Alternative linearized expressions for qP, qS_1 and qS_2 phase velocities in a weakly anisotropic orthorhombic medium

P. F. Daley and E. S. Krebes

ABSTRACT

Alternative linearized approximations of the phase velocities for the quasi–compressional, qP, and two quasi–shear wave types, qS_1 and qS_2, in an orthorhombic medium are presented. Algebraic manipulation of the formulae obtained from the standard linearization technique is done so that the phase velocities are in the form consisting of the most degenerate cases of phase velocities in an orthorhombic medium (ellipsoids) plus anellipsoidal correction (perturbation) terms to compensate for the deviation from the degenerate orthorhombic case. The quantities in the formulae for the phase velocities all have physical interpretations, that is, they can all be associated with some physically realizable and measurable quantity. After obtaining these intermediate linearized expressions for the qP, qS_1 and qS_2 phase velocities, further approximations are made to obtain the equivalent of what are termed fully linearized formulae. This includes the introduction of an isotropic background velocity, α, for the qP and β in the qS_1 and qS_2 equations. A comparison of the intermediate linearized approximations for the phase velocities with the exact formulae are presented in a series of figures.

INTRODUCTION

The specific problem of obtaining linearized approximations of the phase velocities for the quasi-compressional, qP and the two quasi-shear wave modes, qS_1 and qS_2, in an orthorhombic anisotropic medium is addressed. The linearization formulae for these phase velocities in a general anisotropic medium may be found in the works of Backus (1965), Every (1980), Jech and Pšenčík (1989), Mensch and Rasolofosaon, (1997), and Pšenčík, and Gajewski (1998), among others. Once linearized approximations are obtained, a rearrangement of terms is done to put the anisotropic coefficients from the original formulae for the qP and qS_1 and qS_2 phase velocities into alternate configurations. This is done so that each of the terms, or individual collection of terms in the expressions for the phase velocities, has a physical meaning or can be associated with some geometrical formalism. What results are approximations that are less "weak anisotropic" than they are "weak anellipsoidal" approximations.

Exact expressions for the qP, qS_1 and qS_2 phase velocities in an orthorhombic medium are presented in Every (1980), and Schoenberg and Helbig (1997) as the solutions of a cubic equation. An approximation of the above exact equation by Tsvankin (1997) produced a linearized formula for the qP phase velocity in a weakly anisotropic medium which was similar to that obtained by Mensch and Rasolofosaon, (1997) who also treated the qS phase velocities. Sayers (1994) developed an approximate expression for the qP phase velocity using an expansion in spherical harmonics. In the work of Pšenčík, and Gajewski (1998) the case of a general weak anisotropic medium is considered from a linearization perspective based on perturbation methods and arrived at the same results as those presented in Backus (1965). Apart from phase velocities, quantities such as polarization vectors are discussed for qP waves in general, and a number of specific anisotropic media types, including orthorhombic and two transversely isotropic orientation categories are discussed in the paper of Pšenčík and Gajewski (1998).

The difference in the formulae for the qP, qS_1 and qS_2 phase velocities in an orthorhombic medium derived here, when compared to the results obtained from the linearization process, are that they all consist of two parts: an ellipsoidal expression and anellipsoidal correction (perturbation) terms. Each term, or specific collection of terms, within these phase velocity formulae can be associated with some physical or geometrical quantity that is measurable from, for example, travel time data. This modification required that some moderately complicated algebraic manipulations of the linearized expressions be undertaken.

It has been shown by a number of authors (for example; Song and Every, 2000, Pšenčík et al., 2000) that of the 9 anisotropic parameters defining an orthorhombic medium only 3 of these, A_{11}, A_{22} and A_{33}, can be obtained from the inversion of qP phase velocity measurements, together with 3 other quantities related to the deviation of the qP wave surface from the ellipsoidal. To fully specify individual anisotropic parameters within this analysis, the inversion of shear wave data producing the terms A_{44}, A_{55} and A_{66} is required, together with the abovementioned anellipsoidal deviation terms from the qP inversion process, to obtain the anisotropic parameters A_{12}, A_{13} and A_{23}. This facilitates undertakings such as determining methods to pursue for the inversion of phase velocity data to obtain the remaining anisotropic parameters that define the medium. Once these rearranged formulations are obtained, other matters are addressed. This includes ensuring that associated individual quantities that are to be determined in the inversion process are all of the same approximate magnitude.

LINEARIZED qP PHASE VELOCITY

The linearized expression for the qP phase velocity in a general anisotropic medium is given by the apparently simple formula presented in the work of Backus (1965), who showed that the linearization process was a direct consequence of first order perturbation theory, as

$$v_{qP}^2(\mathbf{n}) = a_{ijk\ell} n_i n_j n_k n_\ell \quad (i, j, k, \ell = 1,2,3) \tag{1}$$

where summation over repeated indices is assumed. Others who have pursued a similar method of development for phase velocity determination as well as ancillary quantities, such as polarization vectors, are Červený and Jech (1982), Jech and Pšenčík (1989), and Pšenčík and Gajewski (1989). These related extensions will not be considered here.

The quantities n_i in equation (1) are the components of the unit phase normal vector, which is defined as

$$\mathbf{n} = (n_1, n_2, n_3) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \tag{2}$$

where θ is the polar angle measured from the positive x_3 axis $(0 \le \theta \le \pi)$ and ϕ is the azimuthal angle measured in a positive sense from the x_1 axis $(0 \le \phi \le 2\pi)$.

The density (ρ) normalized anisotropic parameters, $a_{ijk\ell} = c_{ijk\ell}/\rho$, which have the dimensions of velocity squared may be transformed to Voigt notation, A_{mn}, using the scheme $a_{ij|k\ell} \to A_{m|n}$ with the following substitutions

$$11 \to 1 \quad 22 \to 2 \quad 33 \to 3 \quad 31 = 13 \to 5 \quad 21 = 12 \to 6 \quad 32 = 23 \to 4 \tag{3}$$

The linearized expression, from equation (1), for the qP phase velocity in an orthorhombic medium, and introducing Voigt notation, is

$$\begin{aligned} v_{qP}^2(\mathbf{n}) = {} & A_{11}n_1^4 + A_{22}n_2^4 + A_{33}n_3^4 + \\ & 2(A_{12} + 2A_{66})n_1^2 n_2^2 + 2(A_{13} + 2A_{55})n_1^2 n_3^2 + 2(A_{23} + 2A_{44})n_2^2 n_3^2 \end{aligned} . \tag{4}$$

Further treatment of the qP phase velocity will be given in some detail. The same detail will not be afforded to the quasi-shear phase velocities as the methods are similar.

The procedure for obtaining am alternate expression of the linearized qP phase velocity, as an ellipsoid with 3 anellipsoidal correction terms, is initiated by adding and subtracting the expression

$$(A_{11} + A_{22})n_1^2 n_2^2 + (A_{11} + A_{33})n_1^2 n_3^2 + (A_{22} + A_{33})n_2^2 n_3^2 \tag{5}$$

from equation (4). This results in the formula

$$\begin{aligned} v_{qP}^2(\mathbf{n}) = {} & A_{11}n_1^4 + A_{22}n_2^4 + A_{33}n_3^4 + \\ & \left[(A_{11} + A_{22})n_1^2 n_2^2 + (A_{11} + A_{33})n_1^2 n_3^2 + (A_{22} + A_{33})n_2^2 n_3^2\right] \\ & 2\left[(A_{12} + 2A_{66}) - (A_{11} + A_{22})/2\right]n_1^2 n_2^2 + 2\left[(A_{13} + 2A_{55}) - (A_{11} + A_{33})/2\right]n_1^2 n_3^2 + \\ & 2\left[(A_{23} + 2A_{44}) - (A_{22} + A_{33})/2\right]n_2^2 n_3^2 \quad . \end{aligned} \tag{6}$$

Simplifying the above equation using the properties of the vector $\mathbf{n}$, given by equation (2), yields

$$\begin{aligned} v_{qP}^2(\mathbf{n}) = {} & A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 + \\ & 2\left[(A_{12} + 2A_{66}) - (A_{11} + A_{22})/2\right]n_1^2 n_2^2 + \\ & 2\left[(A_{13} + 2A_{55}) - (A_{11} + A_{33})/2\right]n_1^2 n_3^2 + \\ & 2\left[(A_{23} + 2A_{44}) - (A_{22} + A_{33})/2\right]n_2^2 n_3^2 \quad . \end{aligned} \tag{7}$$

In more compact notation equation (7) may be written as

$$v_{qP}^2(\mathbf{n}) = A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 + 2B_{12}n_1^2n_2^2 + 2B_{13}n_1^2n_3^2 + 2B_{23}n_2^2n_3^2 \tag{8}$$

with the anellipsoidal deviation terms given by

$$B_{12} = (A_{12} + 2A_{66}) - (A_{11} + A_{22})/2 \tag{9}$$

$$B_{13} = (A_{13} + 2A_{55}) - (A_{11} + A_{33})/2 \tag{10}$$

$$B_{23} = (A_{23} + 2A_{44}) - (A_{22} + A_{33})/2 . \tag{11}$$

An expression identical to equation (8) was obtained by Song and Every (2000), however, no mathematical formalism was employed. They took the expression for the qP phase velocity in the (x_1, x_3) plane of a transversely isotropic medium, generalized it " by index permutation ", and numerically tested their results against the exact expression for the qP phase velocity.

The above equation for the qP phase velocity is indicative of the shape of the slowness surface, which is the inverse of the phase velocity surface (Musgrave, 1970), of the qP in an orthorhombic medium. All terms in $A_{ii}\,(i=1,2,3)$ in Equation 8 have only a second order dependence on n_i, while other linearized forms that appear in the literature have a fourth order dependence. This suggests, consistent with the approximation used, that the related slowness surface is an ellipsoid with the lengths of the half axes being $(A_{ii})^{-1/2}$ $(i=1,2,3)$. The inclusion of the anellipsoidal deviation terms, B_{ij}, $(ij = 12, 13, 23)$, associated with each of the 3 symmetry planes, completes the approximation. This is in agreement with the result that one could expect for anisotropic media with three orthogonal symmetry planes.

The slowness surface associated with equation (8) can only be considered dimensionally correct in velocity if the deviation terms are omitted or set equal to zero. This would yield the degenerate ellipsoidal qP slowness surface, which for weak anisotropy produces a reasonable trend of the actual slowness surface. This "order two only" dependence of the A_{ii} terms will be shown in subsequent sections to also apply to the two quasi-shear modes $(i = 4,5,6)$.

If mathematically and physically justifiable approximations are made to the exact expression for the qP phase velocity, $B_{ij} = \delta_{ij} - \varepsilon_{ij}$ and subsequently δ_{ij}, defined below, differ from those which appeared in earlier literature on a related topic (for example, Tsvankin, 1997). This discrepancy is due to the linearization method used. In the above, for example, the modified (dimensionless) value

$$\hat{B}_{13} \approx \frac{\left(A_{13} + 2A_{55}\right) - \left(A_{11} + A_{33}\right)/2}{A_{33}} \tag{12}$$

may be compared with the exact dimensionless expression (Gassmann, 1964)

$$\tilde{B}_{13} = \frac{\left(A_{13} + A_{55}\right)^2 - \left(A_{11} - A_{55}\right)\left(A_{33} - A_{55}\right)}{2A_{33}\left(A_{33} - A_{55}\right)}. \tag{13}$$

If this quantity becomes zero, the qP phase velocity surface in the (x_1, x_3) plane degenerates to an ellipse. As the $qP - qS_1$ wave motions are coupled in this plane, the qS_1 phase velocity surface is forced to be a circle in this plane. It should be restated that equation (12) is the linearized form of the exact deviation term in the (x_1, x_3) plane, given by equation (13).

The related linearized quantity $\hat{\delta}_{13}$ has the simplified form

$$\hat{\delta}_{13} \approx \frac{\left(A_{13} + 2A_{55} - A_{33}\right)}{A_{33}} \tag{14}$$

when compared with the exact expression introduced by Thomsen (1986)

$$\delta_{13} = \frac{\left(A_{13} + A_{55}\right)^2 - \left(A_{33} - A_{55}\right)^2}{2A_{33}\left(A_{33} - A_{55}\right)} \tag{15}$$

Defining the parameter $\varepsilon_{13} = \frac{\left(A_{11} - A_{33}\right)}{2A_{33}}$, a measure of ellipticity in the (x_1, x_3) plane, and subtracting this quantity from both Equations 14 and 15 yields

$$\hat{B}_{13} = \hat{\delta}_{13} - \varepsilon_{13} = \frac{\left(A_{13} + 2A_{55} - A_{33}\right)}{A_{33}} - \frac{\left(A_{11} - A_{33}\right)}{2A_{33}} = \frac{\left[\left(A_{13} + 2A_{55}\right) - \left(A_{11} + A_{33}\right)/2\right]}{A_{33}} \tag{16}$$

and

$$\begin{aligned} \tilde{B}_{13} = \delta_{13} - \varepsilon_{13} &= \frac{\left(A_{13} + A_{55}\right)^2 - \left(A_{33} - A_{55}\right)^2}{2A_{33}\left(A_{33} - A_{55}\right)} - \frac{\left(A_{11} - A_{33}\right)}{2A_{33}} \\ &= \frac{\left(A_{13} + A_{55}\right)^2 - \left(A_{11} - A_{55}\right)\left(A_{33} - A_{55}\right)}{2A_{33}\left(A_{33} - A_{55}\right)} \end{aligned} \tag{17}$$

Similar expressions can be derived for both $\hat{B}_{12}$ and $\hat{B}_{23}$, and $\tilde{B}_{12}$ and $\tilde{B}_{23}$. However, it is more useful in possible related applications to retain the related forms B_{13}, B_{12} and B_{23}, equations (9) – (11), which are not normalized to any arbitrary quantity.

Returning to equation (8), it should be recalled that the modified quantities B_{ij} are a measure of the deviation of the qP phase velocity from the ellipsoidal. For a linearized weak anisotropy medium, these values should be quite small relative to the A_{ii} terms. It appears to be common practice (Pšenčík and Gajewski, 1998) to require that the terms in A_{ii} $(i=1,2,3)$ are of about the same order of magnitude as the terms involving B_{ij}. An isotropic background velocity, α, is introduced, essentially putting the A_{ii} terms into their perturbed form, $A_{ii} \approx A_{ii}^0 + \Delta A_{ii}$, or equivalently, $A_{ii} \approx \alpha^2 + \Delta A_{ii}$. This is accomplished by adding and subtracting the quantity

$$\alpha^2\left(n_1^2+n_2^2+n_3^2\right) \tag{18}$$

to equation (8), yielding

$$\begin{aligned} v_{qP}^2(\mathbf{n}) = \alpha^2 + \left(A_{11}-\alpha^2\right)n_1^2 + \left(A_{22}-\alpha^2\right)n_2^2 + \left(A_{11}-\alpha^2\right)n_3^2 + \\ 2B_{12}n_1^2n_2^2 + 2B_{13}n_1^2n_3^2 + 2B_{23}n_2^2n_3^2 \end{aligned} \tag{19}$$

or equivalently

$$\begin{aligned} v_{qP}^2(\mathbf{n}) = \alpha^2\Bigg[1 + \frac{\left(A_{11}-\alpha^2\right)n_1^2}{\alpha^2} + \frac{\left(A_{22}-\alpha^2\right)n_2^2}{\alpha^2} + \frac{\left(A_{33}-\alpha^2\right)n_3^2}{\alpha^2} + \\ \frac{2B_{12}n_1^2n_2^2}{\alpha^2} + \frac{2B_{13}n_1^2n_3^2}{\alpha^2} + \frac{2B_{23}n_2^2n_3^2}{\alpha^2}\Bigg] \end{aligned} \tag{20.a}$$

Taking the square root of both sides of equation (20.a) and expanding the *RHS* in a binomial series, retaining only the leading terms results in

$$\begin{aligned} v_{qP}(\mathbf{n}) \approx \alpha\Bigg[1 + \frac{\left(A_{11}-\alpha^2\right)n_1^2}{2\alpha^2} + \frac{\left(A_{22}-\alpha^2\right)n_2^2}{2\alpha^2} + \frac{\left(A_{33}-\alpha^2\right)n_3^2}{2\alpha^2} + \\ \frac{B_{12}n_1^2n_2^2}{\alpha^2} + \frac{B_{13}n_1^2n_3^2}{\alpha^2} + \frac{B_{23}n_2^2n_3^2}{\alpha^2}\Bigg] \end{aligned} \tag{20.b}$$

It is often the case that one of the $\sqrt{A_{ii}}$, $(i=1,2,3)$ is assumed known and consequently the reference velocity, α, is set equal to $\sqrt{A_{ii}}$ $(i=1,2 \text{ or } 3)$, resulting in the loss of one term in equation (20.b).

These last steps in the linearization of the qP phase velocity in an orthorhombic medium, going from equation (8) to (20.b), have been included to maintain a consistency with what appears in the literature. Additionally, the derivation of equation (20b) was thought to be of importance to show the difference that the introduction of equation (5) makes to the final formula; a perturbed ellipsoid, rather than the quartic equation (4) in which the assigning of a specific physical significance to the coefficients is difficult. It is the linearized approximation given by equation (8) that will be used in a latter section to compare with the exact qP phase velocity in an orthorhombic anisotropic medium.

LINEARIZED QS$_1$ AND QS$_2$ PHASE VELOCITIES

The methodology described in Every and Sachse (1992), which will not be repeated here in any detail, involves the application of orthonormal rotation transformation matrices to the Cristoffel matrix $\mathbf{\Gamma}$ to isolate either $\tilde{\Gamma}_{22}$ or $\tilde{\Gamma}_{33}$. This transformation is given by

$$\tilde{\Gamma}_{rs} = \Gamma_{pq} a_{rp} a_{sq} \tag{21}$$

and has been shown to be equivalent to the multiplication of the matrix $\mathbf{\Gamma}$ by one of two vectors, $\mathbf{e_2}$ or $\mathbf{e_3}$. These vectors are required to be orthonormal to the qP phase velocity vector, $\mathbf{n}$, which is defined by

$$\mathbf{n} = (n_1, n_2, n_3) = \mathbf{e_1} = (e_{11}, e_{12}, e_{13}) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{22}$$

The vectors $\mathbf{e_2}$ and $\mathbf{e_3}$ may be written in terms of the angles θ and ϕ, and in terms of the components of the vector n as

$$\begin{aligned} \mathbf{e_2} &= (e_{21}, e_{22}, e_{23}) = (-\cos\theta\cos\phi, -\cos\theta\sin\phi, \sin\theta) \\ &= \frac{(-n_1 n_3, -n_2 n_3, 1 - n_3^2)}{\sqrt{1 - n_3^2}} \end{aligned} \tag{23}$$

$$\begin{aligned} \mathbf{e_3} &= (e_{31}, e_{32}, e_{33}) = (\sin\phi, -\cos\phi, 0) \\ &= \frac{(n_2, -n_1, 0)}{\sqrt{1 - n_3^2}} \end{aligned} \tag{24}$$

The choice of the vectors $\mathbf{e_2}$ and $\mathbf{e_3}$ is not arbitrary as certain conditions on the orthonormal vector triad

$$(\mathbf{n}, \mathbf{e_2}, \mathbf{e_3}) = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}) \tag{25}$$

must be satisfied (Jech and Pšenčík, 1989). Some aspects of this method were dealt with by Vassiliou (1994).

The definition of the $\mathbf{e}_i$ $(i = 1, 2, 3)$ is given in Figure 1. This schematic indicates that the qP and qS_1 phase velocity vectors are constrained to lie within the plane which is perpendicular to the (x_1, x_2) plane and passing through the x_3 axis. The result is that the qS_2 phase velocity vector is always normal to the plane containing the qP and qS_1 phase velocity vectors.

qS_1 phase velocity in an orthorhombic medium

For the leading shear wave, qS_1,

$$a_{rp} = e_{2p} \tag{26}$$

$$a_{sq} = e_{2q} \tag{27}$$

The leading shear wave phase velocity is dependent on the second eigenvalue of the transformed Cristoffel matrix that has been rotated into the elements $\tilde{\Gamma}_{rs}$. Using the relations between the rotation matrices and the vector $\mathbf{e_2}$ results in

$$\tilde{\Gamma}_{22} = \sum_{p,q} \Gamma_{pq} e_{2p} e_{2q} . \tag{28}$$

Retaining only the first order term, the linearized leading shear wave phase velocity in terms of the density normalized A_{ij} that have the dimensions of velocity squared is obtained from

$$v_{qS_1}^2 = \tilde{\Gamma}_{22} \tag{29}$$

Introducing the vector $\mathbf{e_2}$ results in equation (29) having the form

$$\begin{aligned} v_{qS_1}^2 &= \tilde{\Gamma}_{22} = \sum_{p,q} \Gamma_{pq} e_{2p} e_{2q} \\ &= \Gamma_{11} e_{21} e_{21} + \Gamma_{22} e_{22} e_{22} + \Gamma_{33} e_{23} e_{23} + 2\Gamma_{12} e_{23} e_{22} + 2\Gamma_{13} e_{21} e_{23} + 2\Gamma_{23} e_{22} e_{23} \end{aligned} \tag{30}$$

The non-zero elements Γ_{pq} of the matrix $\mathbf{\Gamma}$ for an orthorhombic medium may be found in numerous works, for example, Musgrave (1970):

$$\Gamma_{11} = A_{11} n_1^2 + A_{66} n_2^2 + A_{55} n_3^2 \tag{31}$$

$$\Gamma_{22} = A_{66} n_1^2 + A_{22} n_2^2 + A_{44} n_3^2 \tag{32}$$

$$\Gamma_{33} = A_{55} n_1^2 + A_{44} n_2^2 + A_{33} n_3^2 \tag{33}$$

$$\Gamma_{23} = \Gamma_{32} = \left(A_{23} + A_{44} \right) n_2 n_3 \tag{34}$$

$$\Gamma_{13} = \Gamma_{31} = \left(A_{13} + A_{55} \right) n_1 n_3 \tag{35}$$

$$\Gamma_{12} = \Gamma_{21} = \left(A_{12} + A_{66} \right) n_1 n_2 \tag{36}$$

Thus the linearized leading shear wave phase velocity may be written as

$$v_{qS_1}^2 = \left[A_{11} n_1^2 + A_{66} n_2^2 + A_{55} n_3^2 \right] e_{21} e_{21} + \left[A_{66} n_1^2 + A_{22} n_2^2 + A_{44} n_3^2 \right] e_{22} e_{22} + \left[A_{55} n_1^2 + A_{44} n_2^2 + A_{33} n_3^2 \right] e_{23} e_{23} + 2\left(A_{12} + A_{66} \right) n_1 n_2 e_{21} e_{22} + 2\left(A_{13} + A_{55} \right) n_1 n_3 e_{21} e_{23} + 2\left(A_{23} + A_{44} \right) n_2 n_3 e_{22} e_{23} \tag{37}$$

Adding and subtracting the term

$$\left[\left(A_{11} + A_{33} \right) n_1 n_3 \right] e_{21} e_{23} + \left[\left(A_{22} + A_{33} \right) n_2 n_3 \right] e_{22} e_{23} + \left[\left(A_{11} + A_{22} \right) n_1 n_2 \right] e_{21} e_{22} - 2 A_{66} n_1 n_2 e_{21} e_{22} - 2 A_{55} n_1 n_3 e_{21} e_{23} - 2 A_{44} n_2 n_3 e_{22} e_{23} \tag{38}$$

from equation (37, the following equation results

$$\begin{aligned} v_{qS_1}^2 = {} & A_{44} \sin^2\phi + A_{55} \cos^2\phi + \\ & 2\left[\left(A_{12} + 2A_{66} \right) - \left(A_{11} + A_{22} \right)/2 \right] \sin^2\theta \cos^2\theta \cos^2\phi \sin^2\phi - \\ & 2\left[\left(A_{13} + 2A_{55} \right) - \left(A_{11} + A_{33} \right)/2 \right] \sin^2\theta \cos^2\theta \cos^2\phi - \\ & 2\left[\left(A_{23} + 2A_{44} \right) - \left(A_{22} + A_{33} \right)/2 \right] \sin^2\theta \cos^2\theta \sin^2\phi \end{aligned} \tag{39.a}$$

or in a more compact form as

$$\begin{aligned} v_{qS_1}^2 = {} & A_{44} \sin^2\phi + A_{55} \cos^2\phi + 2 B_{12} \sin^2\theta \cos^2\theta \cos^2\phi \sin^2\phi - \\ & 2 B_{13} \sin^2\theta \cos^2\theta \cos^2\phi - 2 B_{23} \sin^2\theta \cos^2\theta \sin^2\phi \end{aligned} \tag{39.b}$$

where the B_{ij} $\left(ij = 12, 13, 23 \right)$ were defined in the previous section.

If $\phi = 0$ then

$$v_{qS_1}^2 = A_{55} - 2\left[\left(A_{13} + 2A_{55} \right) - \left(A_{11} + A_{33} \right)/2 \right] \sin^2\theta \cos^2\theta \tag{40.a}$$

$$v_{qS_1}^2 = A_{55} - 2B_{13} \sin^2\theta \cos^2\theta \tag{40.b}$$

and for $\phi = \pi/2$

$$v_{qS_1}^2 = A_{44} - 2\left[\left(A_{23} + 2A_{44} \right) - \left(A_{22} + A_{33} \right)/2 \right] \sin^2\theta \cos^2\theta \tag{41.a}$$

$$v_{qS_1}^2 = A_{44} - 2B_{23} \sin^2\theta \cos^2\theta \tag{41.b}$$

In addition, for both $\theta = 0$ and $\theta = \pi/2$,

$$v_{qS_1}^2 = A_{44} \sin^2\phi + A_{55} \cos^2\phi \tag{42}$$

It is clear from equations (40) – (42) that this mode of quasi–shear propagation is similar to the qS_V mode in a transversely isotropic medium, at least in the $\left(x_1, x_3 \right)$ and $\left(x_2, x_3 \right)$ symmetry planes.

As in the qP case, it is at times convenient to have the coefficients of all the terms in the equation for the qS_1 phase velocity of a similar magnitude. To accomplish this, add and subtract from equation (39.b) the term

$$\beta^2\left(\cos^2\phi+\sin^2\phi\right) \tag{43}$$

to yield

$$\begin{aligned} v_{qS_1}^2 = \beta^2 \Bigg[& 1+\frac{\left(A_{44}-\beta^2\right)}{\beta^2}\sin^2\phi+\frac{\left(A_{55}-\beta^2\right)}{\beta^2}\cos^2\phi+ \\ & \frac{2B_{12}\sin^2\theta\cos^2\theta\cos^2\phi\sin^2\phi}{\beta^2}-\frac{2B_{13}\sin^2\theta\cos^2\theta\cos^2\phi}{\beta^2}- \\ & \frac{2B_{23}\sin^2\theta\cos^2\theta\sin^2\phi}{\beta^2}\Bigg] \end{aligned} \tag{44}$$

The linearization process may be completed by taking the square root of both sides of equation (44) and expanding the *RHS* in a binomial series, retaining only the leading terms, to obtain

$$\begin{aligned} v_{qS_1} \approx \beta \Bigg[& 1+\frac{\left(A_{44}-\beta^2\right)}{2\beta^2}\sin^2\phi+\frac{\left(A_{55}-\beta^2\right)}{2\beta^2}\cos^2\phi+ \\ & \frac{B_{12}\sin^2\theta\cos^2\theta\cos^2\phi\sin^2\phi}{\beta^2}-\frac{B_{13}\sin^2\theta\cos^2\theta\cos^2\phi}{\beta^2}- \\ & \frac{B_{23}\sin^2\theta\cos^2\theta\sin^2\phi}{\beta^2}\Bigg] \end{aligned} \tag{45}$$

It is evident from the above derivation that the linearized qP phase velocity may be obtained in a similar manner. This would involve isolating $\tilde{\Gamma}_{11}$ using rotation matrices associated with the vector $\mathbf{n}=\mathbf{e_1}$, resulting in $v_{qP}^2=\tilde{\Gamma}_{11}$.

qS_2 phase velocity in an orthorhombic medium

From the introduction in the previous section on the qS_1 phase velocity it may be determined (Every and Sachse, 1992) that the expression for the second shear wave phase velocity, v_{qS_2}, in an orthorhombic medium is related to the matrix element $\tilde{\Gamma}_{33}$ in the same manner as v_{qS_1} is related to $\tilde{\Gamma}_{22}$. From equations (21), (26) and (27) in the previous section the following relation, in terms of the vector $\mathbf{e_3}$, is obtained

$$\tilde{\Gamma}_{33} = \sum_{p,q} \Gamma_{pq} e_{3p} e_{3q} \tag{46}$$

Thus the linearized expression for the second quasi-shear phase velocity may be written as

$$\begin{aligned} v_{qS_2}^2 &= \tilde{\Gamma}_{33} = \sum_{p,q} \Gamma_{pq} e_{3p} e_{3q} \\ &= \Gamma_{11} e_{31} e_{31} + \Gamma_{22} e_{32} e_{32} + 2\Gamma_{12} e_{31} e_{32} \end{aligned} . \tag{47}$$

Introducing the expressions for the Γ_{pq} defined in equations (31) – (36) yields

$$\begin{aligned} v_{qS_2}^2 = &\left[A_{11} n_1^2 + A_{66} n_2^2 + A_{55} n_3^2 \right] e_{31} e_{31} + \left[A_{66} n_1^2 + A_{22} n_2^2 + A_{44} n_3^2 \right] e_{32} e_{32} + \\ &2\left(A_{12} + A_{66} \right) n_1 n_2 e_{31} e_{32} \end{aligned} \tag{48}$$

In this case, adding and subtracting the terms

$$\left(A_{11} + A_{22} \right) n_1 n_2 e_{31} e_{32} - 2 A_{66} n_1 n_2 e_{31} e_{32} \tag{49}$$

from equation (48) results in the phase velocity for the second quasi – shear wave being given as

$$\begin{aligned} v_{qS_2}^2 = &A_{44} \cos^2\theta \cos^2\phi + A_{55} \cos^2\theta \sin^2\phi + A_{66} \sin^2\theta - \\ &2\left[\left(A_{12} + 2A_{66} \right) - \left(A_{11} + A_{22} \right)/2 \right] \sin^2\theta \sin^2\phi \cos^2\phi \end{aligned} \tag{50.a}$$

or equivalently, with B_{12} defined by equation (9),

$$\begin{aligned} v_{qS_2}^2 = &A_{44} \cos^2\theta \cos^2\phi + A_{55} \cos^2\theta \sin^2\phi + A_{66} \sin^2\theta - \\ &2B_{12} \sin^2\theta \sin^2\phi \cos^2\phi \end{aligned} \tag{50.b}$$

If $\phi = 0$, then equation (50.b) reduces to

$$v_{qS_2}^2 = A_{44} \cos^2\theta + A_{66} \sin^2\theta , \tag{51}$$

and, if $\phi = \pi/2$,

$$v_{qS_2}^2 = A_{55} \cos^2\theta + A_{66} \sin^2\theta \tag{52}$$

while for $\theta = 0$

$$v_{qS_2}^2 = A_{44} \cos^2\phi + A_{55} \sin^2\phi , \tag{53}$$

and for $\theta = \pi/2$

$$v_{qS_2}^2 = A_{66}. \tag{54}$$

The preceding four equations indicate that this quasi – shear mode of wave propagation behaves like the qS_H mode in a transversely isotropic medium, again, in the (x_1, x_3) and (x_2, x_3) symmetry planes. The behaviour in the (x_1, x_2) plane cannot be placed in any category.

In a manner similar to the previous two sections, the equalization of magnitudes of the coefficients of equation (50.b) is once again achieved by adding and subtracting the term

$$\beta^2\left[\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta\right] \tag{55}$$

from equation (50.b) to obtain

$$\begin{aligned} v_{qS_2}^2 = \beta^2 \Bigg[1 &+ \frac{\left(A_{44} - \beta^2\right)}{\beta^2}\cos^2\theta\cos^2\phi + \frac{\left(A_{55} - \beta^2\right)}{\beta^2}\cos^2\theta\sin^2\phi + \\ &\frac{\left(A_{66} - \beta^2\right)}{\beta^2}\sin^2\theta - \frac{2B_{12}\sin^2\theta\sin^2\phi\cos^2\phi}{\beta^2}\Bigg] \end{aligned} \tag{56}$$

The linearization process requires taking the square root of equation (56) and expanding the *RHS* of the resulting equation in a binomial series, again retaining only the leading terms, to finally obtain

$$\begin{aligned} v_{qS_2} \approx \beta \Bigg[1 &+ \frac{\left(A_{44} - \beta^2\right)}{2\beta^2}\cos^2\theta\cos^2\phi + \frac{\left(A_{55} - \beta^2\right)}{2\beta^2}\cos^2\theta\sin^2\phi + \\ &\frac{\left(A_{66} - \beta^2\right)}{2\beta^2}\sin^2\theta - \frac{B_{12}\sin^2\theta\sin^2\phi\cos^2\phi}{\beta^2}\Bigg] \end{aligned} \tag{57}$$

It has been convenient in the last shear wave cases to retain the (θ, ϕ) notation rather than the (e_{2i}, e_{3j}) notation as it provides more clarity in identifying the wave's behaviour.

As approximate (linearized) expressions have now been obtained for the three modes of elastic waves propagating in an orthorhombic anisotropic medium, a check on their accuracy will be performed using the exact expressions given by Every (1980) or Schoenberg and Helbig (1997). It should be noted that in deriving the exact expressions by solving a cubic equation with 3 real roots, the two shear modes for which exact expressions are given in the works cited above, are correctly named the "fast" and "slow" shear wave modes. There is not necessarily a one to one correspondence to the qS_1 and qS_2 modes discussed above. This is due to the fact that the qS_1 and qS_2 shear wavefront surfaces may intersect one another, while the "fast" shear mode in the exact formulation is always that, the fastest of the two shear modes, and the "slow" shear mode is similar. Kiss singularities occur when these two wave surfaces are about to intersect, resulting in some of the related formulae describing the two wave types becoming undefined at these points.

NUMERICAL RESULTS

The model used is a fairly significant modification of the example given in Pšenčík and Gajewski (1998), with the 21 parameter model downgraded to an orthorhombic input data set. It is defined in terms of the density normalized $(\text{velocity})^2$ parameters, A_{ij}, with dimensions of km^2/s^2 as

$$\mathbf{A}_{Model} = \begin{bmatrix} 19.3 & 0.9 & 1.3 & 0 & 0 & 0 \\ & 17.4 & 0.2 & 0 & 0 & 0 \\ & & 14.1 & 0 & 0 & 0 \\ & & & 5.1 & 0 & 0 \\ & & & & 5.5 & 0 \\ & & & & & 4.6 \end{bmatrix} \tag{58}$$

To put the B_{ij} $(ij = 12,13,23)$ in perspective with the A_{ij}, the following numerical values obtained from equations (9) – (11) follow

$$B_{12} = (A_{12} + 2A_{66}) - (A_{11} + A_{22})/2 = -8.25\,km^2/s^2 \quad (-0.585) \tag{9'}$$

$$B_{13} = (A_{13} + 2A_{55}) - (A_{11} + A_{33})/2 = -4.4\,km^2/s^2 \quad (-0.312) \tag{10'}$$

$$B_{23} = (A_{23} + 2A_{44}) - (A_{22} + A_{33})/2 = -5.35\,km^2/s^2 \quad (-0.379) \tag{11'}$$

The bracketed values are the B_{ij} made dimensionless by dividing each by A_{33}.

The exact values of the qP, qS_1 and qS_2 phase velocities are computed using equations from Every (1980) or Schoenberg and Helbig (1997). The exact (red) results, together with the approximate velocities in black, derived in the text for the qP phase velocity and given by Equation 18 are shown in Figures 2 and 4.

The exact (red) and approximate (black) qS_1 and qS_2 phase velocities obtained from equations (39.b) and (50.b) are presented in Figures 3 and 5 for the same model used for the qP case. The differences between the exact and approximate phase velocities are much more evident in Figures 1 and 2 due to the scale used on the vertical axis of the plots.

CONCLUSIONS

Using the linearized expressions for the qP, qS_1 and qS_2 phase velocities in an orthorhombic anisotropic medium as a starting point, alterations to the formulae were made to put them in a more indicative form. All $(A_{ii}, i = 1,2,\ldots,6)$ terms in the phase velocities are of order 2 in the phase velocity vector components rather than order 4, which the standard linearization procedure produces. The expressions for the phase velocities are all of a form such that they are ellipsoids with anellipsoidal correction terms, which are of order 4 in the phase velocity vector components.

Compared with other expressions for these three phase velocities, which appear in numerous works in the literature, almost without exception in terms of $(\varepsilon_\mu, \mu = x, y, z)$ and $(\delta_\mu, \mu = x, y, z)$, the formulae presented here are much less cumbersome. All quantities within the formulae can be associated with a physically measurable quantity.The results presented display the close match with the exact formulae.

REFERENCES

Backus, G.E., 1965, Possible forms of seismic anisotropy of the uppermost mantle under oceans, Journal of Geophysical Research, **70**, 3429-3439.

Červený, V. and Jech, J. 1982, Linearized solutions of kinematic problems for inhomogeneous anisotropic media – a linearization approach, Contr. Geophys. Inst. Slov. Acad. Sci., **13**, 127-133.

Courant, R. and Hilbert, D., 1962, Methods of Mathematical Physics, Vol. II, Partial Differential Equations: Interscience Publications, New York.

Every, A.G., 1980, General closed-form expressions for acoustic waves in elastically anisotropic solids, Physics Review B, **22**, 1746-1760.

Every, A.G. and Sachse, W., 1992, Sensitivity of inversion algorithms for recovering elastic constants of anisotropic solids from longitudinal wavespeed data, Ultrasonics, **30**, 43-48.

Gassmann, F., 1964, Introduction to seismic travel time methods in anisotropic media, Pure and Applied Geophysics, **58**. 63-112.

Jech, J. and Pšenčík, I., 1989, First order perturbation method for anisotropic media, Geophysical Journal International, **99**, 369-376.

Mensch, T. and Rasolofosaon, P., 1997, Elastic wave velocities in anisotropic media of arbitrary anisotropy – generalization of Thomsen's parameters ε, δ and γ, Geophysical Journal International, **128**, 43-64.

Musgrave, M.J.P., 1970, Crystal Acoustics, Holden-Day, San Francisco

Pšenčík, I., Liu, J-S. and Young, R.P., 2000, Inversion of qP wave travel times from the URL experiment, Seismic Waves in Complex 3-D Structures, Report 9, Department of Geophysics, Charles University, Prague, 83-112.

Pšenčík, I. and Gajewski, D., 1998. Polarization, phase velocity and NMO velocity of qP waves in arbitrary weakly anisotropic media, Geophysics, **63**, 1754-1766.

Sayers, C.M., 1994, P – wave propagation in weakly anisotropic media, Geophysical Journal International, **116**, 799-805.

Schoenberg, M. and Helbig, K., 1997, Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth, Geophysics, **62**, 1954-1974.

Song, L-P. and Every, A.G., 2000, Approximate formulae for acoustic wave group slowness in weakly orthorhombic media, Journal of Physics D: Applied Physics, **33**, L81-L85.

Thomsen, L., 1986, Weak elastic anisotropy, Geophysics, **51**, 1954-1966.

Tsvankin, I.D., 1997, Anisotropic parameters and *P*-wave velocities for orthorhombic media, Geophysics, **62**, 1292-1300.

Vassiliou, A., 1994, Personal communication.

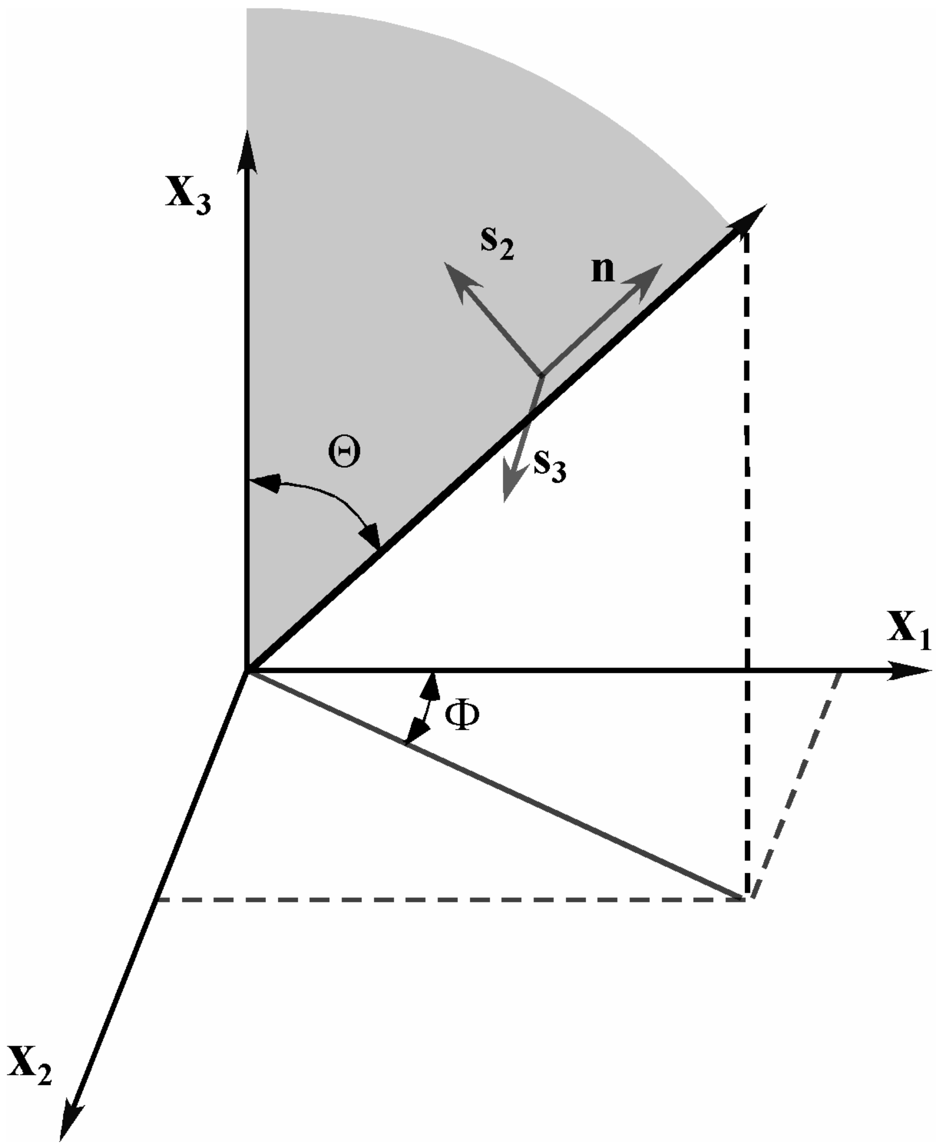

FIG. 1. Geometry of the phase velocity vectors used in the text.

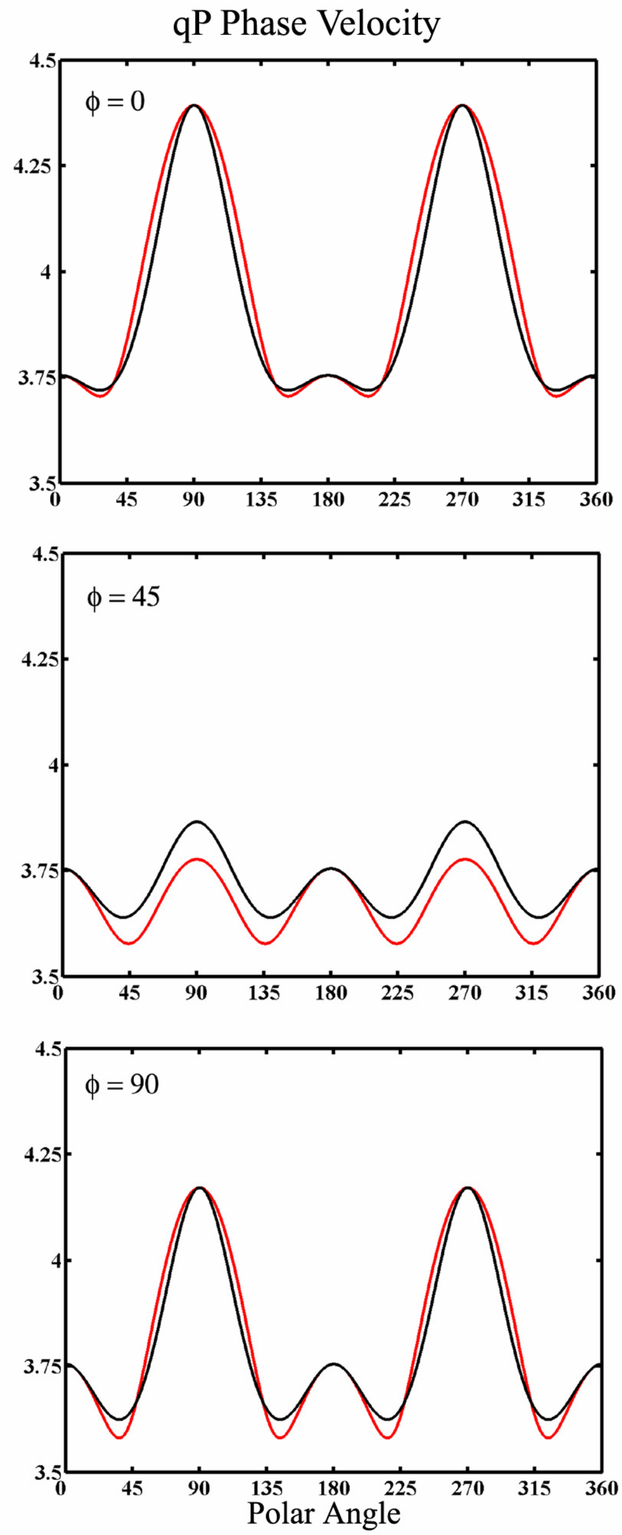

FIG. 2. qP phase velocities – red is exact and black is approximate. Azimuthal angles of 0, $\pi/4$ and $\pi/2$ radians. Vertical scale enhances any mismatch.

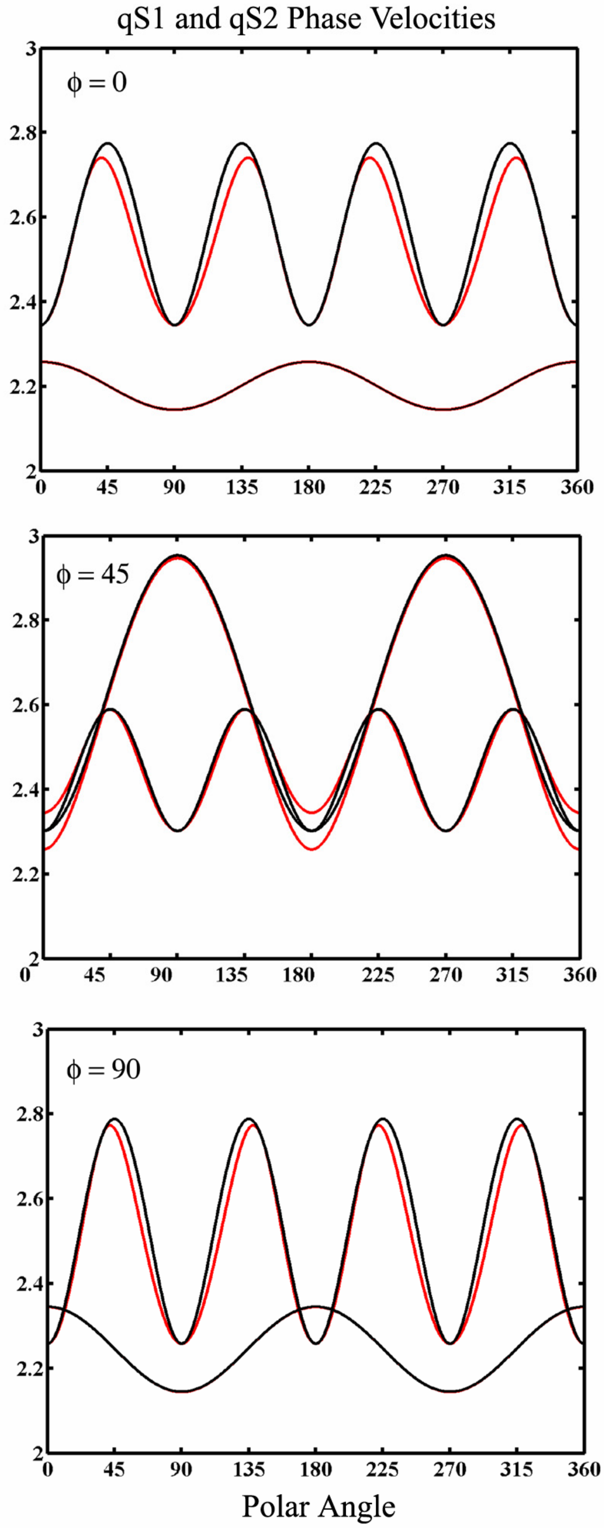

FIG. 3. qS_1 and qS_2 phase velocities – red is exact and black is approximate. Azimuthal angles of 0, $\pi/4$ and $\pi/2$. Vertical scale enhances any mismatch.

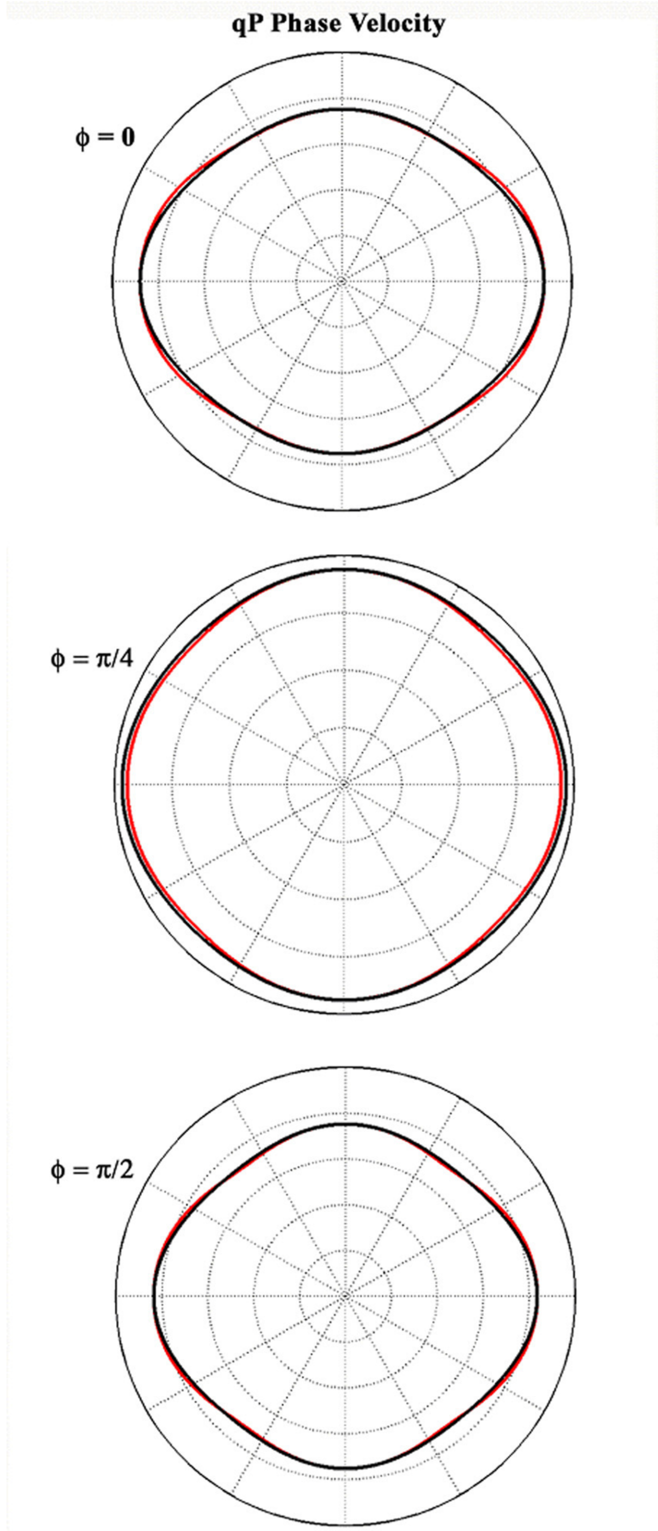

FIG. 4. Polar plot of the same data used to produce the panels in Figure 1.

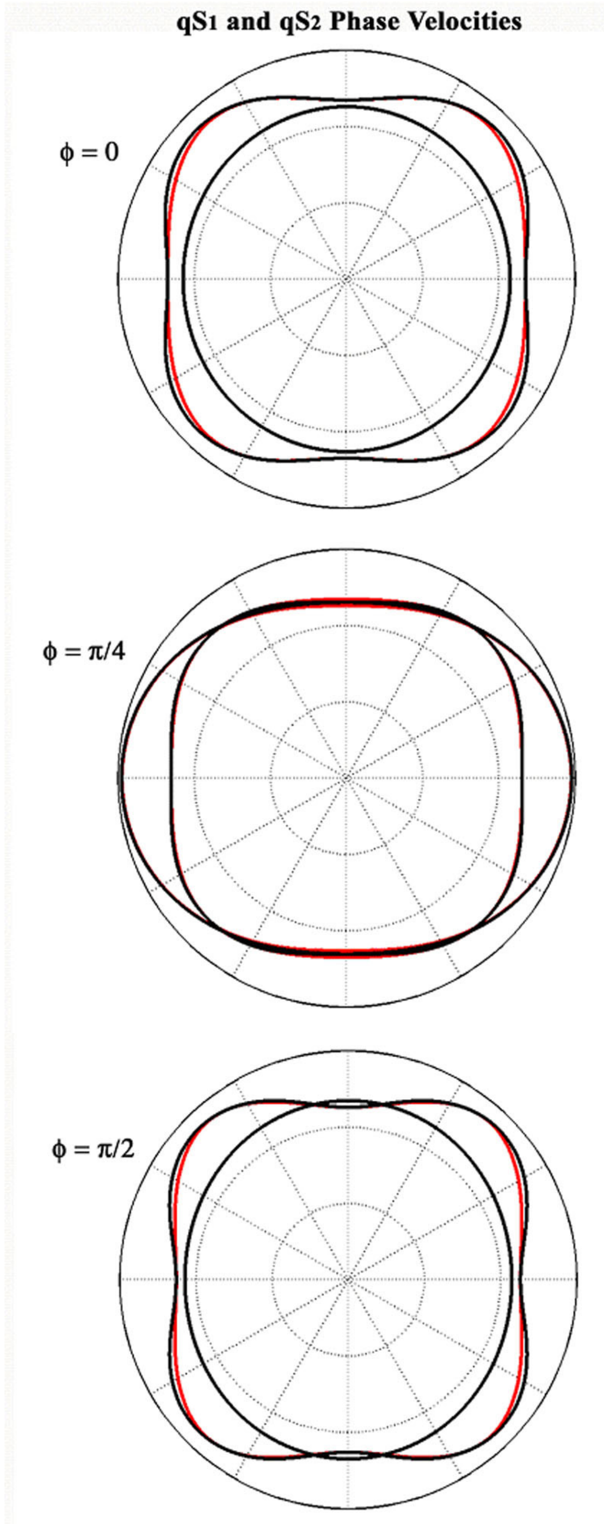

FIG. 5. Polar plot of the same data used to produce the panels in Figure 2.