# Phase velocity approximations in a transversely isotropic medium

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## ABSTRACT

Two approaches and the subsequent results of approximations for phase velocities in transversely isotropic (TI) media are examined in the context of what has been termed weak anisotropy. What is addressed is the range of anisotropic complexity from general or exact, to mild or weak anelliptic and finally to linearized weak and the limits within which the approximations can be assumed to be applicable. In what follows, the term weak anelliptic anisotropy will not be synonymous with linearized weak anisotropy. The phase (wavefront normal) velocity is the quantity selected for this study, as it is perhaps that which is most often chosen as a candidate for approximation, from which other relevant, associated quantities may be derived or computed – including group (ray) velocity, polarization vectors and intermediate values associated with amplitude computations, reflection and transmission coefficients. The quasi-compressional, $qP$, and quasi-shear, $qS_{\mathrm{V}}$, are approximated in such a manner that they are dependent on two parameters – ellipticity and anellipticity. Both of these quantities are physically realizable and measurable.

## INTRODUCTION

In Thomsen's 1986 paper on transversely isotropic (TI) media, weak linearized approximations to the exact $qP$ and $qS_V$ phase (wave front normal) velocities are derived. The prefix q indicates quasi and is used to indicate that the associated polarization vectors are not, in general, aligned with either the ray, which indicates the direction of energy propagation, or the wavefront normal. The simplification of the phase velocities is equivalent to reducing the complexity of the associated eikonal equation. The derivation of the expression for the weak linearized form of the $qP$ phase velocity for an arbitrary anisotropic medium is discussed in Backus (1965), Pšenčík and Gajewski (1998), and Jech and Pšenčík (1989), among others.

The exact eikonal equations for the $qP$ and $qS_V$ modes of wave propagation in a TI medium are quasi-linear partial differential equations. These equations are homogeneous of order 2 in powers of the slowness vector components. As these partial differential equations are obtained from a hyperbolic system of equations (forms of the elastic wave equation) they may be employed to compute the characteristics of the system. In geophysical applications, these characteristics are most often referred to as rays, along which energy travels from a source to some point in a TI medium. This property of rays is a most powerful tool in seismic modeling. However, when an approximation is made such that the eikonal loses the above property of homogeneity of order 2 in powers of components of the slowness vector, the theory of characteristics is no longer applicable. It is convenient at this point to define a "mild " anisotropic approximation to an eikonal to be any approximation which preserves its homogeneous quasi-linear partial differential equation property (Schoenberg and Helbig, 1996). In a transversely isotropic medium this translates into a medium which may have any degree of ellipticity but is somewhat constrained in its anellipticity. A "mild" anisotropic medium is not necessarily a "weak" anisotropic medium.

As with any approximation, some indication of its range of applicability should be established. The advantages and disadvantages of the implementation of an approximation should be viewed in terms of both the intent of the approximation and the possible consequences of employing it. To investigate some of these points further, Thomsen's derivation will be revisited. As well, an extension of the formulae presented in Gassmann (1964) will be looked at in more detail. The motivation for this is to establish the accuracy achieved by the approximations, at various stages in the linearization process, relative to the exact solution.

A slight change in definition of the parameter $\delta$ will be made from Thomsen's 1986 paper to provide modified linearized approximations for the $qP$ and $qS_V$ phase velocities that are in terms of physically realizable quantities. Although it would be preferable to introduce the anisotropic parameters, $C_{ij}$, using the density normalized $A_{ij}$ notation $\left(A_{ij} = C_{ij}/\rho\right)$, this may confuse matters excessively. Thus Thomsen's notation will be retained and an alternative approximation involving $A_{ij}$ that will be presented in a subsequent section will also be put in this notation.

## TI PHASE VELOCITY – THOMSEN'S LINEARIZATION (MODIFIED)

The eikonal equations for the quasi-compressional $(qP)$ and quasi-shear $(qS_V)$ waves are functions of the horizontal and vertical components of the slowness vector, $\mathbf{p} = (p, q)$, which may be written in terms of the phase (wavefront normal) angle, $\theta_j$, an acute angle measured from the vertical axis, and phase (wavefront normal) velocities, $v_j(\theta)$ $\left[j = qP, qS_V\right]$, as follows

$$p = \frac{\sin\theta_j}{v_j(\theta)} \qquad q = \frac{\cos\theta_j}{v_j(\theta)} \tag{1}$$

The exact expressions for the phase velocities, in $A_{ij}$ (Voigt) notation, may be written as (Gassmann, 1964)

$$v_{qP}(\theta) = \left(\left[A_{11}\sin^2\theta + A_{33}\cos^2\theta\right] + \frac{A_\alpha}{2}\left[\left(1 + 4\kappa_D\right)^{1/2} - 1\right]\right)^{1/2} \tag{2}$$

and

$$v_{qS_V}(\theta) = \left( A_{44} - \frac{A_\alpha}{2}\left[ \left(1+4\kappa_D\right)^{1/2} - 1 \right] \right)^{1/2} \tag{3}$$

If the $C_{ij}$ are designated as the anisotropic parameters of a medium, the associated density normalized parameters, having the dimensions of velocity squared, are given as

$$A_{ij} = C_{ij} / \rho \tag{4}$$

$A_\alpha$ and $\kappa_D$, both used above, require a definition.

$$A_\alpha = \left(A_{11} - A_{55}\right)\sin^2\theta + \left(A_{33} - A_{55}\right)\cos^2\theta \tag{5}$$

and

$$\kappa_D = \frac{A_D \sin^2\theta \cos^2\theta}{A_\alpha^2} \tag{6}$$

with $A_D$ being defined as

$$A_D = \left(A_{13} + A_{55}\right)^2 - \left(A_{11} - A_{55}\right)\left(A_{33} - A_{55}\right). \tag{7}$$

$A_D$ is a true measure of the deviation of the wavefront from the ellipsoidal case in a transversely isotropic medium, or simply, the anellipticity. This may be seen by setting $A_D = 0$ in Equations 2 and 3. The phase velocities have been written in an exact form that are such that they are the square root of the sum of the square of the degenerate ellipsoidal velocity plus a term defining the deviation from the ellipsoidal case. As mentioned above, setting $A_D = 0$ in Equations 2 and 3 for the $qP$ and $qS_V$ phase velocities in $A_{ij}$ results in

$$v_{qP}(\theta) = \left(A_{11}\sin^2\theta + A_{33}\cos^2\theta\right)^{1/2} = v_{qP}^{(e)}(\theta) \tag{8}$$

$$v_{qS_V}(\theta) = \sqrt{A_{55}} = v_{qS_V}^{(e)}(\theta) \tag{9}$$

The notation presented in Thomsen (1986) will be introduced here, as it is these parameters that most often appear in the more recent literature. The $qP$ phase velocity along the vertical, *z*, axis is

$$\alpha_0 = \sqrt{A_{33}} \tag{10}$$

while the $qS_V$ phase velocity along both the horizontal and vertical, *x* and *z*, axes:

$$\beta_0 = \sqrt{A_{55}} \tag{11}$$

The measure of ellipticity of the $qP$ wavefront is given as

$$\varepsilon = \frac{A_{11} - A_{33}}{2A_{33}} \tag{12}$$

In the initial and intermediate stages of this derivation, which appears in Thomsen (1986), the quantity $\delta^*$ is defined as

$$\begin{aligned}\delta^* &= \frac{1}{2A_{33}^2}\left[2\left(A_{13} + A_{55}\right)^2 - \left(A_{33} - A_{55}\right)\left(A_{11} + A_{33} - 2A_{55}\right)\right] \\ &= \frac{1}{2A_{33}^2}\left[\left[\left(A_{13} + A_{55}\right)^2 - \left(A_{33} - A_{55}\right)\left(A_{11} - A_{55}\right)\right] + \right. \\ &\qquad\qquad \left.\left[\left(A_{13} + A_{55}\right)^2 - \left(A_{33} - A_{55}\right)^2\right]\right]\end{aligned} \tag{13}$$

It is one of two parameters (the other being $\varepsilon$) used to parameterize the ellipticity and anellipticity of the phase velocity front. At a later point in the derivation, $\delta^*$ is replaced by

$$\delta^* = \left(2\delta - \varepsilon\right)\left(1 - \beta_0^2/\alpha_0^2\right). \tag{14}$$

The quantity $\delta$ is defined in its now well-known form as

$$\delta = \frac{\left(A_{13} + A_{55}\right)^2 - \left(A_{33} - A_{55}\right)^2}{2A_{33}\left(A_{33} - A_{55}\right)} \tag{15}$$

The reason for the choice of the definitions of the dimensionless quantities $\varepsilon$ and $\delta$ to parameterize a TI medium is that $\delta = \varepsilon$ corresponds to elliptical anisotropy. This immediately leads to $\delta - \varepsilon \neq 0$ indicating that the medium is anelliptic and that $\left(\delta - \varepsilon\right)$ is a measure of this.

In Equation 15 the dimensionless quantities $\varepsilon$ and $\delta$ have been normalized to the square of the $qP$ phase velocity at vertical incidence, which will introduce a factor of $A_{33}/A_{55} = \alpha_0^2/\beta_0^2$ into the expression for the $qS_V$ phase velocity. A slightly different definition of $\delta$ will be introduced and for the present be designated as $\hat{\delta}$. This new definition of $\delta^*$ in terms of $\hat{\delta}$ is given by

$$\delta^* = \left(2\hat{\delta} + \varepsilon\right)\left(1 - \beta_0^2/\alpha_0^2\right) \tag{16}$$

where here $\hat{\delta}$ is defined as

$$\hat{\delta} = \frac{\left(A_{13} + A_{55}\right)^2 - \left(A_{11} - A_{55}\right)\left(A_{33} - A_{55}\right)}{2A_{33}\left(A_{33} - A_{55}\right)} \quad . \tag{17}$$

There are very minor differences between Equations 14 and 15, and the modified Equations 16 and 17.

In the *weak* anisotropic limit $\varepsilon$ gives the deviation of the phase velocity surface from the spherical to the ellipsoidal, viz.

$$A_{11} = \left(1 + 2\varepsilon\right) A_{33} \approx \left(1 + \varepsilon\right)^2 A_{33}, \; for\, \varepsilon << 1.0 \tag{18}$$

which produces

$$\alpha_{\pi/2} \approx \left(1 + \varepsilon\right)\alpha_0 \quad , \quad \alpha_{\pi/2} = \sqrt{A_{11}} \quad , \quad \alpha_0 = \sqrt{A_{33}} \quad . \tag{19}$$

The velocity $\alpha_{\pi/2}$ is the $qP$ velocity along the $x$ axis. As previously stated, $\varepsilon$, although a measure of the deviation of the energy (ray) surface from the spherical, is not an exact indication of this. As shown in both Equations 18 and 19, $\varepsilon$ is a reasonable approximation to this deviation only in the weak anisotropic limit.

The exact expressions for the $qP$ and $qS_V$ phase (wave front normal) velocities in Thomsen's notation may be written as

$$v_{qP}^2\left(\theta\right) = \alpha_0^2\left[1 + \varepsilon \sin^2\theta + D^*\left(\theta\right)\right] \tag{20}$$

and

$$v_{qS_V}^2\left(\theta\right) = \beta_0^2\left[1 + \frac{\alpha_0^2}{\beta_0^2}\varepsilon \sin^2\theta - \frac{\alpha_0^2}{\beta_0^2} D^*\left(\theta\right)\right] \tag{21}$$

where

$$D^*\left(\theta\right) = \frac{1}{2}\left(1 - \frac{\beta_0^2}{\alpha_0^2}\right)\left\{\left[1 + \frac{4\delta^*}{\left(1 - \beta_0^2/\alpha_0^2\right)^2}\sin^2\theta\cos^2\theta + \frac{4\left(1 - \beta_0^2/\alpha_0^2 + \varepsilon\right)\varepsilon}{\left(1 - \beta_0^2/\alpha_0^2\right)^2}\sin^4\theta\right]^{1/2} - 1\right\} \tag{22}$$

Approximating $D^*\left(\theta\right)$ by expanding the radical $\left[\left(1 + x\right)^{1/2} \approx 1 + x/2\right]$, under the assumption that $x << 1$, and accordingly retaining only first order terms in (22) yields

$$D^*(\theta) \approx \frac{1}{2}\left(1-\frac{\beta_0^2}{\alpha_0^2}\right)\left\{\frac{2\delta^*}{\left(1-\beta_0^2/\alpha_0^2\right)^2}\sin^2\theta\cos^2\theta + \frac{2\left(1-\beta_0^2/\alpha_0^2+\varepsilon\right)\varepsilon}{\left(1-\beta_0^2/\alpha_0^2\right)^2}\sin^4\theta\right\} \quad (23)$$

A further approximation in which only the linear terms in $\varepsilon$ and $\delta^*$ are retained, specifically deleting the term in $\varepsilon^2$, results in

$$D^*(\theta) \approx \frac{\delta^*}{\left(1-\beta_0^2/\alpha_0^2\right)}\sin^2\theta\cos^2\theta + \varepsilon\sin^4\theta \quad (24)$$

These two approximation steps are equivalent to expanding $D^*(\theta)$ in a Taylor series in $\delta^*$ and $\varepsilon$ about $\left(\varepsilon,\delta^*\right)=(0,0)$ in the following manner

$$D^*\left(\theta:\varepsilon,\delta^*\right) \approx D^*(\theta:0,0) + \left.\frac{\partial D^*}{\partial\varepsilon}\right|_{\varepsilon=\delta^*=0}\varepsilon + \left.\frac{\partial D^*}{\partial\delta^*}\right|_{\varepsilon=\delta^*=0}\delta^* + \cdots \quad (25)$$

This has the effect of removing the condition specified for the approximation to be termed mild as defined earlier in this report. The $qP$ and $qS_V$ phase velocities may now be written as the approximate expressions

$$v_{qP}^2(\theta) \approx \alpha_0^2\left[1+\varepsilon\sin^2\theta + \frac{\delta^*}{\left(1-\beta_0^2/\alpha_0^2\right)}\sin^2\theta\cos^2\theta + \varepsilon\sin^4\theta\right] \quad (26)$$

and

$$v_{qS_V}^2(\theta) \approx \beta_0^2\left[1+\frac{\alpha_0^2}{\beta_0^2}\varepsilon\sin^2\theta - \frac{\alpha_0^2}{\beta_0^2}\left[\frac{\delta^*}{\left(1-\beta_0^2/\alpha_0^2\right)}\sin^2\theta\cos^2\theta + \varepsilon\sin^4\theta\right]\right]. \quad (27)$$

Introducing the expression for $\delta^*$ in terms of $\hat{\delta}$ at this point results in the $qP$ phase velocity having the form

$$v_{qP}^2(\theta) \approx \alpha_0^2\left[1+\varepsilon\sin^2\theta + \left(2\hat{\delta}+\varepsilon\right)\sin^2\theta\cos^2\theta + \varepsilon\sin^4\theta\right] \quad (28)$$

or equivalently

$$v_{qP}(\theta) \approx \alpha_0\left[1+2\left(\hat{\delta}+\varepsilon\right)\sin^2\theta - 2\hat{\delta}\sin^4\theta\right]^{1/2}. \quad (29)$$

Another binomial expansion yields a further (final) linearized approximate expression for the $qP$ phase velocity in the weak anisotropic limit if only the leading terms are retained, i.e., $(1+x)^{1/2} \approx 1+x/2$,

$$v_{qP}(\theta) \approx \alpha_0 \left[ 1 + \left( \hat{\delta} + \varepsilon \right) \sin^2 \theta - \hat{\delta} \sin^4 \theta \right]. \tag{30}$$

For the case of the $qS_V$ wave propagation the following sequence of algebraic steps leads to an analogous form of Thomsen's final formula, in this instance

$$v_{qS_V}^2(\theta) \approx \beta_0^2 \left[ 1 + \frac{\alpha_0^2}{\beta_0^2} \varepsilon \sin^2 \theta - \frac{\alpha_0^2}{\beta_0^2} \left[ \left( 2\hat{\delta} + \varepsilon \right) \sin^2 \theta \cos^2 \theta + \varepsilon \sin^4 \theta \right] \right] \tag{31}$$

and, after some simplification,

$$v_{qS_V}(\theta) \approx \beta_0 \left[ 1 - 2 \frac{\alpha_0^2}{\beta_0^2} \hat{\delta} \sin^2 \theta \cos^2 \theta \right]^{1/2}. \tag{32}$$

A binomial expansion retaining only the two leading terms results in the linearized equation

$$v_{qS_V}(\theta) \approx \beta_0 \left[ 1 - \frac{\alpha_0^2}{\beta_0^2} \hat{\delta} \sin^2 \theta \cos^2 \theta \right]. \tag{33}$$

Equations 30 and 33 differ from the results obtained using $\delta$ (Equation 15) rather than the quantity $\hat{\delta}$ (Equation 17).

It should be pointed out that $\hat{\delta}$ is identical to $\sigma$, defined in Daley and Lines (2004) as

$$\sigma = (\delta - \varepsilon) = \hat{\delta}. \tag{34}$$

This quantity $\sigma$ is the dimensionless anellipticity or deviation from the ellipsoidal of a ray or slowness surface. The dimensionality was removed earlier by introducing the $qP$ related quantity, $A_{33} = \alpha_0^2$, in the denominator. For this reason, the ratio $\alpha_0^2 / \beta_0^2 = A_{33} / A_{55}$ appears in the $qS_V$ phase velocity, and associated quantities. Using the notation given in (34), with Thomsen's 1986 equivalent expressions given for reference purposes, Equations 30 and 33 may be written as

$$v_{qP}(\theta) \approx \alpha_0 \left[ 1 + (\varepsilon + \sigma) \sin^2 \theta - \sigma \sin^4 \theta \right] \tag{35}$$

$$\left\{ v_{qP}(\theta) \approx \alpha_0 \left[ 1 + \delta \sin^2 \theta \cos^2 \theta + \varepsilon \sin^4 \theta \right] \right\}_{\text{Thomsen}} \tag{35'}$$

and

$$v_{qS_V}(\theta) \approx \beta_0 \left[ 1 - \frac{\alpha_0^2}{\beta_0^2} \sigma \sin^2\theta \cos^2\theta \right] \tag{36}$$

$$\left\{ v_{qS_V}(\theta) \approx \beta_0 \left[ 1 + \frac{\alpha_0^2}{\beta_0^2} (\varepsilon - \delta) \sin^2\theta \cos^2\theta \right] \right\}_{\text{Thomsen}} \tag{36'}$$

or equivalently,

$$v_{qS_V}(\theta) \approx \beta_0 \left[ 1 - \frac{\alpha_0^2}{\beta_0^2} \sigma \sin^2\theta + \frac{\alpha_0^2}{\beta_0^2} \sigma \sin^4\theta \right]. \tag{37}$$

Some may prefer this parameterization of the linearized $qP$ and $qS_V$ phase velocities in terms of $\varepsilon$ and $\sigma$, as both are physically realizable quantities. Formulae involving them are more indicative of the dependencies on ellipticity and anellipticity. An example is the expression for the $qS_V$ phase velocity, which in reality is not dependent on the ellipticity, $\varepsilon$, while the $qP$ phase velocity, as indicated above, is dependent on both ellipticity and anellipticity. This is clear from viewing Equations 35 – 37. If the $(\varepsilon, \delta)$ parameterization had been used, this would not be immediately apparent.

## PHASE VELOCITY APPROXIMATION (GASSMANN)

In this section the $qP$ and $qS_V$ phase velocities will be simplified in a slightly different manner. Weak anisotropy will be assumed initially, and certain approximations consistent with that assumption will be made before the linearizing process by expansion in a Taylor series is undertaken. Using Equations 2 and 3 as the definitions of the exact expressions of the $qP$ and $qS_V$ phase velocities and, as before, defining the degenerate ellipsoidal phase velocities as

$$v_{qP}^{(e)} = \left[ A_{11} \sin^2\theta + A_{33} \cos^2\theta \right]^{1/2} \tag{38}$$

for the $qP$ case and

$$v_{qS_V}^{(e)} = \sqrt{A_{55}} \tag{39}$$

for $qS_V$ wave propagation. By manipulating certain quantities in the expressions for the exact expression for the $qP$ phase velocity, the following quantity arises

$$\kappa_\alpha = \sqrt{1 + \frac{A_\alpha \left( \sqrt{1 + 4\kappa_D} - 1 \right)}{2 \left[ v_{qP}^{(e)} \right]^2}} - 1 \tag{40}$$

where the following three terms require definition (Equations 4, 5 and 6) are restated here

$$A_\alpha = \left(A_{11} - A_{55}\right)\sin^2\theta + \left(A_{33} - A_{55}\right)\cos^2\theta \tag{41}$$

$$\kappa_D = \frac{A_D \sin^2\theta \cos^2\theta}{A_\alpha^2} \tag{42}$$

$$A_D = \left(A_{13} + A_{55}\right)^2 - \left(A_{11} - A_{55}\right)\left(A_{33} - A_{55}\right) \tag{43}$$

and $v_{qP}^{(e)}$ is defined in Equation 38. The exact expression for the $qP$ phase velocity may be written in the more compact form

$$v_{qP}\left(\theta\right) = v_{qP}^{(e)}\left(1 + \kappa_\alpha\right), \tag{44}$$

or equivalently,

$$v_{qP}^2\left(\theta\right) = \left[v_{qP}^{(e)}\right]^2 \left(1 + \kappa_\alpha\right)^2 . \tag{45}$$

For $A_D$ sufficiently small (a weakly anelliptic TI medium), the following approximations to $\kappa_\alpha$ may be made

$$\kappa_\alpha \approx \sqrt{1 + \frac{A_\alpha \kappa_D}{\left[v_{qP}^{(e)}\right]^2}} - 1 \approx \frac{A_\alpha \kappa_D}{2\left[v_{qP}^{(e)}\right]^2} = \frac{A_D \sin^2\theta \cos^2\theta}{2A_\alpha\left[v_{qP}^{(e)}\right]^2} . \tag{46}$$

In the $qS_V$ case the term comparable to $\kappa_\alpha$ is

$$\kappa_\beta = \sqrt{1 - \frac{A_\alpha\left(\sqrt{1 + 4\kappa_D} - 1\right)}{2\left[v_{qS_V}^{(e)}\right]^2}} - 1, \tag{47}$$

and it follows that the $qS_V$ phase velocity may be written as

$$v_{qS_V}\left(\theta\right) = v_{qS_V}^{(e)}\left(1 + \kappa_\beta\right), \tag{48}$$

or again as in the $qP$ case

$$v_{qS_V}^2\left(\theta\right) = \left[v_{qS_V}^{(e)}\right]^2 \left(1 + \kappa_\beta\right)^2 \tag{49}$$

where, as in Equation 41, $\kappa_\beta$ may, for $A_D$ sufficiently small, be estimated as

$$\kappa_\beta \approx \sqrt{1-\frac{A_\alpha \kappa_D}{\left[v_{qS_V}^{(e)}\right]^2}}-1 \approx -\frac{A_\alpha \kappa_D}{2\left[v_{qS_V}^{(e)}\right]^2} = -\frac{A_D \sin^2\theta \cos^2\theta}{2A_\alpha \left[v_{qS_V}^{(e)}\right]^2} \tag{50}$$

where all terms in Equation 50 have been previously defined. The reason for including Equations 45 and 49 will be discussed later in this section.

The linearization process will now be continued using Equations 44 and 48. It is required to expand the $qP$ phase velocity expression (Equation 44) in a Taylor series for small angles, $x^2 = \sin^2\theta \sim 0$. As this step is for comparison purposes the first three terms will be retained to show equivalence with those equations rederived in the previous section.

$$v_{qP}(\theta) = v_{qP}^{(e)}(1+\kappa_\alpha) \approx v_{qP}(x^2)\Big|_{x^2=0} + \frac{dv_{qP}}{d(x^2)}\bigg|_{x^2=0} x^2 + \frac{1}{2}\frac{d^2 v_{qP}}{d(x^2)^2}\bigg|_{x^2=0} x^4 + \ldots \tag{51}$$

After some algebraic manipulation and, as in Equation 24 of the previous section, neglecting terms in $\varepsilon^2$, the weak anisotropic approximation is given as

$$v_{qP}(\theta) \approx \sqrt{A_{33}} + \left[\frac{(A_{11}-A_{33})}{2\sqrt{A_{33}}} + \frac{A_D}{2\sqrt{A_{33}}(A_{33}-A_{55})}\right]x^2 + \left[\frac{A_D}{2\sqrt{A_{33}}(A_{33}-A_{55})}\right]x^4 + \ldots \tag{52}$$

Introducing the definitions of $\alpha_0$, $\varepsilon$ and $\delta$ results in

$$v_{qP}(\theta) \approx \alpha_0\left[1+\left(\frac{(A_{11}-A_{33})}{2A_{33}} + \frac{2A_{33}(A_{33}-A_{55})(\delta-\varepsilon)}{2A_{33}(A_{33}-A_{55})}\right)x^2 - \left(\frac{2A_{33}(A_{33}-A_{55})(\delta-\varepsilon)}{2A_{33}(A_{33}-A_{55})}\right)x^4\right]. \tag{53}$$

Finally, using $(\varepsilon,\sigma)$ $\left[(\varepsilon,\hat{\delta})\right]$ notation and replacing $x^2$ by $\sin^2\theta$, the following equation is obtained

$$v_{qP}(\theta) \approx \alpha_0\left[1+(\varepsilon+\sigma)\sin^2\theta - \sigma\sin^4\theta\right] . \tag{54}$$

In a similar manner it can be shown that, starting with

$$v_{qS_V}(\theta) = v_{qS_V}^{(e)}(1+\kappa_\beta) = \sqrt{A_{55}}(1+\kappa_\beta), \tag{55}$$

with $v_{qS_V}^{(e)} = \sqrt{A_{55}}$ in this case and expanding in a Taylor series about $x^2 = \sin^2\theta \approx 0$, retaining only the first three terms, yields

$$v_{qS_V}(\theta) \approx \beta_0\left[1-\left(\alpha_0^2/\beta_0^2\right)\sigma\sin^2\theta+\left(\alpha_0^2/\beta_0^2\right)\sigma\sin^4\theta\right] \quad . \tag{56}$$

Equations 54 and 56 are identical, to terms of second order, with those presented in the previous section.

Returning now to the exact Equations 45 and 49 and implementing the weak anellipticity approximations to $\kappa_\alpha$ and $\kappa_\beta$ $\left(|\kappa_\alpha| \text{ and } |\kappa_\beta| << 1\right)$ derived in Equations 46 and 50 results in the following expressions for the $qP$ and $qS_V$ phase velocities

$$v_{qP}^2(\theta) \approx \left[v_{qP}^{(e)}\right]^2\left(1+2\kappa_\alpha\right) = \left[v_{qP}^{(e)}\right]^2 + \frac{A_D\sin^2\theta\cos^2\theta}{A_\alpha} \tag{57}$$

and

$$v_{qS_V}^2(\theta) \approx \left[v_{qS_V}^{(e)}\right]^2\left(1+2\kappa_\beta\right) = \left[v_{qS_V}^{(e)}\right]^2 - \frac{A_D\sin^2\theta\cos^2\theta}{A_\alpha} \tag{58}$$

where all terms in the equations have been previously defined in this section. Rewriting these two equations in terms of the vertical and horizontal components of slowness, $p$ and $q$, employing the definitions given in terms of the phase velocities as $p=\sin\theta_j/v_j(\theta)$ and $q=\cos\theta_j/v_j(\theta)$, $\left[j=qP, qS_V\right]$ results in the $qP$ and $qS_V$ eikonal equations, $G_{qP/qS_V}=1$, for a mild anisotropic medium, having the forms

$$G_{qP}(p,q) = A_{11}p^2 + A_{33}q^2 + \frac{A_D p^2 q^2}{\left(A_{11}-A_{55}\right)p^2+\left(A_{33}-A_{55}\right)q^2} = 1 \tag{59}$$

and

$$G_{qS_V}(p,q) = A_{55}\left(p^2+q^2\right) - \frac{A_D p^2 q^2}{\left(A_{11}-A_{55}\right)p^2+\left(A_{33}-A_{55}\right)q^2} = 1 \quad . \tag{60}$$

These two eikonals satisfy the definition given in a previous section for a mildly anisotropic medium in that they are both homogeneous of powers of 2 in terms of the slowness vector components. This allows these eikonal equations to be used in derivations in which the theory of characteristics is incorporated, such as the determination of ray trajectories from a point within a medium. The form of the above two eikonal equations is indicative of the associated slowness surfaces: ellipsoids of revolution and spheres with an additional term, linear in the anellipticity $A_D \propto \sigma$ $\left(\sigma=\left(\delta-\varepsilon\right)\right)$, that results in anelliptic deformation. Although these eikonals are useful from a tutorial perspective, in practice, the exact eikonal equations are not much more difficult to implement in the solutions of related problems that are usually less analytical than computational. To summarize, "mild" anisotropy does not, in general, mean "weak" anisotropy. However, "mild" anisotropy does indicate weak-anellipticity.

## NUMERICAL RESULTS AND DISCUSSION

The models selected for the computation and comparison of phase velocities using differing degrees of approximations are Dog Creek shale and Mesaverde clay - shale. The first may be classified as a weak anisotropic material while the second stretches this definition somewhat. However, it will be retained to act as an indicator of the accuracy of the formulae presented. The anisotropic parameters describing these two media are taken from Thomsen (1986) and given as

**Dog Creek Shale**

$$\alpha_0 = 1875\, m/s \quad \beta_0 = 826\, m/s$$
$$\varepsilon = 0.225$$
$$\delta = 0.100$$
$$\sigma = -0.125$$
$$\rho = 2.000\, g/cm^3 .$$

**Mesaverde Clayshale**

$$\alpha_0 = 3928\, m/s \quad \beta_0 = 2055\, m/s$$
$$\varepsilon = 0.334$$
$$\delta = 0.730$$
$$\sigma = 0.396$$
$$\rho = 2.590\, g/cm^3 .$$

These examples were chosen to illustrate the differences between a weak anelliptic (mild anisotropic) approximation and a linearized weak anisotropic TI approximation. Figures 1 and 2 use the weak linearized expressions for the $qP$ and $qS_V$ phase velocities given by Equations 35 and 37 and are compared to the exact expressions from Equations 2 and 3 for the Dog Creek shale model. The weak anelliptic approximation (Equations 44, 48 with the approximations (Equations 46 and 50) implemented) for this model are not shown as they are almost indistinguishable from the exact phase velocities.

The weak linearized anisotropic approximations to the $qP$ and $qS_V$ phase velocities for the Mesaverde clay - shale model are shown in Figures 4 and 6, where they are compared with the exact expressions. The weak anelliptic approximations are compared with the exact expressions are presented in Figures 3 and 5.

It becomes clear from viewing this last set of four figures that there is a significant difference between the weak anelliptic and linearized weak approximations to the phase velocities. As the linearized weak velocities are often used in obtaining group velocities and angles, the possible consequences of utilizing approximations of any level of complexity should be stated, as indicated by this limited study of the matter.

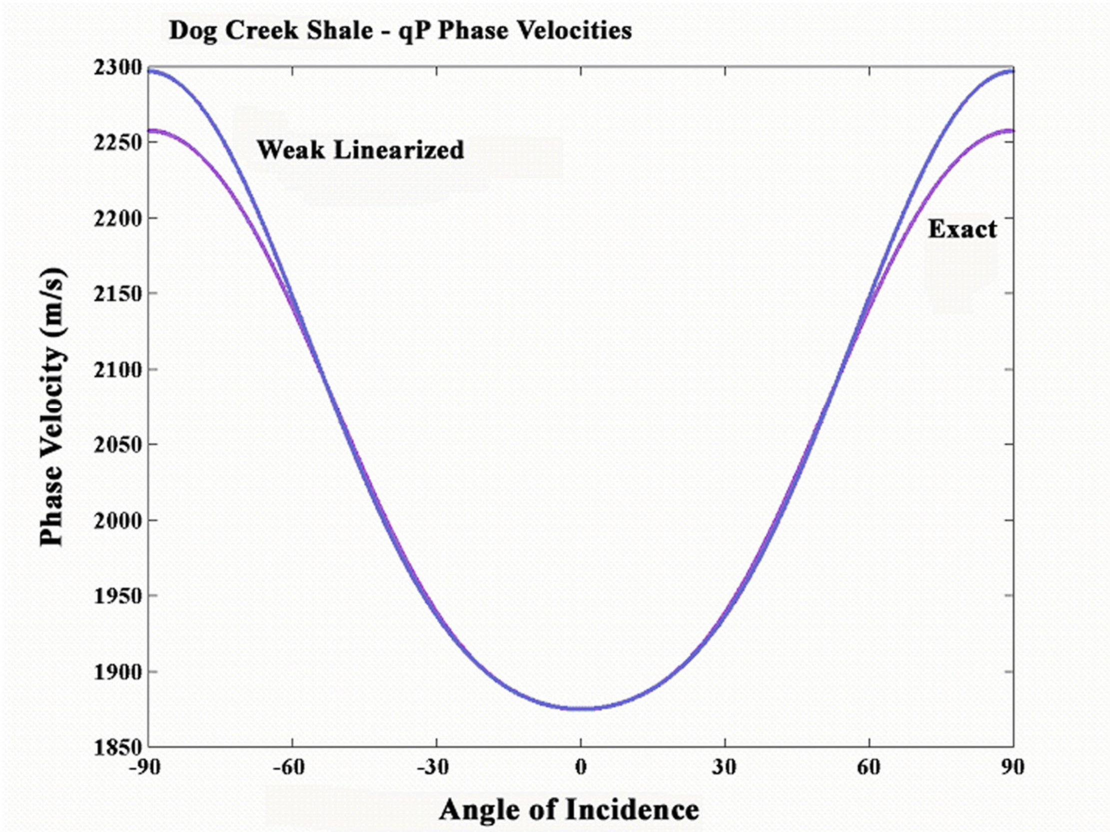


FIG. 1. A comparison of the weak linearized anisotropic and exact $qP$ phase velocities for Dog Creek shale.

## CONCLUSIONS

The derivation of approximations to the $qP$ and $qS_V$ phase velocities for a transversely isotropic medium is revisited. Starting with the exact expressions for these quantities, two alternate approaches are taken to arrive at the same results. The linearization method employed by Thomsen (1986) is reproduced with a modification that puts the final equations in a form that is dependent on the physically realizable quantities associated with the ellipticity and anellipticity of the phase velocity surfaces (the $qS_V$ phase velocity being independent of the ellipticity). The dependence on the previously used parameter $\delta$ was removed - because of its categorization as "intuitively inaccessible" it was reasoned that an alternative was desirable to be proposed. To complement the above derivation, the equations presented in Gassmann (1964) were used to produce *weak anelliptic* (mild anisotropic) and *weak linearized* anisotropic approximations for the $qP$ and $qS_V$ phase velocities in a transversely isotropic medium. Two physical models of different degrees of anisotropy were chosen to graphically show the differences of the approximations when compared with the exact phase velocities.

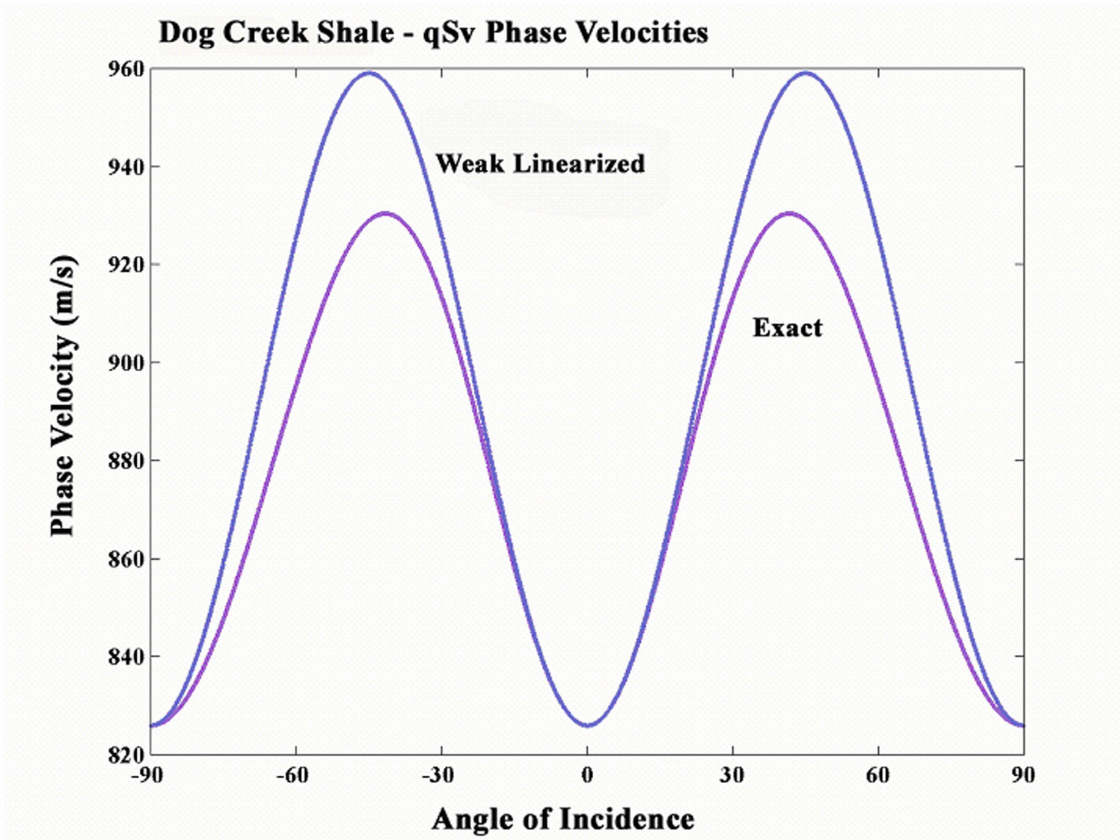


FIG. 2. A comparison of the weak linearized anisotropic and exact $qS_V$ phase velocities for Dog Creek shale.

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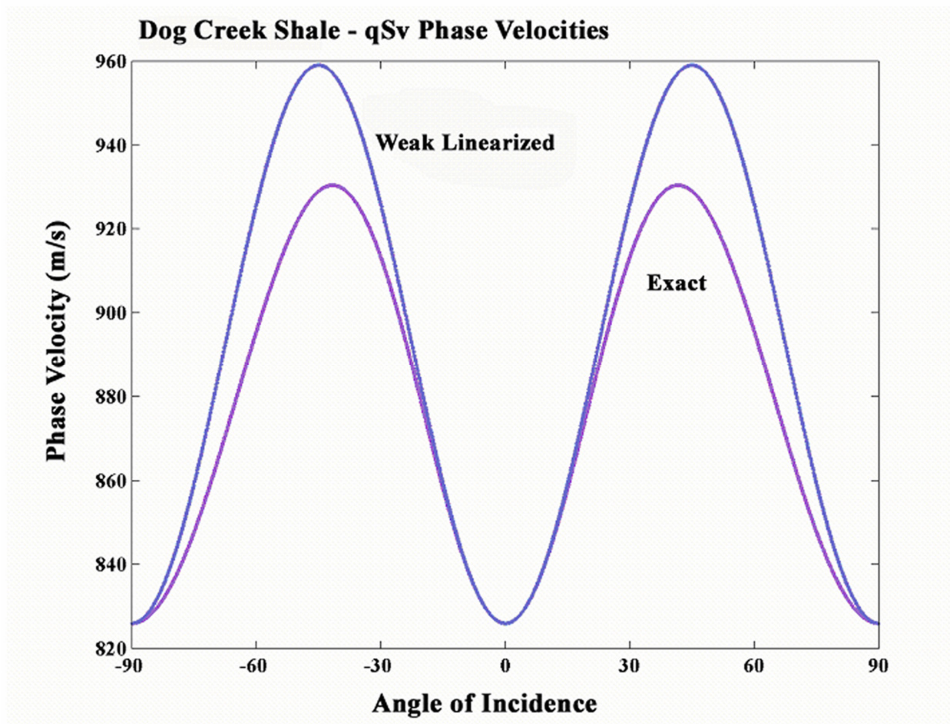


FIG. 3. Mesaverde clay – shale: A comparison of the weak anelliptic and exact $qP$ phase velocities.

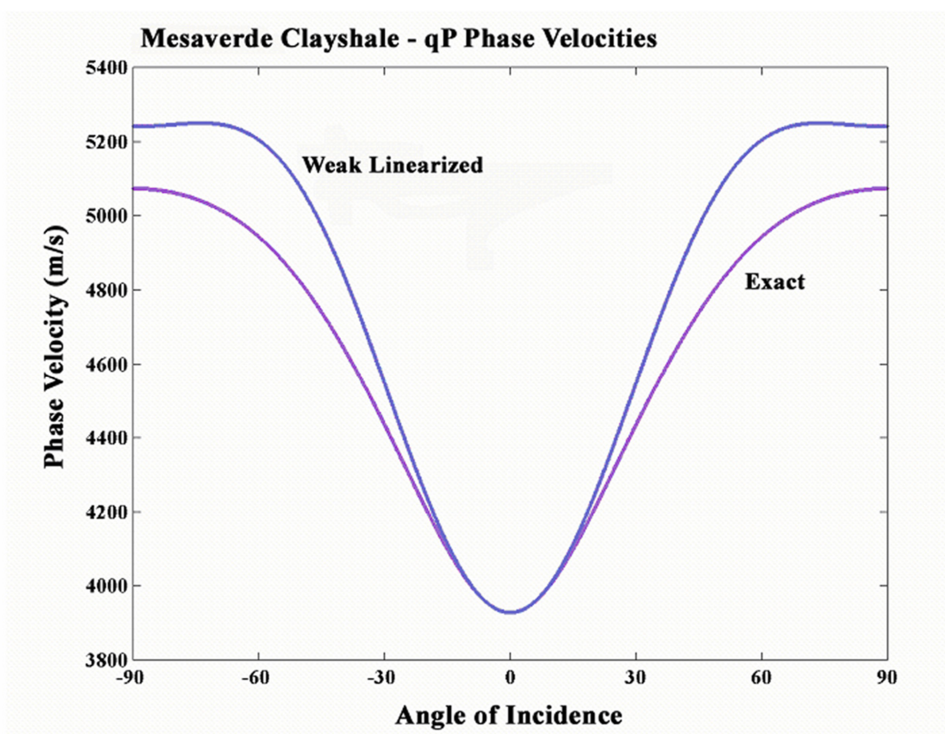


FIG. 4. Mesaverde clay – shale: A comparison of the weak linearized anisotropic and exact $qP$ phase velocities.

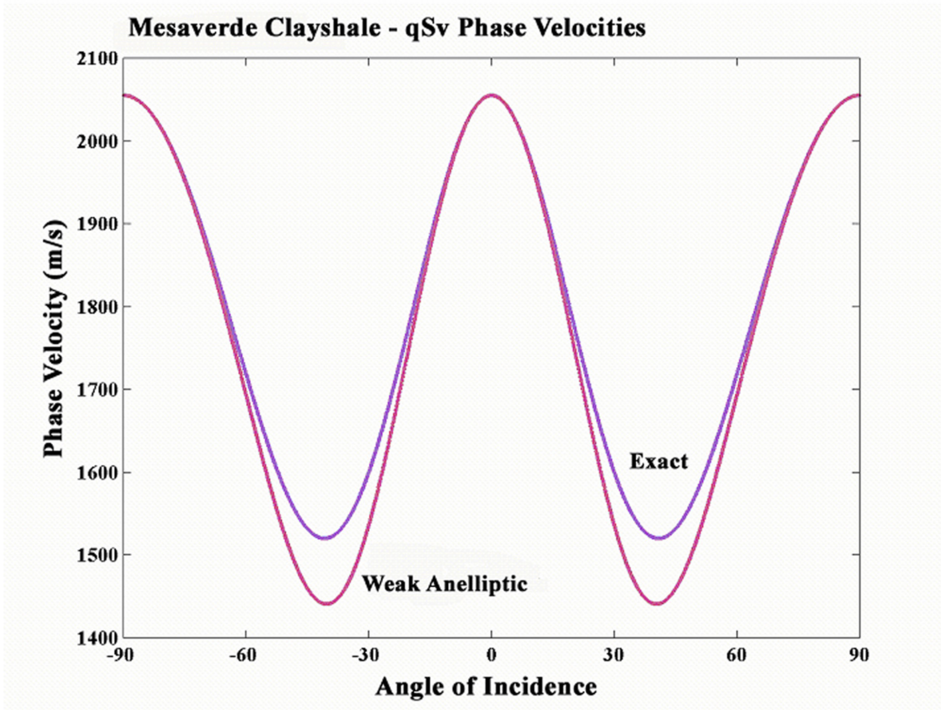


FIG. 5. Mesaverde clay – shale: A comparison of the weak anelliptic and exact $qS_V$ phase velocities.

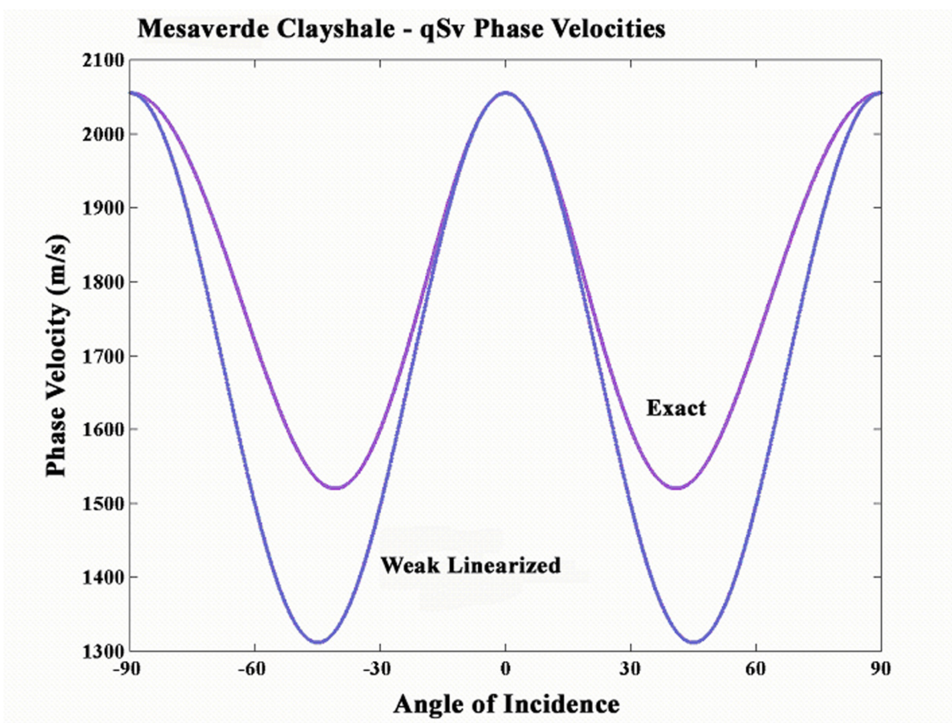


FIG. 6. Mesaverde clay – shale, comparison of the weak linearized anisotropic and exact $qS_V$ phase velocities.