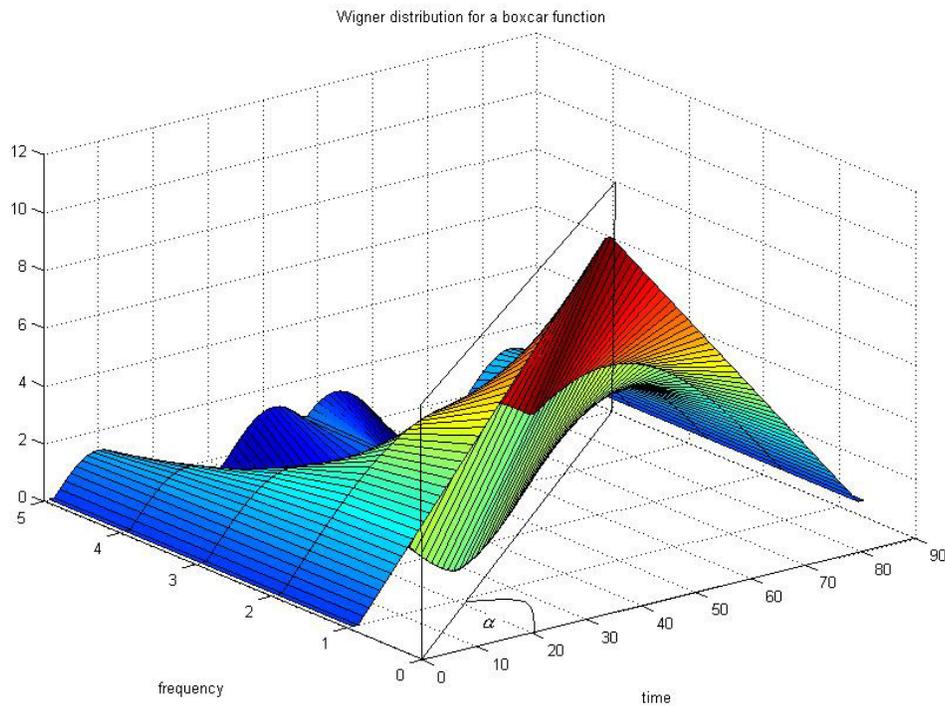


FIG. 2. Wigner distribution for a boxcar function. Its projection (Radon transform) onto a plane making an angle α with respect to the time axis equals the squared module of the fractional Fourier transform of order $\alpha = a \frac{\pi}{2}$. Time frequency MATLAB toolbox (Auger, 1999) was used.

The Fourier transform breaks down a signal in harmonic elementary signals. In a similar way the fractional Fourier transform splits a signal in elementary signals called Hermite-Gauss functions, shown in Figure 3, also well known as chirp functions. Harmonic and delta functions are extreme cases of Hermite-Gauss functions.

The computational cost of the discrete fractional transform has the same order of magnitude as the fast Fourier transform, i.e., $(n \cdot \log n)$, where n is the number of samples

in the trace. Additivity of orders is a property of relevant importance in the computing of the discrete fractional Fourier transform. It establishes that to take consecutively transforms of order b and c is exactly the same as to take the transform of order $b+c$.

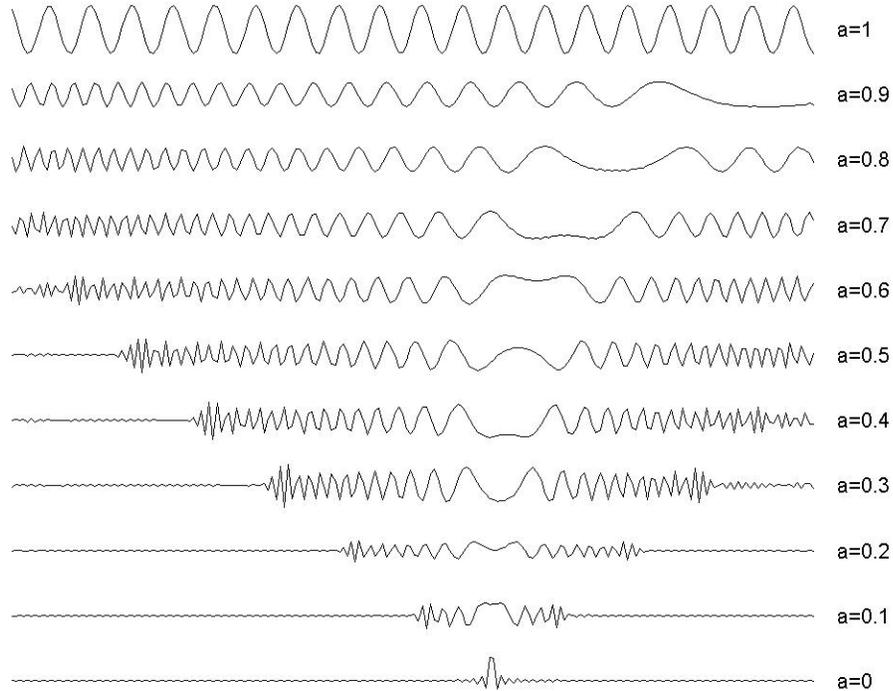


FIG 3. Elementary Hermite-Gauss functions for a given frequency (30 Hz) in which the fractional Fourier transform breaks down a signal. The fractional order is indicated to the left of each signal. As special cases, when $a=1$, the Hermite Gauss function corresponds to a sine function; when a is zero, the Hermite-Gauss function is a delta function.

Predictive filtering

A series $\mathbf{C}=[c_1 \ c_2 \ c_3 \dots \ c_n]$ can be obtained as a result of the filtering process of the series $\mathbf{B}=[b_1 \ b_2 \ b_3 \dots \ b_n]$, defined by the following matrix multiplication

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n+m-1} \end{bmatrix} = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_1 & 0 & 0 \\ b_3 & b_2 & b_1 & \vdots \\ \vdots & b_3 & b_2 & b_1 \\ b_n & b_n & b_3 & b_2 \\ 0 & b_n & \vdots & b_3 \\ \vdots & \dots & b_n & \vdots \\ 0 & 0 & \dots & b_n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} \quad (4)$$

where $\mathbf{F}=[f_1 \ f_2 \ \dots \ f_m]$ is called the filter series. Equation (4) can be written in operator notation as

$$\mathbf{C} = \mathbf{B}\mathbf{F} . \quad (5)$$

In prediction filtering the desired output \mathbf{D} is an advanced version of the series \mathbf{B} , $\mathbf{B}_{ADV}=[b_m \ b_{m+1} \ b_{m-2} \ \dots \ b_{m+n}]$. The goal is to determine the optimum filter \mathbf{F} for which the sum-squared difference between the actual and the desired output $\mathbf{E} = \mathbf{D} - \mathbf{B}_{ADV}$, which is called the error, is minimized. The solution is usually found using a least squared method (Claerbout, 1985). The least-squares minimization ends up with the solution of the normal equations

$$\mathbf{R}\mathbf{F} = \mathbf{G} , \quad (8)$$

where

$$\mathbf{R} = \mathbf{B}^{T*}\mathbf{B} \text{ and } \mathbf{G} = \mathbf{B}^{T*}\mathbf{D} . \quad (9)$$

The examples shown in this paper were obtained from the solution of equation (8) using a biconjugate gradient solver (Claerbout, 1992).

Spatial filtering in the frequency domain (f-x)

Spatial prediction in the frequency domain (Canales, 1984), is performed by windowing data into small tiles where the events can be considered approximately linear. Linear events in the f-x domain are periodic in x direction, which allows the prediction process to be performed on individual frequencies (Gulunay, 1986). Traces from frequency to frequency are highly uncorrelated and no significant improvement can be achieved by including neighboring frequencies in the prediction process (Hornbostel, 1991). The filter is obtained from the least-squares solution of equation (10),

$$\begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n+1} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ x_3 & x_2 & x_1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_n & x_{n-1} & x_{n-2} & x_{n-3} \\ 0 & x_n & x_{n-1} & x_{n-2} \\ 0 & 0 & x_n & x_{n-1} \\ 0 & 0 & 0 & x_n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}, \quad (10)$$

which must be solved for each frequency. The number of points in the filter is the only additional parameter to be chosen. Too low of a value for this parameter results in poor filter performance whereas too high of a value increases the computational cost without achieving any additional benefit. Equation (10) is what is normally called a “forward” prediction filter. A “backward” prediction filter is obtained using $[x_0 \ x_1 \ x_2 \ \dots \ x_3]$ in the LHS of equation (10). The average of these two predictions is usually better than just forward prediction.

Spatial filtering in the time domain (t-x)

In the t-x domain a trace is predicted from its neighboring traces using a 2D filter (Abma, 1995). In contrast to the f-x case, traces are correlated both in vertical and horizontal directions. The number of traces used to build the filter is an important parameter that largely affects the quality of the performance and the computational cost. It depends on the noise level and the degree of structure. A typical value for this parameter in t-x prediction is approximately 5 to 7. The filter length is chosen according to the dip of the events. The prediction process, both in t-x and f-x domains, is carried out forwards and backwards in the x direction and the results averaged to ensure symmetric application. Vertical overlapping of the windows with subsequent averaging could be applied as well. Equation (11) has to be solved by least-squares to determine the filter, in the two-neighbor traces case.

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-2} \\ z_{n-1} \\ z_n \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 0 & 0 \\ x_2 & y_2 & x_1 & y_1 \\ x_3 & y_2 & x_1 & y_2 \\ \vdots & \vdots & \vdots & \vdots \\ x_{n-2} & y_{n-2} & x_{n-3} & y_{n-3} \\ x_{n-1} & y_{n-1} & x_{n-2} & y_{n-2} \\ x_n & y_n & x_{n-1} & y_{n-1} \\ 0 & 0 & x_n & y_n \end{bmatrix} \begin{bmatrix} f_{1,1} \\ f_{1,2} \\ f_{2,1} \\ f_{2,2} \end{bmatrix}. \quad (11)$$

$\mathbf{X}=[x_1 \ x_2 \ x_3 \ \dots \ x_n]$ is the left side neighboring trace, $\mathbf{Y}=[y_1 \ y_2 \ y_3 \ \dots \ y_n]$ is the right side neighboring trace and $\mathbf{Z}=[z_1 \ z_2 \ z_3 \ \dots \ z_n]$ the central trace to be predicted (Claerbout, 1985).

Spatial filtering in the fractional domains (u-x)

The trace in the fractional domain is an intermediate representation between the signal and its Fourier transform. It should be possible to extend the spatial prediction method to the fractional domains. The u-x section is obtained taking the α th fractional Fourier transform of the x-t section in the t direction. Three alternatives appear at first sight to extend the prediction process to u-x domain: 1) to use equation (10), as in the f-x method, where $[x_1 \ x_2 \ x_3 \ \dots \ x_n]$ is now a horizontal layer of the u-x section; 2) to use equation (11) where \mathbf{X} , \mathbf{Y} and \mathbf{Z} are now the α th order fractional Fourier transform of three consecutive traces; or 3) to develop a new algorithm based on the theory of optimal fractional Fourier domain filter (Kutay, 1995). At this point only the first two approaches will be evaluated. After the prediction process is applied to the u-x section, the α -th fractional Fourier transform of the u-x section will generate the filtered dataset.

RESULTS

Spatial prediction, in the three types of domains: t-x, f-x and u-x, is tested on a synthetic section consisting of dipping and hyperbolic crossing events. The synthetic data used for the test is made up of crossing dipping and hyperbolic events. Random noise is added to the dataset using the MATLAB CREWES toolbox command *rnoise*. The algorithms for t-x (Abma, 1995) and f-x (Gulunay, 1986) were coded in MATLAB to generate Figures 4 and 5, on which the best results achieved are shown. The datasets transformed to u-x were subjected to spatial prediction filtering using the same MATLAB functions and then inversely fractional transformed. The results for fractional orders close to one using equation (10) are comparable to those achieved f-x domain using the same equation, Figure 6. For fractional orders close to zero using equation (11) are similar to those obtained in t-x domain using the same equation, as seen in Figures 7. In other cases the result are much worse as in the example shown in Figure 8 for $\alpha=0.5$.

The fractional Fourier transform in the examples shown in this paper was computed using the algorithm proposed by Ozaktas (2001).

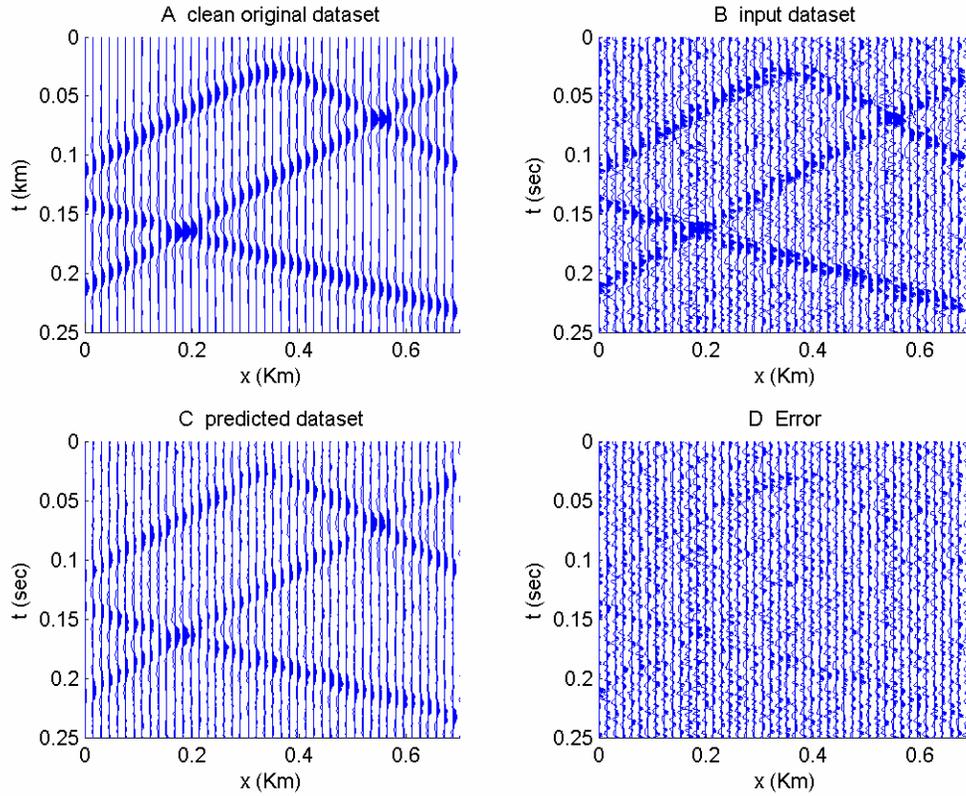


FIG. 4. Filtering in the f - x domain, (Gulunay,1986). A is the original synthetic dataset with 2 msec. sample rate. B is the original dataset contaminated with random noise, signal to noise ratio equal to one. C is the predicted filtered output data. D is the random noise removed by the filter. This is the best result obtained among the different tests carried out, regarding cleanness and character signal conservation, though some strength has been missed in the amplitude.

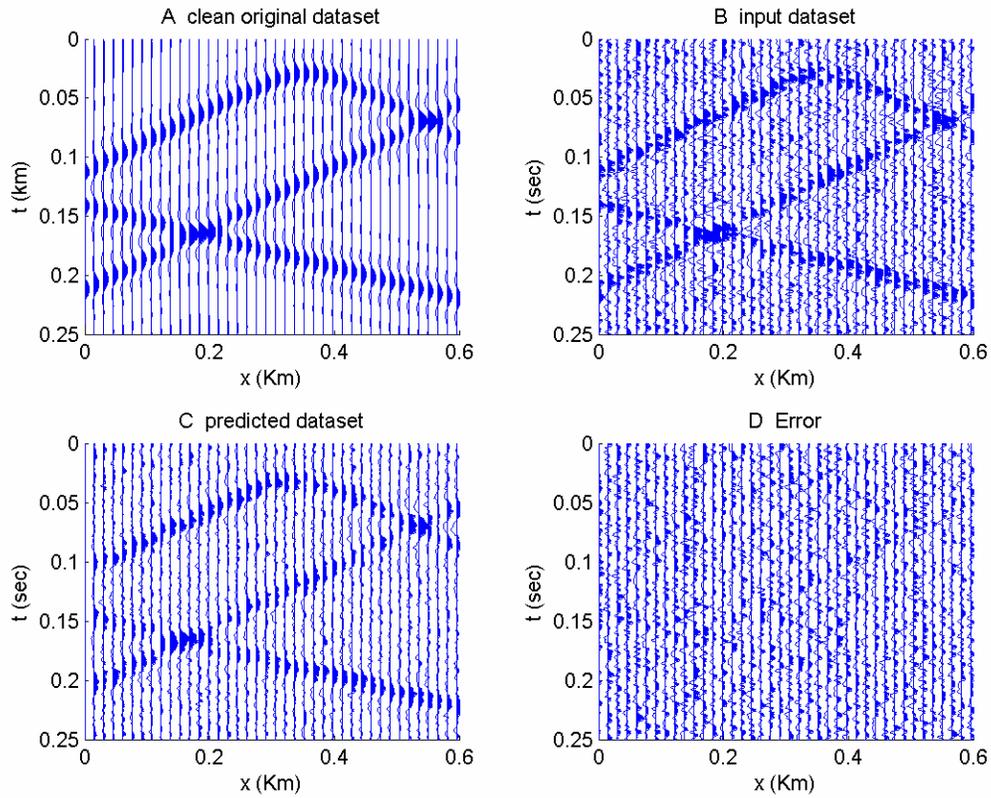


FIG 5. Spatial prediction filter using t-x approach (Abma, 1995). The t-x filter uses 5 traces, 128 time samples and 5 points. A is the original synthetic dataset with 2 msec. sample rate. B is the original dataset contaminated with random noise, signal to noise ratio equal to one. C is the predicted filtered output data. D is the random noise removed by the filter. The output dataset is not as clearer as in the previous case. The wave-shape of the events is as well more affected than in the f-x case.

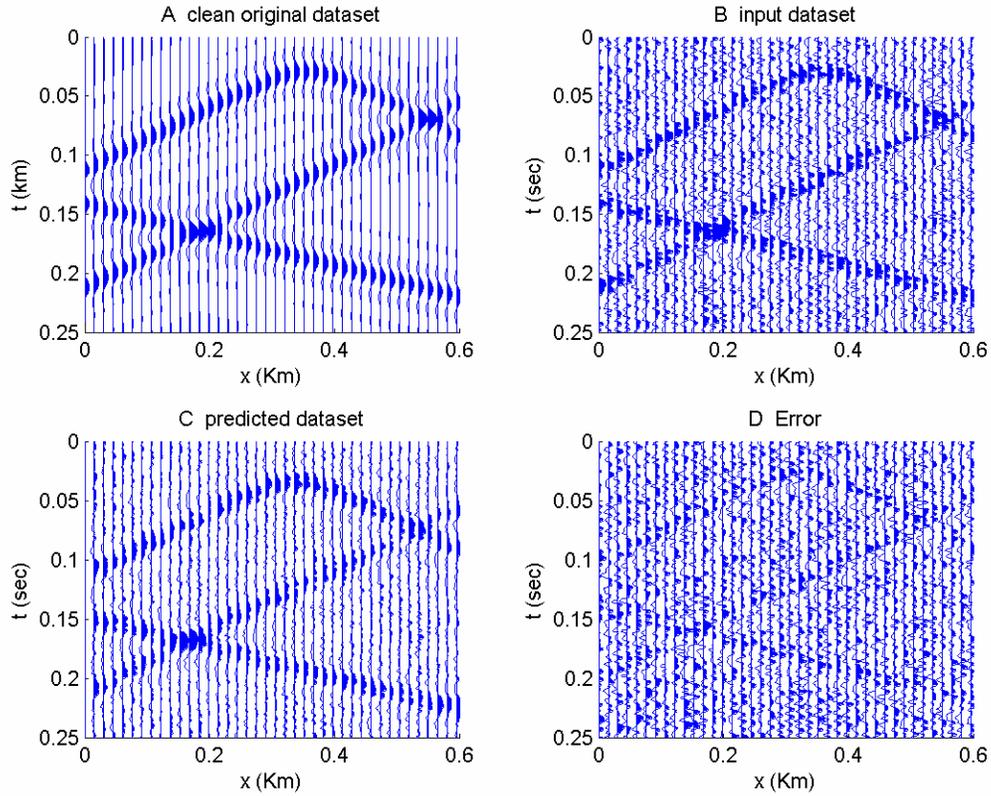


FIG 6. Spatial prediction filtering using u-x (parameter fractional $\alpha=0.001$) approach according to equation (11). A is the original synthetic dataset with 2 msec. sample rate. B is the original dataset contaminated with random noise, signal to noise ratio equal to one. C is the predicted filtered output data. D is the random noise removed by the filter. Filter parameters: 5 points, 5 traces and 128 time samples. Similar results as in the t-x approach are obtained when the value of the fractional parameter moves closer to zero.

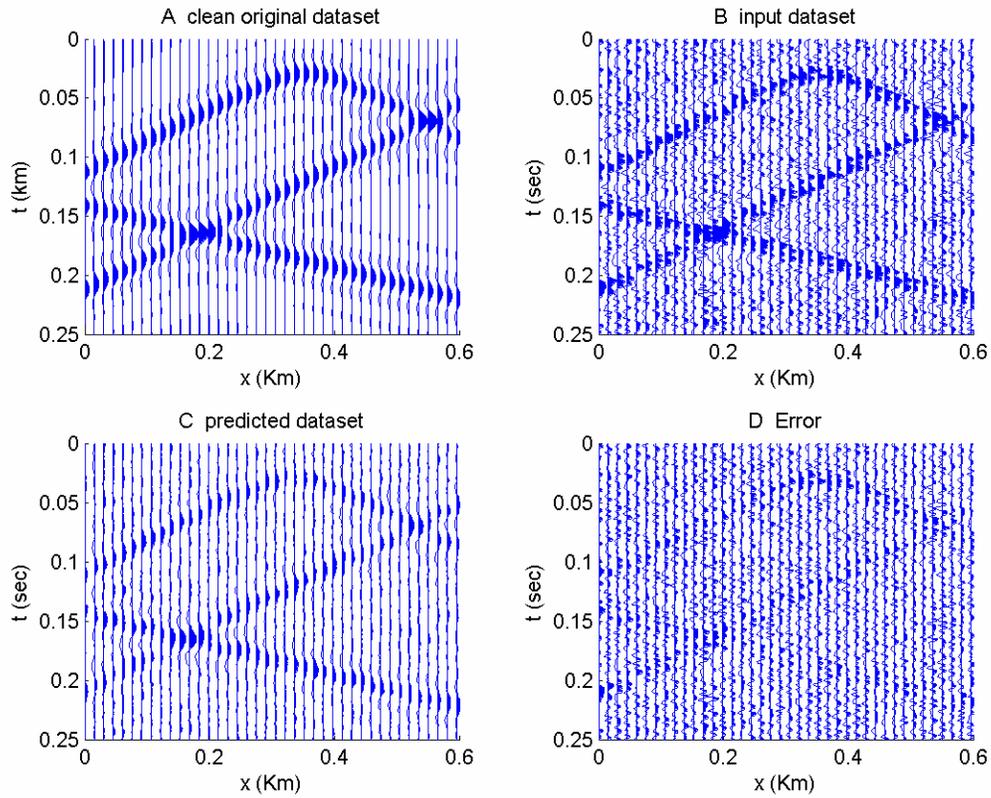


FIG. 7. Spatial prediction filter in u-x domain, $\alpha=0.9$, using equation (10). A is the original synthetic dataset with 2 msec. sample rate. B is the original dataset contaminated with random noise, signal to noise ratio equal to one. C is the predicted filtered output data. D is the random noise removed by the filter. Filter parameter: 5 points. When the value of the fractional parameter moves closer to 1, the results are similar to those obtained with the f-x method.

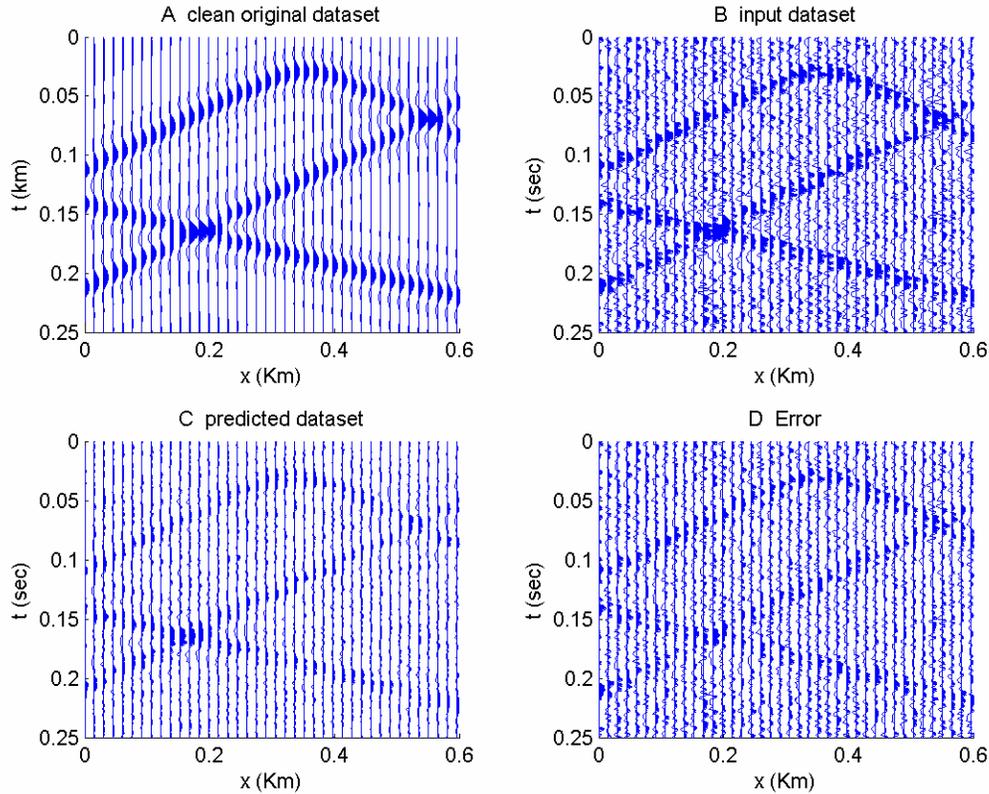


FIG. 8. Spatial prediction filter in fractional Fourier domain, $\alpha=0.5$, using f-x algorithm. A is the original synthetic dataset with 2 msec. sample rate. B is the original dataset contaminated with random noise, signal to noise ratio equal to one. C is the predicted filtered output data. D is the random noise removed by the filter. Filter parameter: 5 points. The quality of the results in the u-x domain makes worse for values far away from 0 or 1. Too much signal is taking away.

CONCLUSIONS

Two important aspects of the fractional Fourier transform theory are expected to influence the results of the spatial u-x prediction: 1) the decomposition of a signal into a set of chirp functions, which could have an important impact on vibroseis datasets and 2) the configuration of the signal and the noise mixing. A first approach to prediction filtering in u-x domains has been achieved on synthetic data, using algorithms designed for t-x or f-x domains. The results obtained are comparable, though not superior, to those obtained in t-x domain when the fractional parameter moves closer to 0 and the same algorithm as in the t-x approach is used. When the fractional parameter moves closer to 1 the results are similar to those obtained with the f-x method if the same algorithm as in the f-x approach is used. Best results are expected from the implementation of the theory of optimum filtering in the fractional Fourier domain.

FUTURE WORK

We will refine the MATLAB functions developed to perform the tests shown in this paper, so that they can be applied to real datasets. Also we will evaluate the influence of the filter parameters in the performance of a spatial prediction filter in fractional domains. We will apply the optimal Wiener filter in fractional Fourier domains theory to an algorithm that allows taking advantage of the potentialities of the fractional Fourier representation of a signal. We will evaluate the feasibility of performing spatial prediction filtering in other time-frequency domains, as the Gabor and wavelet transforms and the Wigner distribution.

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