Energy-preserving windows for non-stationary filtering

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ABSTRACT

We demonstrate the construction of window functions for processing time-variant signals such as seismic data logs, which preserve both local amplitudes and local energies. This is a key step in nonstationary filtering that allows the localization of signals without introducing spurious energy artifacts.

INTRODUCTION

A key step in the processing of a time-variant signal such as a seismic log is to localize the signal, in order to extract or modify time-dependent features of the signal. Such features might include spectral content, energy, noise levels, and the resulting modifications could include time-variant filtering (see Margrave, 1998), nonstationary deconvolution (see Margrave et al., 2002) and Q-correction (see Grossman et al., 2002). A common approach is to break-down a signal $s(t)$ into many individual slices, $s_n(t)$, $n = 1, 2, 3, \ldots$, each of which are made short enough in duration as to be effectively stationary in their spectral properties. The spectral nature of each individual slice $s_n(t)$ is then taken as being indicative of the instantaneous spectrum of the full signal $s(t)$ at the corresponding moment of time.

Typically, the signal is decomposed, or localized, through the use of a sliding window function. In this case, each slice $s_n(t)$ is defined as the product of the original signal $s(t)$ with some specific windowing function $w_n(t)$,

$$s_n(t) \;=\; w_n(t)s(t), \tag{1}$$

where each windowing function is simply a translation in time of a single *mother-window*, $w(t)$:

$$w_n(t) \;=\; w(t - nT). \tag{2}$$

The positive constant T is called the *window spacing*; throughout this article we will take $T = 1$.

In many cases, the window functions are designed without any detailed knowledge of the signal to be analyzed, and chosen with generic features to enhance the signal processing steps. For instance, the mother-window is generally real-valued, smooth, symmetric about $t = 0$, and with support in some interval $[-N/2, N/2]$ which has a length of N times the window spacing $T = 1$. Hamming, Hanning, and Blackwell windows are familiar choices (as in Rabiner and Gold (1975) or Karl (1989)). With the support of the mother-window in the interval $[-1, 1]$, only immediately neighbouring signal slices would exhibit any overlap between each other.

A common requirement is that the superposition of slices should recover the full signal

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itself:

$$\sum_{n\in\mathbb{Z}} s_n(t) \;=\; s(t) \tag{3}$$

for all $t \in \mathbb{R}$. This translates into the following *partition of unity* condition on the windowing functions:

$$\sum_{n\in\mathbb{Z}} w_n(t) \;=\; 1, \qquad \text{for all } t \in \mathbb{R}. \tag{4}$$

In terms of translates of a mother-window, the condition can be rewritten as

$$\sum_{n\in\mathbb{Z}} w(t-nT) \;=\; 1, \qquad \text{for all } t \in \mathbb{R}. \tag{5}$$

We call this the *amplitude criterion*.

In the applications we are interested in, it is useful to require that the superposition of the *energy* of the slices should recover the original energy; that is

$$\sum_{n\in\mathbb{Z}} |s_n(t)|^2 \;=\; |s(t)|^2, \qquad \text{for all } t \in \mathbb{R}. \tag{6}$$

In terms of the windowing functions, this requires

$$\sum_{n\in\mathbb{Z}} |w_n(t)|^2 \;=\; 1, \qquad \text{for all } t \in \mathbb{R}. \tag{7}$$

For translates of a mother-window, this amounts to the condition

$$\sum_{n\in\mathbb{Z}} |w(t-nT)|^2 \;=\; 1, \qquad \text{for all } t \in \mathbb{R}. \tag{8}$$

We will refer to this condition as the *energy criterion*.

When only two windows overlap, the amplitude and energy conditions give two equations in two unknowns,

$$w_n(t) + w_{n+1}(t) = 1, \qquad w_n^2(t) + w_{n+1}^2(t) = 1, \tag{9}$$

which have only the trivial solutions $w_n(t) = 1, w_{n+1}(t) = 0$ and $w_n(t) = 0, w_{n+1}(t) = 1$. These correspond to the boxcar windows, which is a simple and somewhat uninteresting case.

In this article, we demonstrate the construction of continuous mother-window functions with compact support which meet both the amplitude and energy criteria. We demonstrate the possible solutions for overlaps of size $N = 3, 4, 6$ and discuss the general construction.

AN ASYMMETRIC SOLUTION WITH OVERLAP *N = 3*

In the case where three adjacent windows may overlap, we have two equations in three unknowns,

$$w_n(t) + w_{n+1}(t) + w_{n+2}(t) \;=\; 1 \tag{10}$$

$$w_n^2(t) + w_{n+1}^2(t) + w_{n+2}^2(t) \;=\; 1 \tag{11}$$

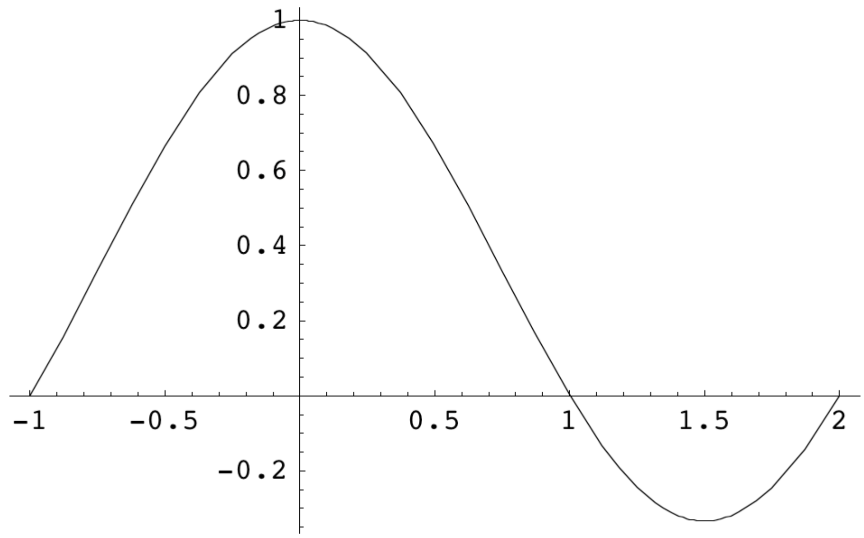

FIG. 1. A 3-overlap, energy-preserving window.

which can be recognized as the intersection of a plane $x + y + z = 1$ with a sphere $x^2 + y^2 + z^2 = 1$, and thus is a circle embedded in three dimensions. A reasonable solution is to parameterize the circle to produce the mother-window,

$$w(t) = \tfrac{1}{3} + \tfrac{2}{3}\cos(\tfrac{2\pi}{3}t), \text{ for } -1 \leq t \leq 2, \tag{12}$$

which is displayed in Figure 1. To demonstrate the amplitude and energy-preserving conditions, we show in Figure 2 that the translates of the window function sum to one, and in Figure 3, that the translates squared also sum to one.

It is also possible to re-parameterize the window to give a smoother version, which is also differentiable at the cutoff. For instance, we may set

$$w(t) \;=\; \tfrac{1}{3} \;+\; \tfrac{2}{3}\cos(\tfrac{2\pi}{3}u(t)) \tag{13}$$

where

$$u(t) \;=\; t - \tfrac{1}{2\pi}\sin(2\pi t). \tag{14}$$

The resulting window function is shown in Figure 4.

SYMMETRIC SOLUTIONS WITH OVERLAP *N = 4, 6*

Allowing for four windows to overlap, we can obtain a symmetric mother-window function, supported on the interval $[-2, 2]$, of the form

$$w(t) = \tfrac{1}{4}\left(1 + (1 + \sqrt{3/2})\cos(\tfrac{\pi}{2}t) + \cos(\pi t) + (1 - \sqrt{3/2})\cos(\tfrac{3\pi}{2}t)\right), \tag{15}$$

which is shown in Figure 5. A second symmetric solution is obtained by reversing the signs on the $\sqrt{3/2}$ in the parameters above.

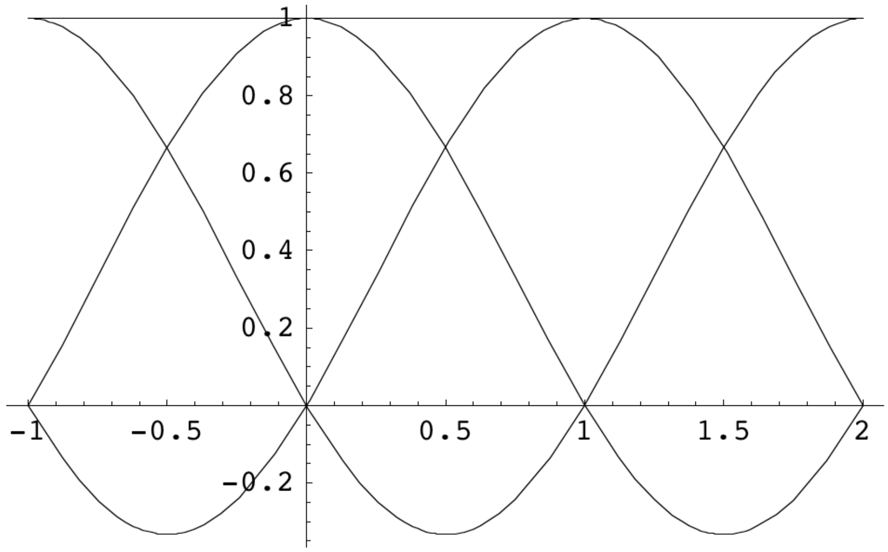

FIG. 2. Window translates summing to one.

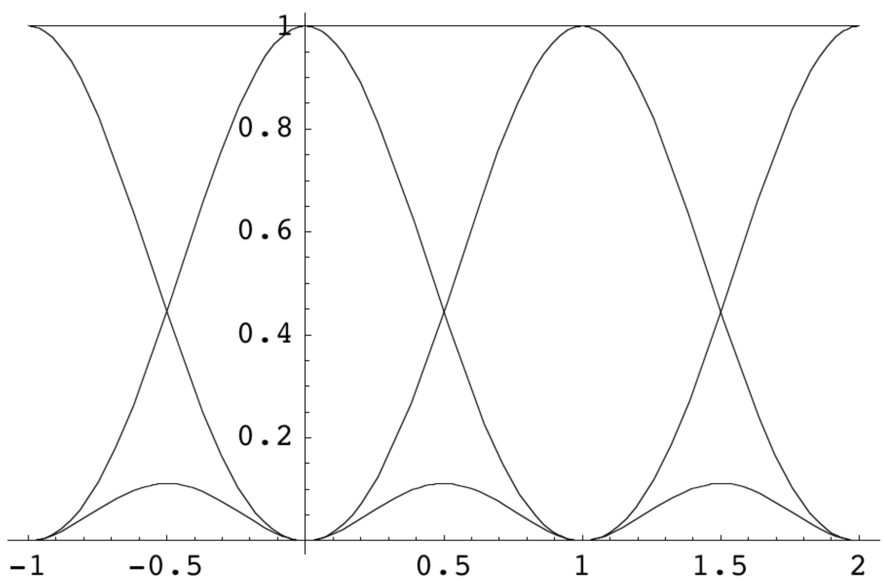

FIG. 3. Window squares summing to one.

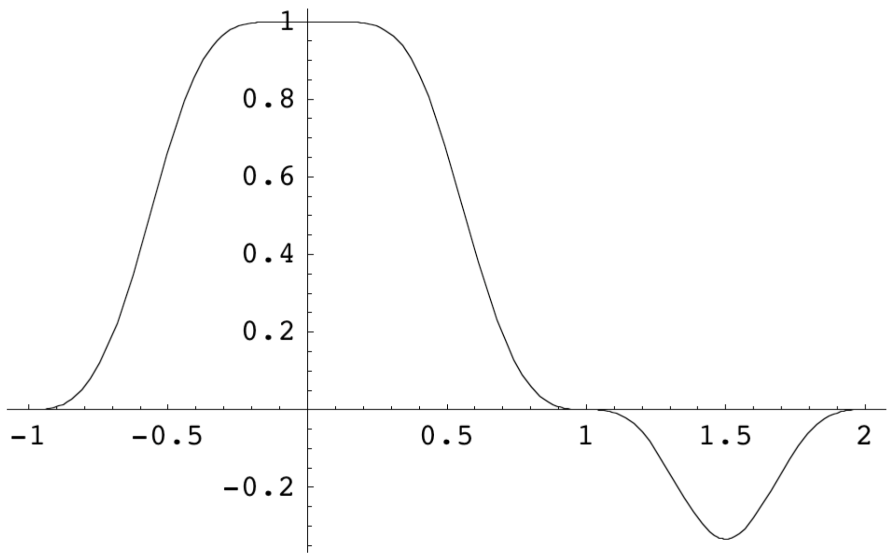

FIG. 4. A smoothed 3-overlap window.

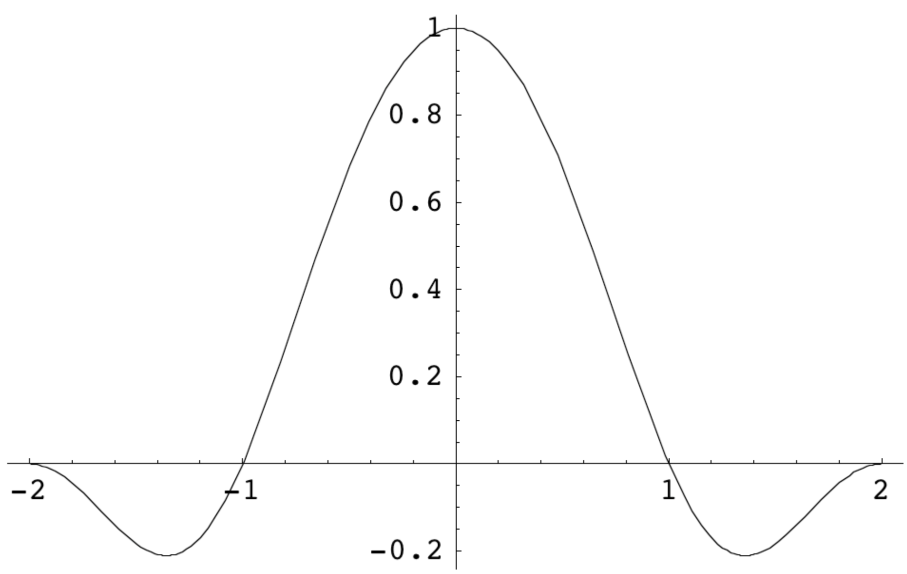

FIG. 5. A 4-overlap, symmetric, energy-preserving window.

For general even integer N, a symmetric mother-window supported on interval $[-N/2, N/2]$ could take the form

$$w(t) = \sum_{k=0}^{N-1} a_k \cos(\frac{2k\pi}{N}t), \qquad \text{for } -N/2 \leq t \leq N/2, \tag{16}$$

for some choice of parameters $a_0, a_1, \ldots a_{N-1}$. The amplitude and energy condition reduce to the equations

$$\begin{aligned} Na_0 &= 1 \\ N(a_0^2 + \frac{a_1^2 + a_2^2 + \ldots + a_{N-1}^2}{2}) &= 1 \\ a_1 a_{N-1} + a_2 a_{N-2} + \ldots + a_{N/2-1} a_{N/2+1} + \frac{1}{2} a_{N/2}^2 &= 0 \end{aligned} \tag{17}$$

while a peak condition $w(0) = 1$ and continuity condition $w(N/2) = 0$ are equivalent to

$$\begin{aligned} a_0 + a_1 + a_2 + \ldots + a_{N-1} &= 1 \\ a_0 - a_1 + a_2 - \ldots - a_{N-1} &= 0. \end{aligned} \tag{18}$$

For $N = 6$, this is five equations in six unknowns, which we can solve as a one-parameter family of solutions, say:

$$\begin{aligned} a_0 &= 1/6 \\ a_1 &= (1 + \sqrt{5/2}\cos\theta)/6 \\ a_2 &= (1 + \sqrt{5/2}\sin\theta)/6 \\ a_3 &= 1/6 \\ a_4 &= (1 - \sqrt{5/2}\sin\theta)/6 \\ a_5 &= (1 - \sqrt{5/2}\cos\theta)/6, \end{aligned} \tag{19}$$

where the parameter θ can be chosen arbitrarily. For instance, Figure 6 shows the window obtained with $\theta = .6$ radians. This window has an advantage over the $N = 4$ solution in that the negative side lobes are smaller.

GENERAL SOLUTIONS

The particular solutions described above are simply special cases of a more general construction, which involves finding a parameterization of the intersection of a hyperplane with a hypersphere in N dimensional space. By assuming our mother-window is symmetric and supported in the interval $[-NT/2, NT/2]$, we may define the *periodic window functions* as the superposition of non-overlapping windows:

$$\widetilde{w}_n(t) = \sum_{k\in\mathbb{Z}} w_{n+kN}(t) = \sum_{k\in\mathbb{Z}} w(t - nT - kNT) \tag{20}$$

for $n = 0, 1, 2, \ldots, N-1$. Note that each of these periodic window functions consists simply of repeated copies of the mother-window, spaced far enough apart so as not to

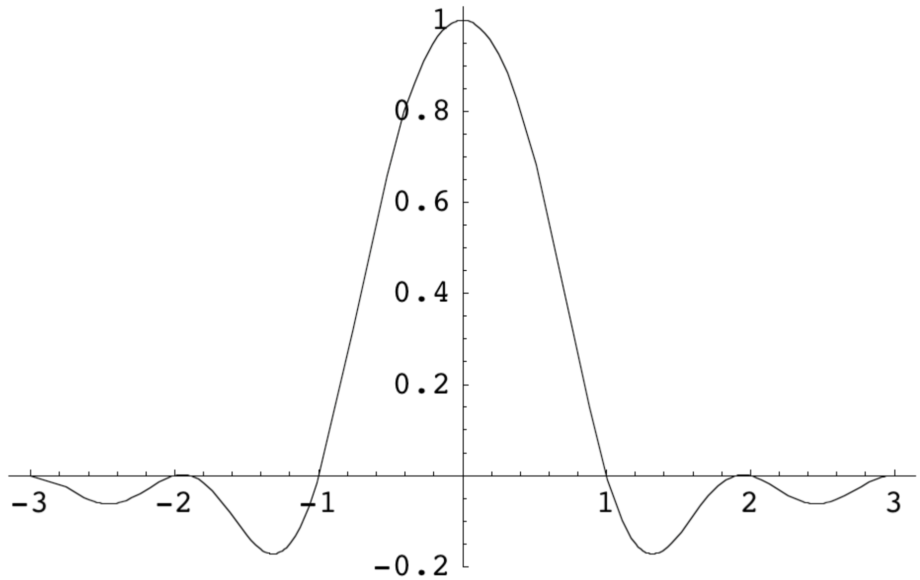

FIG. 6. A 6-overlap, symmetric, energy-preserving window.

overlap each other, and shifted to the right by an amount nT. The amplitude and energy criteria of Equations (4) and (7) can be restated in terms of the finite sums:

$$\sum_{n=0}^{N-1} \widetilde{w}_n(t) \;=\; 1 \tag{21}$$

and

$$\sum_{n=0}^{N-1} |\widetilde{w}_n(t)|^2 \;=\; 1 \tag{22}$$

for all $t \in \mathbb{R}$.

Define a parametrized curve, $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^N$, called the *window curve* using the periodic window functions as the individual coordinates:

$$\vec{\gamma}(t) \;=\; [\; \widetilde{w}_0(t) \;\; \widetilde{w}_1(t) \;\; \widetilde{w}_2(t) \;\; \ldots \;\; \widetilde{w}_{N-1}(t) \;]^T. \tag{23}$$

Note that $\vec{\gamma}(t)$ is a closed curve and exhibits an N-fold degree of symmetry since $\vec{\gamma}(t+T) = P\vec{\gamma}(t)$, where P is the permutation matrix generated by circularly shifting all the columns of the $N \times N$ identity matrix one step to the right:

$$P \;=\; \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{24}$$

The amplitude and energy criterion tell us the window curve $\vec{\gamma}(t)$ lies on the intersection of a hyperplane in $\mathbb{R}^N$ with a hypersphere, and thus is a parameterization of an $N-2$

dimensional sphere. Picking an appropriate curve and parameterization gives a suitable window function. Of course, in higher dimensions, there is more freedom in the choice of curves. Particular solutions are given in the sections above.

SUMMARY

We have shown a construction of window functions on the real line which preserve amplitude and energy in local reconstructions of time-sliced signals such as seismic data. Applications of these windows include signal processing methods that require conservation of these key quantities.

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