Approximate QL phase and group velocities in weakly orthorhombic anisotropic media

P. F. Daley and E. S. Krebes

ABSTRACT

The linearized (approximate) expressions for the quasi-longitudinal (QL) phase and group velocities for an orthorhombic medium, which appeared in a recent work, are discussed. The equivalence of two apparently different phase velocity approximations is established. Using the method of characteristics, the exact group velocity is derived for the degenerate elliptical transversely isotropic (TI) case from the eikonal equation. The exact eikonal, if it can be obtained in an analytic form, is homogeneous in powers of 2 in the components of the (phase) slowness vector. The linearization process introduces quartic terms into the eikonal equation. The combination of quadratic and quartic terms in an eikonal presents difficulties that require estimations to reduce the quartic terms to quadratic. These are introduced to obtain an expression for the perturbed TI case. Next, an anisotropic medium displaying orthorhombic symmetry is considered. As in the TI problem, the degenerate ellipsoidal case is treated first to obtain an insight into the more complex perturbed ellipsoidal problem. The 3 symmetry plane perturbation terms are treated collectively to obtain an approximate group velocity or group slowness for an orthorhombic medium.

INTRODUCTION

For an anisotropic medium of even moderate complexity, the eikonal equations associated with all three modes of wave propagation, specifically the quasi-longitudinal QL wave, but including the two shear modes, QT_1 and QT_2, may be quite unwieldy expressions. For each eikonal equation there is a corresponding related phase velocity that is equally cumbersome. The analytic expression for the exact group velocity, which may be obtained from the exact eikonal, becomes at least as complex to derive. For these reasons approximations are sought, usually in the form of linearized expressions obtained using perturbation or related methods (Song et al., 2001 and the relevant references cited therein).

A recent paper by Song and Every (2000) will be referred to frequently here. In that work, identical results to those derived here were presented, and were "… not established … by rigorous derivation but we were lead to [them] by plausibility arguments that are backed up by the numerical results …" Both the phase and group velocity formulae presented in that paper are probably as accurate as any that can be obtained, given the type of approximation used. It is shown here that the expression obtained for the QL phase velocity is, apart from some minor algebraic manipulations, identical to what is termed the "weak anisotropic" approximation. A further motivation is to introduce some mild mathematical formalism to obtain the same QL group velocity expression reasoned by an intuitive process.

Any analytic expression for the eikonal equation of a specific mode of wave propagation in an arbitrary anisotropic medium is homogeneous of order two in powers of components of the slowness vector (Červený, 2001). The components of the slowness vector are partial derivatives of the phase function, $\tau(x_j)$, which describes the wavefront propagation through an elastic medium. The eikonal equation of any mode of propagation is a function of the components of the slowness vector, $s_i = \partial\tau(x_j)/\partial x_i$, and is related to the transport of energy in the medium. Obtaining useful expressions for the group velocity, which is in a form such that it is dependent on the group velocity angles, is difficult for an anisotropic medium of the complexity of an orthorhombic medium.

The eikonal equation is a quasi-linear partial differential equation obtained from the elastodynamic equation, which is hyperbolic. The solution for the rays or characteristics along which energy propagates in an elastic medium is related to the eikonal equation, which lends itself quite readily for use by the method of characteristics (Courant and Hilbert, 1962, Červený, 2001). Solving for the characteristics produces an expression for the group velocity, the phase velocity having been obtained directly from the eikonal equation.

The eikonal equations are obtained from the solution of a cubic polynomial, which is the determinant of a symmetric matrix, indicating that each of the three roots (the eikonal equations) of the cubic polynomial are real.

When an orthorhombic medium is considered, analytic solutions for the eikonals corresponding to the 3 modes of propagation may be obtained. However, they are of such complexity that anything but a numerical analysis does not provide insight into the actual form.

It was thought that it might be useful to include here a brief overview of the theory of characteristics upon which depends much of what follows. In a generally inhomogeneous anisotropic medium, the equations of motion describing the propagation of elastic waves in a Cartesian coordinate system, which is assumed to be aligned with both model and crystal axes, may be written as (Červený, 2001)

$$\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right) - \rho\frac{\partial^2 u_i}{\partial t^2} = 0. \tag{1}$$

where t is time, ρ is the density, u_j are the components of the displacement vector $\mathbf{u}(\mathbf{x},t)$ and c_{ijkl} are the anisotropic elastic parameters. The elastic parameters, c_{ijkl}, and density, ρ, and their first derivatives are assumed to be continuous functions of position.

Three types of waves, one quasi-longitudinal, QL, and two quasi-transverse, QT_1 and QT_2, propagate within a medium of this symmetry. Assuming that all three waves have been generated at some point source within an infinite medium, the associated wavefronts may be described by a phase function $\tau(x_i)$. In general, each propagation mode has a unique phase function describing its propagation through the medium at some time $t = \tau(x_i)$. This propagation is dependent on the spatial variation of the anisotropic parameters and varies with direction. The fronts propagate independently of one another, and they may intersect, particularly the two shear modes. A receiver located in the medium at some distance from the source would record the superposition of the displacements associated with each of these fronts.

The standard geometrical optics asymptotic solution, for problems of this type, has been discussed by numerous authors (see, for example, Červený, 2001) and its form is

$$\mathbf{u}(x_j,t)=\left[u_1(x_j,t),u_2(x_j,t),u_3(x_j,t)\right]=\sum_{n=0}^{\infty}\mathbf{A}_n(x_j)f_n\left(t-\tau(x_j)\right) \tag{2}$$

where $\mathbf{A}_n(x_j)$ is a vector amplitude term dependent only on spatial coordinates, $\tau(x_j)$ is the phase function which has been discussed previously, and $f_n(\xi)$ is some generalized function with the property

$$\frac{df_n(\xi)}{d\xi}=f_{n-1}(\xi) \tag{3}$$

and is subject to the constraint that $f_n(\xi)\equiv 0,(n<0)$. The most common form that $f_n(\xi)$ takes in engineering, physics and related applications is

$$f_n\left(t-\tau(x_j)\right)=\frac{\exp\left[i\omega\left(t-\tau(x_j)\right)\right]}{(i\omega)^n}. \tag{4}$$

The following notation changes and definitions are introduced to simplify this discussion. The second of these, $a_{ijk\ell}$, refers to the density normalized anisotropic elastic parameters. They are introduced because they have the dimensions of velocity squared, and this feature makes the discussion that follows easier to implement as velocity is the central topic here.

$$\Gamma_{jk}=a_{ijkl}s_is_\ell \quad , \quad a_{ijkl}=c_{ijkl}/\rho \quad , \quad s_i=\partial\tau/\partial x_i\ . \tag{5}$$

The vector $\mathbf{s}=(s_1,s_2,s_3)$ is the phase slowness vector, which by its definition above is normal to the phase surface. The order 3, rank 4 tensor, a_{ijkl}, with 21 generally independent members, will be replaced by the 6×6 symmetric matrix A_{mn} (Musgrave, 1970) as follows:

$$a_{ij|kl}\rightarrow A_{mn}. \tag{6}$$

This simplification is made subject to the following scheme

$$11\rightarrow 1,\quad 22\rightarrow 2,\quad 33\rightarrow 3,\quad 23\rightarrow 4,\quad 13\rightarrow 5,12\rightarrow 6. \tag{7}$$

Substitution of Equation 2 into 1 yields, apart from systems of equations for dynamic (amplitude) quantities, the following problem results

$$\left[\Gamma_{jk} - \delta_{jk}\right] A_0^k = 0 . \tag{8}$$

Under the assumption that $\mathbf{A}_0$ from Equation 2 is not identically zero, for a solution to exist, the following must hold

$$\det\left[\Gamma_{jk} - G_M\,\delta_{jk}\right]_{G=1} = 0 \qquad \left(M = \mathrm{QL}, \mathrm{QT}_1 \text{ or } \mathrm{QT}_2\right). \tag{9}$$

This leads to the eigenvalue problem, involving the matrix $\boldsymbol{\Gamma}$, to be satisfied

$$G^3 - \mathrm{tr}\left(\boldsymbol{\Gamma}\right) G^2 + \mathrm{tr}\left[\mathrm{cof}\left(\boldsymbol{\Gamma}\right)\right] G - \det\left(\boldsymbol{\Gamma}\right) = 0 . \tag{10}$$

As $\boldsymbol{\Gamma}$ is symmetric and positive definite, its eigenvalues are real, positive and generally have distinct values. It should be noted here that the analytic expression for G_M, however complex, is homogeneous of second order in powers of s_j, so that

$$s_j \frac{\partial G_M}{\partial s_j} = 2G_M \qquad \left(M = \mathrm{QL}, \mathrm{QT}_1, \mathrm{QT}_2\right). \tag{11}$$

As a consequence of the known feature (Červený, 2001) that $\boldsymbol{\Gamma}$ is positive definite, the resulting qualities of the three eigenvalues, equation (1) is often referred to as “the irreducible case” when discussing cubic equations (Abramowitz and Stegun, 1980). There is, however, a variable change (for example: Every, 1980 or Schoenberg and Helbig, 1997), which allows for the determination of analytic expressions for the three eigenvalues. These are quite awkward and more suitable to a numerical rather than analytical implementation, a necessary usage for checking the accuracy of approximate solutions. They will not, however, be considered in more detail here.

If an analytic expression for the eikonal equation may be found, the direction and velocity of energy transport along a ray within an anisotropic medium may be determined using the method of characteristics (Courant and Hilbert, 1962). The velocity of energy transport, or group velocity, is obtained using the appropriate eikonal equation of the three that may be obtained from the solution of equation (9), i.e. $G_M\left(x_j, s_j\right) = 1$, $\left(M = \mathrm{QL}, \mathrm{QT}_1, \mathrm{QT}_2\right)$. This is accomplished by obtaining a solution, given the initial conditions at some time t_0, $\mathbf{x}\left(t_0\right) = \mathbf{x}_0$ and $\mathbf{s}\left(t_0\right) = \mathbf{s}_0$, of the fully specified initial value problem for the system of equations

$$\frac{dx_j}{dt} = \frac{1}{2}\frac{\partial G_M}{\partial s_j} \tag{12}$$

$$\frac{ds_j}{dt} = -\frac{1}{2}\frac{\partial G_M}{\partial x_j} . \tag{13}$$

The group or ray velocity vector is given as

$$\frac{d\mathbf{x}}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt} \right), \tag{14}$$

with the magnitude defined as

$$\left| \frac{d\mathbf{x}}{dt} \right| = \left[\left(\frac{dx_1}{dt} \right)^2 + \left(\frac{dx_2}{dt} \right)^2 + \left(\frac{dx_2}{dt} \right)^2 \right]^{1/2}, \tag{15}$$

and the azimuthal angle, Φ, $\left(0 \le \Phi < 2\pi\right)$ which may be expressed as

$$\tan\Phi = \left[\frac{dx_2}{dx_1} \right] = \left[\frac{dx_2/dt}{dx_1/dt} \right]. \tag{16}$$

Defining the projection of the 3D group velocity vector onto the (x_1, x_2) plane as

$$\frac{dr}{dt} = \left[\left(\frac{dx_1}{dt} \right)^2 + \left(\frac{dx_2}{dt} \right)^2 \right]^{1/2}, \tag{17}$$

the group polar angle, Θ, $\left(0 \le \Theta \le \pi\right)$ is given by

$$\tan\Theta = \left[\frac{dr}{dx_3} \right] = \left[\frac{dr/dt}{dx_3/dt} \right]. \tag{18}$$

Before beginning the determination of an approximate expression for the group velocity in an orthorhombic medium, the linearized phase velocity of that media type will be discussed. The group velocity derivation is highly dependent on the phase velocity, or, probably more correctly, the related eikonal equation.

PHASE VELOCITY

The expression given in the work of Song and Every (2000) for the QL phase velocity obtained for a weakly orthorhombic anisotropic medium differs from that obtained by standard linearization techniques (for example, Backus, 1965). The major discrepancy between the two expressions for the QL phase velocity in a weakly orthorhombic medium in the above papers is that the A_{ii} terms, $(i = 1,2,3)$, which are defined below, are quadratic in the components of the phase velocity direction vector **n** in Song and Every (2000) rather than quartic as in Backus (1965). This apparent inconsistency will be examined. The density normalized anisotropic parameters A_{ij}, which will be used here, have the dimensions velocity squared and have been defined in Equations 5 through 7.

The linearized formula for the QL phase velocity, for a weak orthorhombic anisotropic medium using Backus' (1965) method being the standard that appears in the literature for a range of areas of study, is

$$v_{qL}^2 = A_{11}n_1^4 + A_{22}n_2^4 + A_{33}n_3^4 + \\ 2\left(A_{12} + 2A_{66}\right)n_1^2n_2^2 + 2\left(A_{13} + 2A_{55}\right)n_1^2n_3^2 + 2\left(A_{23} + 2A_{44}\right)n_2^2n_3^2 \,. \tag{19}$$

This equation is identical to that presented in Every and Sachse (1992), apart from the use of A_{ij} rather than C_{ij} $\left(A_{ij} = C_{ij}/\rho\right)$. The phase velocity direction vector $\mathbf{n}$ is defined as

$$\mathbf{n} = \left(n_1, n_2, n_3\right) = \left(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta\right). \tag{20}$$

Here, θ is the polar angle measured from the positive x_3 axis $\left(0 \le \theta \le \pi\right)$ and ϕ is the azimuthal angle measured in a positive sense from the x_1 axis $\left(0 \le \phi < 2\pi\right)$. In equation (19) the A_{ii} terms, $(i = 1,2,3)$, are clearly fourth order in powers of the components of $\mathbf{n}$.

The phase velocity formula for this medium type given the in paper by Song and Every (2000) may be obtained from Equation 19 by adding to it and subtracting from it the terms

$$n_1^2n_2^2\left(A_{11} + A_{22}\right) + n_1^2n_3^2\left(A_{11} + A_{33}\right) + n_2^2n_3^2\left(A_{22} + A_{33}\right) \tag{21}$$

doing some minor algebraic manipulations and utilizing the properties of $\mathbf{n}$ to obtain

$$\begin{aligned} v_{qL}^2 = {} & A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 + \\ & 2\left[\left(A_{12} + 2A_{66}\right) - \left(A_{11} + A_{22}\right)/2\right]n_1^2n_2^2 + \\ & 2\left[\left(A_{13} + 2A_{55}\right) - \left(A_{11} + A_{33}\right)/2\right]n_1^2n_3^2 + \\ & 2\left[\left(A_{23} + 2A_{44}\right) - \left(A_{22} + A_{33}\right)/2\right]n_2^2n_3^2 \quad . \end{aligned} \tag{22}$$

In a more compact form the above may be written as

$$v_{qP}^2 = A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 + B_{12}n_1^2n_2^2 + B_{13}n_1^2n_3^2 + B_{23}n_2^2n_3^2 \tag{23}$$

with

$$B_{12} = 2\left(A_{12} + 2A_{66}\right) - \left(A_{11} + A_{22}\right) \tag{24}$$

$$B_{13} = 2\left(A_{13} + 2A_{55}\right) - \left(A_{11} + A_{33}\right) \tag{25}$$

$$B_{23} = 2\left(A_{23} + 2A_{44}\right) - \left(A_{22} + A_{33}\right). \tag{26}$$

This is the same result as that developed by Song and Every (2000). The only difference between the linearized expression (equation 19) and their approximation, equation (23), is that a quantity has been added and then subtracted so they must be equivalent. Further, equation (23) is more conceptually indicative of the physics of the situation. The phase velocity surface is an ellipsoid with three "anellipsoidal" correction terms, one corresponding to each of the 3 symmetry planes. In combination, these comprise an approximation of second order accuracy. A fairly wide survey of the literature, covering many disciplines of physics, especially geophysics, suggests that equation (1) appears to be a more popular representation than equation (23) due to the introduction in that research area of the alternate $(\varepsilon, \delta, \gamma, \chi)$ notation (see for example Pšenčík and Gajewski, 1998, Jech and Pšenčík, 1989 and the numerous references cited therein).

GROUP VELOCITY

Transversely isotropic medium

Using equation (22), the problem of determining an expression for the quasi-longitudinal QL group velocity in an orthorhombic anisotropic medium will now be considered.

Although the derivation presented here is not mathematically rigorous, it does incorporate some theoretical aspects not found in Song and Every (2000), while relying on that work for a certain amount of direction.

As an introductory problem, the group velocity for a QL wave in a transversely isotropic (TI) medium, specifically the degenerate elliptical case, will be solved in an exact manner and the results used to develop a linearized expression for the more complex anelliptic TI case. The exact eikonal equation for any mode of propagation in an elastic medium is a quasi-linear partial differential equation homogeneous of order 2 in powers of the phase slowness vector components, $\mathbf{s} = (s_1, s_2, s_3)$, $s_j = \partial\tau(x_i)/\partial x_j$ (Musgrave, 1970). The phase function, $\tau(x_j)$, a function of position, describes the wavefront propagation through the medium, (Musgrave, 1970).

The eikonal equation for the simple elliptical QL problem is investigated using the theory of characteristics or rays (Courant and Hilbert, 1962). The major obstacle to be overcome in more complex anisotropic media types in obtaining expressions for the QL group velocity is that the linearized phase velocities, and hence eikonal equations, do not retain the homogeneity of order 2 in powers of slowness vector components.

The elliptical form of the QL eikonal equation in a transversely isotropic medium is given as

$$G(s_1, s_3) = A_{11}s_1^2 + A_{33}s_3^2 = 1, \quad (27)$$

and the eikonal equation for a weakly anisotropic TI medium is

$$G(s_1, s_3) = A_{11}s_1^2 + A_{33}s_3^2 + \hat{B}_{13}s_1^2 s_3^2 = 1 \quad (28)$$

where the A_{ii} are as previously defined, and $\hat{B}_{13}$ is the anellipticity factor explicitly contained in the exact QL eikonal equation. This quantity, defined below, must be temporarily introduced, replacing B_{13} in equation (13) to preserve the dimensionality of the eikonal equation (28), as "$s_1^2 s_3^2$" has dimensions of velocity to the negative fourth power while the dimension of B_{13}, given by equation (25), is only velocity squared. The quantity $\hat{B}_{13}$ (in distinctive notations) is defined (Gassmann, 1965, Every, 1980, Schoenberg and Helbig, 1997) as

$$\hat{B}_{13} = \left(A_{13} + A_{55}\right)^2 - \left(A_{11} - A_{55}\right)\left(A_{33} - A_{55}\right). \tag{29}$$

The $s_j, \left(j = 1,3\right)$ are the horizontal and vertical components of the slowness vector, $\mathbf{s} = \left(s_1, s_3\right)$ for this two dimensional problem in slowness space. Equation (10) is equivalent to the following expression for the related phase velocity as a function of the phase or wavefront normal angle, θ,

$$v^2\left(\theta\right) = A_{11}\sin^2\theta + A_{33}\cos^2\theta \tag{30}$$

when taken together with the definition of the slowness vector components in terms of the phase velocity and angle which are

$$s_1 = \frac{\sin\theta}{v\left(\theta\right)}, \quad s_3 = \frac{\cos\theta}{v\left(\theta\right)}. \tag{31}$$

The initial value problem specified previously for the general case of an anisotropic medium in equations (11) and (12) can be obtained for the specific problem of an elliptical QL wavefront in a TI medium. The components of the group (ray) velocity vector are given as

$$\frac{dx_1}{dt} = \frac{1}{2}\frac{\partial G}{\partial s_1} = A_{11}s_1 \tag{32}$$

$$\frac{dx_3}{dt} = \frac{1}{2}\frac{\partial G}{\partial s_3} = A_{33}s_3. \tag{33}$$

Under the assumption that the anisotropic parameters, A_{ij}, are spatially independent, the ds_j/dt terms are all identically zero so that $\mathbf{s_0} = \mathbf{s} = \text{constant}$.

The group velocity vector has the form

$$\frac{d\mathbf{x}}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_3}{dt}\right) \tag{34}$$

with magnitude

$$V_e\left(\Theta\right) = \left|\frac{d\mathbf{x}}{dt}\right| = \left[\frac{d\mathbf{x}}{dt}\cdot\frac{d\mathbf{x}}{dt}\right]^{1/2} = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2}. \tag{35}$$

The subscript "e" indicates the elliptical case, and Θ is the angle between the ray along which energy propagates, and the vertical axis. The magnitude of the group velocity is expressed in terms of A_{11}, A_{33}, and the components of the slowness vector. Introducing equations (32) and (33) into (35), the QL group velocity for the elliptical TI case is given in terms of phase angle and velocity as

$$V_e(\Theta) = \left[A_{11}^2 s_1^2 + A_{33}^2 s_3^2 \right]^{1/2} . \qquad (36)$$

Using a trigonometric relationship between the vector components of the group velocity, the angle, Θ, at which the group velocity associated with a specified phase angle, θ, travels with respect to the vertical, x_3, axis is obtained from the relationship

$$\tan\Theta = \left[\frac{dx_1/dt}{dx_3/dt} \right] = \left[\frac{dx_1}{dx_3} \right] = \frac{A_{11}}{A_{33}} \tan\theta . \qquad (37)$$

Inserting equations (36) and (37) in equations (30) and (31), the components of the slowness vector may be written solely in terms of the group velocity and angle as (Daley and Hron, 1979 and Appendix A)

$$s_1 = \frac{V_e(\Theta)\sin\Theta}{A_{11}} \quad \text{and} \quad s_3 = \frac{V_e(\Theta)\cos\Theta}{A_{33}} \qquad (38)$$

Using equation (38) and the related eikonal equation (27) yields, for the elliptical case,

$$1 = \frac{V_e^2(\Theta)\sin^2\Theta}{A_{11}} + \frac{V_e^2(\Theta)\cos^2\Theta}{A_{33}} \qquad (39)$$

so that the QL group velocity as a function of group angle in this instance has the form

$$\frac{1}{V_e^2(\Theta)} = \frac{\sin^2\Theta}{A_{11}} + \frac{\cos^2\Theta}{A_{33}} \qquad (40)$$

Now assume that some small dimensionless perturbation, $a_{13} << 1$, is introduced in the following manner such that the elliptical group velocity surface is deformed in the (x_1, x_3) plane

$$\frac{1}{V^2} = \frac{1}{V_e^2\left[1 + a_{13}\right]} \approx \frac{\left[1 - a_{13}\right]}{V_e^2} = \frac{1}{V_e^2} - \frac{a_{13}}{V_e^2} \qquad (41)$$

It is understood that a_{13} should be related to a similar quantity that appears in equation (28) for the weakly anisotropic eikonal for this media type. Using the paper of Song and Every (2000), together with an approximation to the exact eikonal equation for a QL wave propagating in a TI medium as points of reference, the choice of $a_{13} = \hat{B}_{13} s_1^2 s_3^2$ would not seem to be unreasonable. This quantity is quartic and not quadratic in powers of components of the slowness vector, s_j. However, at this point it is dimensionally correct, a_{13} being dimensionless, which is a requirement for equation (41) to be valid.

Before proceeding further, equation (28) will be put in a form that is homogeneous of order 2 in powers of slowness vector components, by making certain approximations. One method of accomplishing this is to write the anelliptic term in Equation 28 as

$$\hat{B}_{13}s_1^2s_3^2 = \hat{B}_{13}s_1\left[s_1s_3\right]s_3 \tag{42}$$

and approximate the square bracketed term in some fashion. Plausible estimated values, based on physical principals, would be to approximate s_1 and s_3 as $s_1 \approx \sin\Theta\big/\sqrt{A_{11}}$ and $s_3 \approx \cos\Theta\big/\sqrt{A_{33}}$. The values of these terms correspond to those they would have if the medium type lay somewhere between isotropic and elliptically anisotropic. Introducing these approximations into equation (42) yields

$$a_{13} = \hat{B}_{13}s_1^2s_3^2 = \hat{B}_{13}s_1\left[s_1s_3\right]s_3 \approx \hat{B}_{13}s_1\left[\frac{\sin\Theta}{\sqrt{A_{11}}}\frac{\cos\Theta}{\sqrt{A_{33}}}\right]s_3 \tag{43}$$

Approximating $\left[s_1s_3\right]$ as above results in that at $\Theta = 0$ or $\left|\Theta\right| = \pi/2$, $\left[s_1s_3\right] \equiv 0$ and that within the range of angles $0 < \left|\Theta\right| < \pi/2$, $\left[s_1s_3\right]$ attains its maximum values. This angle may be obtained for the general TI case by minimizing the part of the exact eikonal equation that characterizes the anellipticity and is discussed in greater detail in Gassmann (1965). Due to the symmetry of the phase velocity surface, the above arguments are valid in the range of the angle Θ, $(\pi/2 < \Theta < \pi = -\pi/2)$.

The approximation given in Equation 43 requires the eikonal to be quadratic in terms of the components of the slowness vector, $s_j\,(j = 1,3)$. Using equations (38) as trial solutions for s_1 and s_3 in equation (43) results in equation (41) becoming

$$\frac{1}{V^2(\Theta)} \approx \frac{\sin^2\Theta}{A_{11}} + \frac{\cos^2\Theta}{A_{33}} - \frac{\hat{B}_{13}}{V_e^2}\left[\frac{V_e\sin\Theta}{A_{11}}\right]\left[\frac{\sin\Theta}{\sqrt{A_{11}}}\frac{\cos\Theta}{\sqrt{A_{33}}}\right]\left[\frac{V_e\cos\Theta}{A_{33}}\right], \tag{44}$$

or, after some simplification,

$$\frac{1}{V^2(\Theta)} \approx \frac{\sin^2\Theta}{A_{11}} + \frac{\cos^2\Theta}{A_{33}} - \frac{\hat{B}_{13}}{\sqrt[3]{A_{11}A_{33}}}\sin^2\Theta\cos^2\Theta. \tag{45}$$

This defines the approximate group velocity for a slightly perturbed ellipse. Rearranging equation (45), which includes reintroducing the linearized deviation term B_{13} to replace $\hat{B}_{13}$, yields

$$\frac{1}{V^2(\Theta)} \approx \frac{\sin^2\Theta}{A_{11}} + \frac{\cos^2\Theta}{A_{33}} - \frac{B_{13}\sin^2\Theta\cos^2\Theta}{A_{11}A_{33}} \tag{46}$$

The substitution $\hat{B}_{13} \to B_{13}$ above required the dimension of the denominator of the anelliptic term in Equation 46 to be reduced by a factor of velocity squared. This was accomplished by dividing the denominator by $\sqrt{A_{11}A_{33}}$, a reasonable approach based on its definition and the approximation introduced for $[s_1 s_3]$ to force dimensional correctness and homogeneity of slowness vector components in equation (43). The use of $V_e(\Theta_e)$ as defined in equation (40) with Θ_e given by equation (37) is replaced in equation (44)) by $V_e(\Theta)$ where Θ is now some arbitrary quantity. Apart from the use of density normalized anisotropic coefficients, the above equation is the same as that obtained by Song and Every (2000).

The derivation of equation (46) required that some possibly questionable mathematical assumptions were made. These were felt to be justified to keep the procedure as uncomplicated as possible and yet introduce some suggestion as to the mathematical direction to take. Also, when compared with methods where approximations were made to the exact QL eikonal equation for a TI medium, the results are quite consistent (Gassmann, 1965). Other more complex approaches could have been pursued to obtain the equivalent of equation (46) but at some point an approximation similar to that introduced in equation (43) must be made. Starting with the exact eikonal for a TI medium produces more accurate intermediate results. However, to arrive at equation (43), approximations related to linearization must be made. The correspondence, up to some normalizing constants, of B_{13} and $\hat{B}_{13}$ is the most important example of this.

Orthorhombic medium

Modifying the expression for the phase velocity given by equation (23) in a manner similar to that used to obtain the linearized eikonal equations (27) and (28) in the TI case, the degenerate (ellipsoidal) QL eikonal for an orthorhombic medium is given by

$$G_e(s_1, s_2, s_3) = A_{11}s_1^2 + A_{22}s_2^2 + A_{33}s_3^2 = 1\,, \tag{47}$$

and the linearized eikonal for a weak anisotropic medium of this type has the form

$$G(s_1, s_2, s_3) = A_{11}s_1^2 + A_{22}s_2^2 + A_{33}s_3^2 + \hat{B}_{12}s_1^2 s_2^2 + \hat{B}_{13}s_1^2 s_3^2 + \hat{B}_{23}s_2^2 s_3^2 = 1 \tag{48}$$

with $\hat{B}_{13}$ given by equation (29) and $\mathbf{s} = (s_1, s_2, s_3)$ being the phase slowness vector. The quantities $\hat{B}_{12}$ and $\hat{B}_{23}$ required, together with $\hat{B}_{13}$, to maintain proper spatial dimensionality are defined (Every, 1980 or Schoenberg and Helbig, 1997) as

$$\hat{B}_{12} = (A_{12} + A_{66})^2 - (A_{11} - A_{66})(A_{22} - A_{66}) \tag{49}$$

$$\hat{B}_{23} = (A_{23} + A_{44})^2 - (A_{22} - A_{44})(A_{33} - A_{44})\,. \tag{50}$$

As in the elliptical TI case, the phase slowness vector components may also be expressed solely in terms of group velocity related parameters. The method of characteristics is again used for this purpose, at least in the degenerate QL ellipsoidal case. The algebra becomes more complicated and the derivation of relevant quantities is described in Appendix B.

Assuming that the degenerate medium type above is homogeneous, having partially independent A_{ij}, the rays are straight lines from the source to the receiver and the three components of the group velocity vector are given by

$$\frac{dx_1}{dt} = s_1 A_{11} \tag{51}$$

$$\frac{dx_2}{dt} = s_2 A_{22} \tag{52}$$

$$\frac{dx_3}{dt} = s_3 A_{33} \tag{53}$$

with $ds_j/dt \equiv 0, (\, j = 1,2,3)$ due to the assumption of the spatial invariance of the A_{ij}. The addition of the initial conditions, $\mathbf{x}(t_0)$ and $\mathbf{s}(t_0)$ at some reference time, t_0, fully specifies the initial value problem in this case. As shown in the introduction, and for a similar problem in the TI elliptical case, the group velocity vector is given as

$$\frac{d\mathbf{x}}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt} \right) = \left(A_{11}s_1, A_{22}s_2, A_{33}s_3 \right), \tag{54}$$

the magnitude of which is

$$\begin{aligned} V_e(\Theta,\Phi) &= \left| \frac{d\mathbf{x}}{dt} \right| = \left[\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt} \right]^{1/2} = \left[\left(\frac{dx_1}{dt} \right)^2 + \left(\frac{dx_2}{dt} \right)^2 + \left(\frac{dx_3}{dt} \right)^2 \right]^{1/2} \\ &= \left(A_{11}^2 s_1^2 + A_{22}^2 s_2^2 + A_{33}^2 s_3^2 \right) \end{aligned} \tag{55}$$

The angles Θ and Φ that describe the orientation of the group velocity vector with respect to the Cartesian system in 3D space are derived in Appendix B for this case. The unit vector along the group velocity vector is given by

$$\mathbf{N} = (\, N_1, N_2, N_3 \,) = (\, \sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta \,) \tag{56}$$

with Θ being the polar angle of the direction of energy (ray) propagation measured from the vertical or x_3 axis $\left(0 \le \Theta \le \pi\right)$, and Φ the azimuthal angle measure in a positive sense from the x_1 axis $\left(0 \le \Phi < 2\pi\right)$.

Introducing equations (B.14) – (B.16) into equation (55) results in the group slowness, $S_e(\Theta,\Phi)$, and velocity, $V_e(\Theta,\Phi)$ defined using

$$S_e^2(\Theta,\Phi) = \frac{1}{V_e^2(\Theta,\Phi)} = \frac{\sin^2\Theta\cos^2\Phi}{A_{11}} + \frac{\sin^2\Theta\sin^2\Phi}{A_{22}} + \frac{\cos^2\Theta}{A_{33}} \tag{57}$$

for the degenerate ellipsoidal orthorhombic problem, which in terms of the unit vector **N** has the form

$$S_e^2(\Theta,\Phi) = \frac{1}{V_e^2(\Theta,\Phi)} = \frac{N_1^2}{A_{11}} + \frac{N_2^2}{A_{22}} + \frac{N_3^2}{A_{33}}. \qquad (58)$$

Under the assumption that a method similar to that employed in the comparable problem in a TI medium is valid in this context, the perturbed group velocity, with the inclusion of the three symmetry plane anellipsoidal terms, $a_{ij} << 1$, can be written as

$$\begin{aligned} \frac{1}{V^2(\Theta,\Phi)} &= \frac{1}{V_e^2(\Theta,\Phi)[1+a_{13}+a_{12}+a_{23}]} \\ &\approx \frac{[1-a_{13}-a_{12}-a_{23}]}{V_e^2(\Theta,\Phi)} = \frac{1}{V_e^2(\Theta,\Phi)} - \frac{1}{V_e^2(\Theta,\Phi)}[a_{13}+a_{12}+a_{23}] \end{aligned} .(59)$$

Arguments similar to those used in the perturbed TI case may be employed in converting from phase to group slowness in the deviation terms, $a_{mn} = \hat{B}_{mn} s_m^2 s_n^2$, $(mn = 12,13,23)$. This approach is reasonable as there are no terms in the eikonal equation that contain expressions with a dependency on slowness vector components on other quartic terms involving odd powers of s_j, $(j = 1,2,3)$, or higher order terms such as "$s_1^2 s_2^2 s_3^2$".

It is shown in Appendix B that the phase slowness vector components s_j, $(j = 1,2,3)$ can be written completely in terms of group velocity parameters. An intermediate point in the approximation of the QL group velocity derivation, similar to equation (44) in the TI case, is

$$\begin{aligned} \frac{1}{V^2(\Theta,\Phi)} = \frac{N_1^2}{A_{11}} + \frac{N_2^2}{A_{22}} + \frac{N_3^2}{A_{33}} &- \frac{\hat{B}_{13}}{V_e^2}\left[\frac{V_e N_1}{A_{11}}\right]\left[\frac{N_1}{\sqrt{A_{11}}}\frac{N_3}{\sqrt{A_{33}}}\right]\left[\frac{V_e N_3}{A_{33}}\right] - \\ &\frac{\hat{B}_{12}}{V_e^2}\left[\frac{V_e N_1}{A_{11}}\right]\left[\frac{N_1}{\sqrt{A_{11}}}\frac{N_2}{\sqrt{A_{22}}}\right]\left[\frac{V_e N_2}{A_{22}}\right] - \\ &\frac{\hat{B}_{23}}{V_e^2}\left[\frac{V_e N_2}{A_{22}}\right]\left[\frac{N_2}{\sqrt{A_{22}}}\frac{N_3}{\sqrt{A_{33}}}\right]\left[\frac{V_e N_3}{A_{33}}\right]. \end{aligned} \qquad (60)$$

The result, after only a marginally more complicated derivation than for the TI case, using a similar approach, is the following group velocity approximation for a QL wave in an orthorhombic anisotropic medium

$$\frac{1}{V^2(\Theta,\Phi)} \approx \frac{N_1^2}{A_{11}} + \frac{N_2^2}{A_{22}} + \frac{N_3^2}{A_{33}} - \frac{B_{13}N_1^2N_3^2}{A_{11}A_{33}} - \frac{B_{12}N_1^2N_2^2}{A_{11}A_{22}} - \frac{B_{23}N_2^2N_3^2}{A_{22}A_{33}}, \qquad (61)$$

where the B_{ij} are defined in equations (24) – (26), and the components of **N** by equation (54). The use of $\hat{B}_{13}$, $\hat{B}_{12}$ and $\hat{B}_{23}$ in this problem at certain stages, to replace B_{13}, B_{12} and B_{23}, is for the same reason as in the TI case – to retain proper velocity dimensionality.

In Every and Song (2000) the B_{ij} are referred to as “longitudinal elastic constants”. In fact, the linearized expressions for both the QT_1 and QT_2 quasi-transverse waves contain these anellipsoidal terms. However, what is of importance is that Equation 44 is the same as the expression for the QL group slowness (velocity) for an orthorhombic medium presented in the above paper where the authors " were lead to it by plausibility arguments that are backed up by numerical results” rather than by “rigorous derivation". The derivation presented here is far from rigorous but does attempt to add some mathematical formalism to the derivation process.

CONCLUSIONS

Linearized approximations for the phase and group velocities of the quasi-longitudinal, QL, wave in an orthorhombic anisotropic medium are derived or presented. The equivalence of two forms of the linearized phase velocity for this media type was mathematically shown. The eikonal equation related to the second of these approximations, given in Song and Every (2000), was a most useful starting point for using the method of characteristics for the determination of the linearized QL group velocity. The method of characteristics was used to solve for the QL group velocities in the degenerate ellipsoidal QL cases in weak TI and orthorhombic media. Perturbation terms were introduced into both the TI and orthorhombic degenerate ellipsoidal QL group velocity expressions, using the perturbation terms from the linearized phase velocities, to construct solutions for anellipsoidal QL group velocities in both media types (the TI case being a subset of the orthorhombic case).

The solutions obtained in a fairly formal mathematical manner were identical to those obtained by Song and Every (2000) by intuitive methods. The agreement of the expressions derived here with those intuited by them is an indication of the reliability of their formulae. This coupled with the agreement of the approximate solutions with the exact solution, graphically shown in the above paper, should demonstrate that the QL group velocity approximation presented here and in that paper are acceptable within the constraints of weak anisotropy.

REFERENCES

Abramowitz, M. and Stegun, I., 1980, Handbook of Mathematical Functions: Dover Publications Inc., New York

Backus, G.E., 1965, Possible forms of seismic anisotropy of the uppermost mantle under oceans, Journal of Geophysical Research, **70**, 3429-3439.

Cerveny, V., 2001, Seismic Ray Theory, Cambridge Univ. Press, Cambridge.

Courant, R. and Hilbert, D., 1962, Methods of Mathematical Physics, Vol. II: Partial Differential Equations: Interscience Publications, New York.

Daley. P.F. and Hron, F., 1979, SH waves in layered transversely isotropic media – an asymptotic expansion approach, Bulletin of the Seismological Society of America, **69,** 689-711.

Every, A.G,1980, General closed-form expressions for acoustic waves in elastically anisotropic solids, Phys. Rev. B, **22** 1746-1760

Every, A.G. and Sachse, W., 1992, Sensitivity of inversion algorithms for recovering elastic constants of anisotropic solids from longitudinal wavespeed data, Ultrasonics, **30**, 43-48.

Gassmann, F., 1964. Introduction to seismic travel time methods in anisotropic media: Pure and Applied Geophysics, **58**, 63-112.

Jech, J. and Pšenčík, I., 1989, First order perturbation method for anisotropic media, Geophysical Journal International, **99**, 369-376.

Musgrave, M J P, 1970, Crystal Acoustics, Holden-Day, San Francisco.
Pšenčík, I. and Gajewski, D., 1998. Polarization, phase velocity and NMO velocity of qP waves in arbitrary weakly anisotropic media, Geophysics, **63**, 1754-1766.
Schoenberg, M. and Helbig, K., 1996. Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth: Geophysics, **62**, 1954-1974.
Song, L-P. and Every, A.G., 2000, Approximate formulae for acoustic wave group slowness in weakly orthorhombic media, Journal of Physics D: Applied Physics, **33**, L81-L85.
Song, L-P. Every, A. G. and Wright, C., 2001, Linearized approximations for phase velocities of elastic waves in weakly anisotropic media, Journal of Physics D: Applied Physics, **34**, 2052-2062.

Appendix A: Slowness Vector Components – Elliptical TI Medium

In terms of the group angle of propagation, Θ, the anisotropic parameters A_{11} and A_{33}, and the components of the slowness vectors, s_1 and s_3, the group velocity for QT ray propagation in an elliptically anisotropic medium may be written as (Equation 36)

$$V_e(\Theta) = \left[A_{11}^2 s_1^2 + A_{33}^2 s_3^2 \right]^{1/2}. \tag{A.1}$$

The slowness vector components, s_1 and s_3, are expressed as functions of the phase angle, θ, and phase velocity, $v(\theta)$, in the form

$$s_1 = \frac{\sin\theta}{v(\theta)} \tag{A.2}$$

and

$$s_3 = \frac{\cos\theta}{v(\theta)} \tag{A.3}$$

where

$$v_e(\theta) = \left[A_{11} \sin^2\theta + A_{22} \cos^2\theta \right]^{1/2}. \tag{A.4}$$

Using the relationship obtained from characteristic theory,

$$\tan\Theta = \left[\frac{dx_1/dt}{dx_3/dt} \right] = \left[\frac{dx_1}{dx_3} \right] = \frac{A_{11}}{A_{33}} \tan\theta = F\tan\theta, \tag{A.5}$$

the quantities $\sin\theta$ and $\cos\theta$ may be written in terms of A_{11} and A_{33} and the group angle Θ as

$$\sin\theta = \frac{\sin\Theta}{\left(F^2\cos^2\Theta + \sin^2\Theta\right)^{1/2}}, \tag{A.6}$$

and

$$\cos\theta = \frac{F\cos\Theta}{\left(F^2\cos^2\Theta + \sin^2\Theta\right)^{1/2}} . \tag{A.7}$$

Substituting these expressions for $\sin\theta$ and $\cos\theta$ into expressions for s_1 and s_3 yields, after some algebraic manipulations,

$$s_1 = \frac{V_e(\Theta)\sin\Theta}{A_{11}} \tag{A.8}$$

and

$$s_3 = \frac{V_e(\Theta)\cos\Theta}{A_{33}} \tag{A.9}$$

where $V_e(\Theta)$ is defined in terms of the group angle, Θ, and the anisotropic parameters A_{11} and A_{33} as

$$\frac{1}{V_e(\Theta)} = \left[\frac{\sin^2\Theta}{A_{11}} + \frac{\cos^2\Theta}{A_{33}}\right]^{1/2} . \tag{A.10}$$

Appendix B: Slowness Vector Components - Ellipsoidal Orthorhombic Medium

The azimuthal angle Φ in an ellipsoidal orthorhombic medium is obtained in the following manner from equation (15)

$$\tan\Phi = \frac{s_2 A_{22}}{s_1 A_{11}} = \frac{A_{22}}{A_{11}}\tan\phi = H\tan\phi \qquad \text{with} \quad H = \frac{A_{22}}{A_{11}} \tag{B.1}$$

such that

$$\sin\phi = \left[\frac{\sin^2\Phi}{H^2\cos^2\Phi + \sin^2\Phi}\right]^{1/2} \tag{B.2}$$

and

$$\cos\phi = \left[\frac{H^2\cos^2\Phi}{H^2\cos^2\Phi + \sin^2\Theta}\right]^{1/2} . \tag{B.3}$$

The radial component of the group velocity, which is the projection of the group velocity onto the (x_1, x_2) – plane, is given by

$$\frac{dr}{dt} = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2\right]^{1/2} = \left[\, s_1^2 A_{11}^2 + s_2^2 A_{22}^2 \,\right]^{1/2} . \tag{B.4}$$

The polar angle Θ may be expressed in terms of the radial and vertical components of the group velocity as

$$\tan\Theta = \frac{(dr/dt)}{(dz/dt)} = \tan\theta\left[\frac{A_{11}^2\cos^2\phi + A_{22}^2\sin^2\phi}{A_{33}^2}\right]^{1/2} \qquad \text{(B.5)}$$

such that

$$\tan\Theta = \frac{\tan\theta}{A_{33}}\left[\frac{\cos^2\Phi}{A_{11}^2} + \frac{\sin^2\Phi}{A_{22}^2}\right]^{-1/2} = K(\Phi)\tan\theta \qquad \text{(B.6)}$$

where

$$K(\Phi) = \frac{1}{A_{33}}\left[\frac{\cos^2\Phi}{A_{11}^2} + \frac{\sin^2\Phi}{A_{22}^2}\right]^{-1/2}. \qquad \text{(B.7)}$$

This results in

$$\sin\theta = \left[\frac{\sin^2\Theta}{K^2\cos^2\Theta + \sin^2\Theta}\right]^{1/2}, \qquad \text{(B.8)}$$

$$\cos\theta = \left[\frac{K^2\cos^2\Theta}{K^2\cos^2\Theta + \sin^2\Theta}\right]^{1/2}. \qquad \text{(B.9)}$$

The phase velocity of the ellipsoidal orthorhombic problem has the form

$$v_e^2(\theta,\phi) = A_{11}\sin^2\theta\cos^2\phi + A_{22}\sin^2\theta\sin^2\phi + A_{33}\cos^2\theta. \qquad \text{(B.10)}$$

In terms of the two group angles, the phase velocity may be expressed, using equations (B.2), (B.3), (B.8) and (B.9) as

$$\begin{aligned} v_e^2(\Theta,\Phi) = {} & A_{11}\frac{\sin^2\Theta}{\left(K^2\cos^2\Theta + \sin^2\Theta\right)}\frac{H^2\cos^2\Phi}{\left(H^2\cos^2\Phi + \sin^2\Phi\right)} + \\ & A_{22}\frac{\sin^2\Theta}{\left(K^2\cos^2\Theta + \sin^2\Theta\right)}\frac{\sin^2\Phi}{\left(H^2\cos^2\Phi + \sin^2\Phi\right)} + \\ & A_{33}\frac{\cos^2\Theta}{\left(K^2\cos^2\Theta + \sin^2\Theta\right)}\frac{\left(H^2\cos^2\Phi + \sin^2\Phi\right)}{\left(H^2\cos^2\Phi + \sin^2\Phi\right)} \quad . \end{aligned} \qquad \text{(B.11)}$$

Thus the first horizontal component of slowness, s_1, may be expressed in terms of group angles and velocities as

$$s_1^2 = \frac{\sin^2\Theta\cos^2\Phi}{A_{11}^2\left[\frac{\sin^2\Theta\cos^2\Phi}{A_{11}} + \frac{\sin^2\Theta\sin^2\Phi}{A_{22}} + \frac{\cos^2\Theta}{A_{33}}\right]} \qquad \text{(B.12)}$$

where the ellipsoidal group velocity is defined as

$$\frac{1}{V_e^2(\Theta,\Phi)} = \frac{\sin^2\Theta\cos^2\Phi}{A_{11}} + \frac{\sin^2\Theta\sin^2\Phi}{A_{22}} + \frac{\cos^2\Theta}{A_{33}} \tag{B.13}$$

so that the required form for s_1 is

$$s_1 = \frac{V_e(\Theta,\Phi)\sin\Theta\cos\Phi}{A_{11}} = \frac{V_e(\Theta,\Phi)N_1}{A_{11}} \tag{B.14}$$

In a similar manner, the expressions for the second horizontal component of slowness and vertical component of slowness are obtained as

$$s_2 = \frac{V_e(\Theta,\Phi)\sin\Theta\sin\Phi}{A_{22}} = \frac{V_e(\Theta,\Phi)N_2}{A_{22}} \tag{B.15}$$

$$s_3 = \frac{V_e(\Theta,\Phi)\cos\Theta}{A_{33}} = \frac{V_e(\Theta,\Phi)N_3}{A_{33}}. \tag{B.16}$$