# Reflected $P_1S_1$ and critically refracted $P_1P_2S_1$ arrivals near a branch point – higher order approximations in terms of special functions

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## ABSTRACT

The high frequency solution to the problem of a $P_1S_1$ reflected wave at a plane interface between two isotropic homogeneous halfspaces, designated 1 and 2, in the vicinity of the branch point $p = p_2 = \alpha_2^{-1}$, is developed. In reflection seismology, sub-critical reflections are often all that is required or wanted for data processing. However, there is a possibility, when a high velocity contrast exists between two layers, resulting in the branch points due to the *P* and possibly even *S* – wave velocities in the second layer, that both may influence the recorded seismic traces at relatively small offsets. Apart from a distorted (compared to the zero order geometrical optics approximation to the arrival) wavelet, the presence of critically refracted (head) waves may possibly be seen on the traces. If the time duration of the wavelet is $t_{pulse}$, the reflected and critically refracted waves are collectively designated as the *interference wave* in the corresponding offset range. Only the occurrence of a critically refracted *P* – wave will be considered here. The possibility, given the proper velocity distribution, of a critically refracted wave due to the branch point at $p = p_4 = \beta_2^{-1}$ may be derived in a similar manner.


## INTRODUCTION

Both the acoustic and elastic cases of a saddle point in the vicinity of a branch point for a $P_1P_1$ reflected wave have been treated in a comprehensive manner in the literature, for example, Červený and Ravindra (1970) and Brekhovskikh (1980). In each of these works a different conformal mapping is introduced so as to incorporate a real parameter in the solutions by high frequency methods involving the evaluation of integrals of the Sommerfeld type. After stating this, it should be noted that the motivation for considering the $P_1S_1$ reflected arrival is partly due to the fact that in the $P_1P_1$ reflected case the saddle point location may be obtained analytically, while in the $P_1S_1$ case this must be done numerically.

Another reason for revisiting this problem type is the increasingly apparent fact that anelasticity plays a significant role in seismic wave propagation in media related to hydrocarbon deposits. To obtain analytical expressions for wave types, such as the reflected $P_1P_1$, $P_1S_1$, and related critically refracted waves, a thorough understanding of the formalism required to obtain solutions for these problems in the elastic case is an important and useful precondition. In the elastic case the saddle points and branch point lie on the real axis of the complex *p*-plane, while in the anelastic problem, they may be located anywhere in that angular part, $\zeta$, of the first quadrant of the *p*-plane such that $0 \le \zeta < \pi/4$.

## SADDLE POINT APPROXIMATION NEAR A BRANCH POINT

The reflected $P_1S_1$ potential at an interface between two isotropic halfspaces may be written as (Aki and Richards, 1980)

$$\psi(r,z,\omega)=i\omega A(\omega)\int_0^{\infty}\left(\frac{1}{i\omega p}\frac{\beta_1}{\alpha_1}R_{P1S1}\right)J_0(\omega pr)\exp\left[i\omega(\eta_1 h+\xi_1 z)\right]\frac{p\,dp}{\eta_1} \qquad (1)$$

The point source of $P$ waves is located at a point $h$ above the interface in medium 1 while the receiver is also located in the upper half space a distance $z$ above the interface (Figure 1). The horizontal distance between the source and receiver is $r$, $p$ - the horizontal component of the slowness vector, which is used as the integration variable, $\omega$ - the circular frequency resulting from a Fourier time transform of the original scalar potential, and $J_n(\kappa)$ - the Bessel function of order $n$. The term $R_{P1S1}$ is the displacement reflection coefficient obtained from using displacement potentials, while the $(\cdot)$ quantity containing it is the reflection coefficient. With the definitions $p_1=\alpha_1^{-1}$, $p_2=\alpha_2^{-1}$, $p_3=\beta_1^{-1}$ and $p_4=\beta_2^{-1}$ the radicals, $\eta_j$ and $\xi_j$, $j$=1,2, may be written as

$$\eta_j=\left(p_j^2-p^2\right)^{1/2},\quad (j=1,2.)\quad \mathrm{Im}\left[\eta_j\right]\geq 0 \qquad (2)$$

$$\xi_j=\left(p_{j+2}^2-p^2\right)^{1/2}=\left(\beta_j^{-2}-p^2\right)^{1/2},\quad (j=1,2.)\quad \mathrm{Im}\left[\xi_j\right]\geq 0 \qquad (3)$$

The particle displacement is obtained from the $P_1S_1$ reflected displacement potential using the relation

$$\mathbf{u}(r,z,\omega)=(u,0,w)=\nabla\times\nabla\times(0,0,\psi), \qquad (4)$$

(Aki and Richards, 1980) or, in rotationally invariant cylindrical coordinates, as

$$\mathbf{u}(r,z,\omega)=(u,w)=\left(\frac{\partial^2\psi}{\partial r\partial z},\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)\right) \qquad (5)$$

yielding the horizontal component of particle displacement as

$$u(r,z,\omega)=i\omega A(\omega)\int_0^{\infty}\left(\frac{1}{i\omega p}\frac{\beta_1}{\alpha_1}R_{P1S1}\right)\left(-i\omega^2 p\xi_1\right)J_1(\omega pr)\exp\left[i\omega(\eta_1 h+\xi_1 z)\right]\frac{p\,dp}{\eta_1}, \qquad (6)$$

and the vertical component:

$$w(r,z,\omega)=i\omega A(\omega)\int_0^{\infty}\left(\frac{1}{i\omega p}\frac{\beta_1}{\alpha_1}R_{P1S1}\right)\left(\omega^2 p^2\right)J_0(\omega pr)\exp\left[i\omega(\eta_1 h+\xi_1 z)\right]\frac{p\,dp}{\eta_1}. \qquad (7)$$

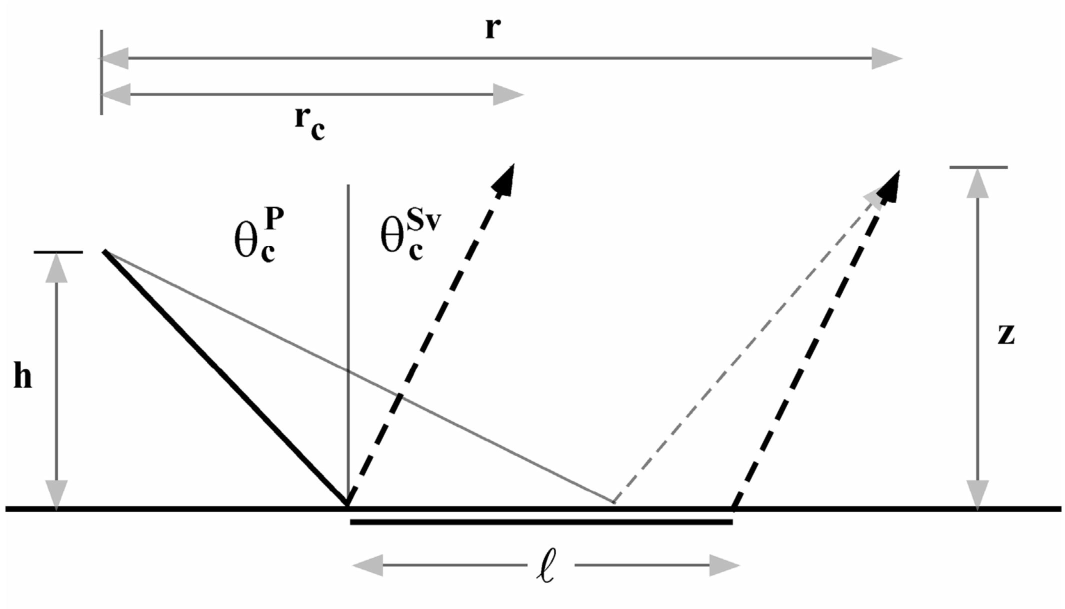


Figure 1. A schematic of a general reflected $P_1S_1$ arrival (light), the point at which critical refraction begins, $r_c$ and a general critically refracted arrival, $P_1P_2S_1$, in the first (upper) medium where the ray has travelled a distance $\ell$ along the interface in the second (lower) medium before returning to be recorded as a disturbance in the upper medium. The quantities $\theta_c^P$ and $\theta_c^{S_V}$ are the critical $P$ and $S_V$ angles.

Only the horizontal component, $u(r,z,\omega)$ will be considered here in detail as the vertical component, $w(r,z,\omega)$, follows the same derivation route. Modifying the integral by replacing the first order Bessel function in equation (4) by the Hankel function of order 1 and type 1, equation (6) becomes (Abramowitz and Stegun, 1980)

$$u(r,z,\omega) = -i\omega^2\left(\frac{\beta_1}{\alpha_1}\right)\frac{A(\omega)}{2}\int_{-\infty}^{\infty} R_{P1S1}(p)\,H_1^{(1)}(\omega pr)\exp\left[i\omega(\eta_1 h+\xi_1 z)\right]\frac{p\,\xi_1\,dp}{\eta_1} \qquad (8)$$

Introducing the asymptotic expansion for $H_1^{(1)}(\omega pr)$ and retaining only the first term in that expansion,

$$H_1^{(1)}(\omega pr) \sim \sqrt{\frac{2}{\pi\omega p r}}\,\exp\left[i(\omega pr - 3\pi/4)\right], \qquad (9)$$

results in equation (8) having the form

$$u(r,z,\omega)=\omega^{3/2}\left(\frac{\beta_1}{\alpha_1}\right)\frac{A(\omega)}{\sqrt{2\pi r}}e^{-i\pi/4}\int_{s.p.} R_{P1S1}(p)\,\exp\left[i\omega\left(rp+\eta_1 h+\xi_1 z\right)\right]\frac{p^{1/2}\,\xi_1\,dp}{\eta_1} \tag{10}$$

where "$s.p.$" indicates that the integral will be evaluated at the saddle point (Figure 2), given as the solution of

$$\frac{df(p)}{dp}=0 \text{ at } p=p_0, \quad \text{with} \quad f(p)=rp+\eta_1 h+\xi_1 z \tag{11}$$

so that

$$\left.\frac{df(p)}{dp}\right|_{p=p_0}=r-p_0\left(\frac{h}{\tilde{\eta}_1}+\frac{z}{\tilde{\xi}_1}\right)=0\,. \tag{12}$$

A quantity superscripted with a tilde is evaluated at the saddle point, $p_0$. It is convenient to introduce the change of variable describing a saddle point contour, in terms of the real variable $y$,

$$\left(p_1^2-p^2\right)^{1/2}=\left(p_1^2-p_0^2\right)^{1/2}-ye^{-i\pi/4}, \quad y-\text{real}, \quad -\infty<y<\infty\,. \tag{13}$$

so that

$$\frac{p\,dp}{\eta_1}=dy\,e^{-i\pi/4}\,. \tag{14}$$

This mapping is a variation of that used by Červený and Ravindra (1970) under the assumption that the major contribution to the saddle point contour integral occurs at $p=p_0\ (y=0)$. Employing the standard saddle point procedure and expanding $f(p)$ in a Taylor series about $p=p_0\ \ (y=0)$ results in

$$f(p)\approx f(p_0)+\left[\frac{df}{dp}\frac{dp}{dy}\right]_{y=0} y+\left[\frac{d^2 f}{dp^2}\left(\frac{dp}{dy}\right)^2+\frac{df}{dp}\frac{d^2 p}{dy^2}\right]_{y=0} y^2+\cdots \tag{15}$$

Since $df/dp\equiv 0$ at the saddle point, equation (15) reduces to

$$f(p)\approx f(p_0)+\left[\frac{d^2 f}{dp^2}\left(\frac{dp}{dy}\right)^2\right]_{y=0} y^2+\cdots \tag{16}$$

where the quantities requiring definition are

$$\left.\frac{d^2 f(p)}{dp^2}\right|_{p=p_0} = -\left(\frac{h}{\alpha_1^2\tilde{\eta}_1^3} + \frac{z}{\beta_1^2\tilde{\xi}_1^3}\right) \tag{17}$$

and

$$\left(\frac{dp}{dy}\right)^2_{p=p_0} = \left(\frac{\tilde{\eta}_1}{p_0}\right)^2 e^{-i\pi/2} \tag{18}$$

The result of this is that (15) may be written as

$$i\omega f(p) \approx i\omega f(p_0) - \omega\left[\left(\frac{h}{\alpha_1^2\tilde{\eta}_1^3} + \frac{z}{\beta_1^2\tilde{\xi}_1^3}\right)\left(\frac{\tilde{\eta}_1}{p_0}\right)^2\right]_{y=0} y^2 + \cdots \tag{19}$$

or in a simplified notation

$$i\omega f(p) \approx i\omega\tau_{PS} - \omega a^2 y^2 + \cdots . \tag{20}$$

Here $\tau_{PS}$ is the travel time of the $P_1S_1$ reflected wave from source to receiver $\left(\tau_{PS} = f(p_0)\right)$ and $a^2$ is implied from equation (19). As previously mentioned a tilde over any quantity indicates that it is to be evaluated at the saddle point. Introducing the modified $P_1S_1$ reflection coefficient $R_{P1S1}(p)$ from equation (A.19) together with the definition of $\eta_2$ from equation (A.31) and $\varepsilon_\pm$ from (A.29) into equation (10) has

$$\begin{aligned} u(r,z,\omega) = \omega^{3/2}\left(\frac{\beta_1\tilde{\xi}_1}{\alpha_1}\right)\frac{A(\omega)}{\sqrt{2\pi r p_0}} e^{i\omega\tau_{PS}} e^{-i\pi/2}\Bigg[ C_1(p_0)\int_{-\infty}^{\infty} e^{-\omega a^2 y^2}dy + \\ C_2(p_0)\varepsilon_+\int_{-\infty}^{\infty}\left(\varepsilon_-^2 + ye^{-i\pi/4}\right)^{1/2} e^{-\omega a^2 y^2}dy + \Bigg] \end{aligned} \tag{21}$$

This equation may be partially evaluated as

$$\begin{aligned} u(r,z,\omega) = \omega^{3/2}\left(\frac{\beta_1\tilde{\xi}_1}{\alpha_1}\right)\frac{A(\omega)}{\sqrt{2\pi r p_0}} e^{i\omega\tau_{PS}} e^{-i\pi/2}\Bigg[ C_1(p_0)\left(\frac{\pi}{\omega a^2}\right)^{1/2} + \\ C_2(p_0)\varepsilon_+\int_{-\infty}^{\infty}\left(\varepsilon_-^2 + ye^{-i\pi/4}\right)^{1/2} e^{-\omega a^2 y^2}dy \Bigg] \end{aligned} \tag{22}$$

Consider the integral in equation (22), $I_{s.p.}$, separately as

$$I_{s.p.} = \varepsilon_+\int_{-\infty}^{\infty}\left(\varepsilon_-^2 + ye^{-i\pi/4}\right)^{1/2} e^{-\omega a^2 y^2}dy . \tag{23}$$

Introduce the variable change

$$s\,e^{-i\pi/4} = \left(\varepsilon_-^2 + y e^{-i\pi/4}\right) \tag{24}$$

so that

$$dy = ds \tag{25}$$

and

$$\exp\left[-\omega a^2 y^2\right] = \exp\left[-\omega a^2\left(s^2 - 2s\varepsilon_-^2 e^{i\pi/4}\right)\right]\exp\left[-i\omega a^2 \varepsilon_-^4\right]. \tag{26}$$

Thus $I_{s.p.}$ may be written as

$$I_{s.p.} = \varepsilon_+ \exp\left[-i\omega a^2\varepsilon_-^4\right] e^{-3i\pi/8} \int_{-\infty}^{\infty} (is)^{1/2} \exp\left[-\omega a^2 s^2 + i2s\omega a^2 \varepsilon_-^2 e^{-i\pi/4}\right] ds \tag{27}$$

whose solution may be obtained in terms of the Parabolic Cylinder Function of order $"1/2"$ as (Gradshteyn and Rhyzik, 1980)

$$I_{s.p.} = \varepsilon_+ e^{-3i\pi/8}\left[\frac{\pi^2}{2\omega^3 a^6}\right]^{1/4} \exp\left[-\frac{i\omega a^2\varepsilon_-^4}{2}\right] D_{1/2}\left[\sqrt{\omega}a\varepsilon_-^2(1-i)\right]. \tag{28}$$

The horizontal component of the $P_1S_1$ reflected displacement may then be written as

$$\begin{aligned} u(r,z,\omega) = \omega\left(\frac{\beta_1\tilde{\xi}_1}{\alpha_1}\right)\frac{A(\omega)}{\sqrt{2r\,p_0 a^2}} e^{i\omega\tau_{PS}} e^{-i\pi/2}\Big[C_1(p_0) + \\ C_2(p_0)\varepsilon_+ e^{-3i\pi/8}\frac{1}{\left(2\omega a^2\right)^{1/4}}\exp\left[-\frac{i\omega a^2\varepsilon_-^4}{2}\right] D_{1/2}\left[\sqrt{\omega}a\varepsilon_-^2(1-i)\right]\Big] \end{aligned} \tag{29}$$

Adding and subtracting the term $C_2(p_0)\tilde{\eta}_2$ from equation (29) yields

$$\begin{aligned} u(r,z,\omega) = \omega\left(\frac{\beta_1\tilde{\xi}_1}{\alpha_1}\right)\frac{A(\omega)}{\sqrt{2r\,p_0 a^2}} e^{i\omega\tau_{PS}} e^{-i\pi/2}\Big[R_{P1S1} + \\ C_2(p_0)\varepsilon_+\left\{e^{-3i\pi/8}\frac{1}{\left(2\omega a^2\right)^{1/4}}\exp\left[-\frac{i\omega a^2\varepsilon_-^4}{2}\right] D_{1/2}\left[\sqrt{\omega}a\varepsilon_-^2(1-i)\right] + i\varepsilon_-\right\}\Big] \end{aligned} \tag{30}$$

The above analysis of the $PS_V$ reflection from a plane solid – solid interface in welded contact is accomplished using a modified saddle point approach for the special case of a saddle point near a branch point. Apart from the zero order geometrical optics contribution to the reflected arrival there is an additional correction term to account for the saddle point being in the vicinity of a branch point. The effect of the correction term may be seen at both pre- and post- critical offsets and becomes less of a factor at offsets removed from the critical distance. If the high frequency approximation $D_{1/2}(z) \sim z^{1/2} e^{-z^2/4}$ is introduced into equation (30) the $[\cdot]$ term is zero, leaving only the zero order term in the reflected wave solution.

In addition, for offsets past the critical point there is another arrival; the critically refracted $P_1P_2S_1$ wave, which will be considered in the next section. For a wavelet of finite time duration there is a related range of offsets in which the reflected and critically refracted wave interfere with one another. This offset range is termed the interference zone. In this region the two arrivals are often treated as a single arrival.

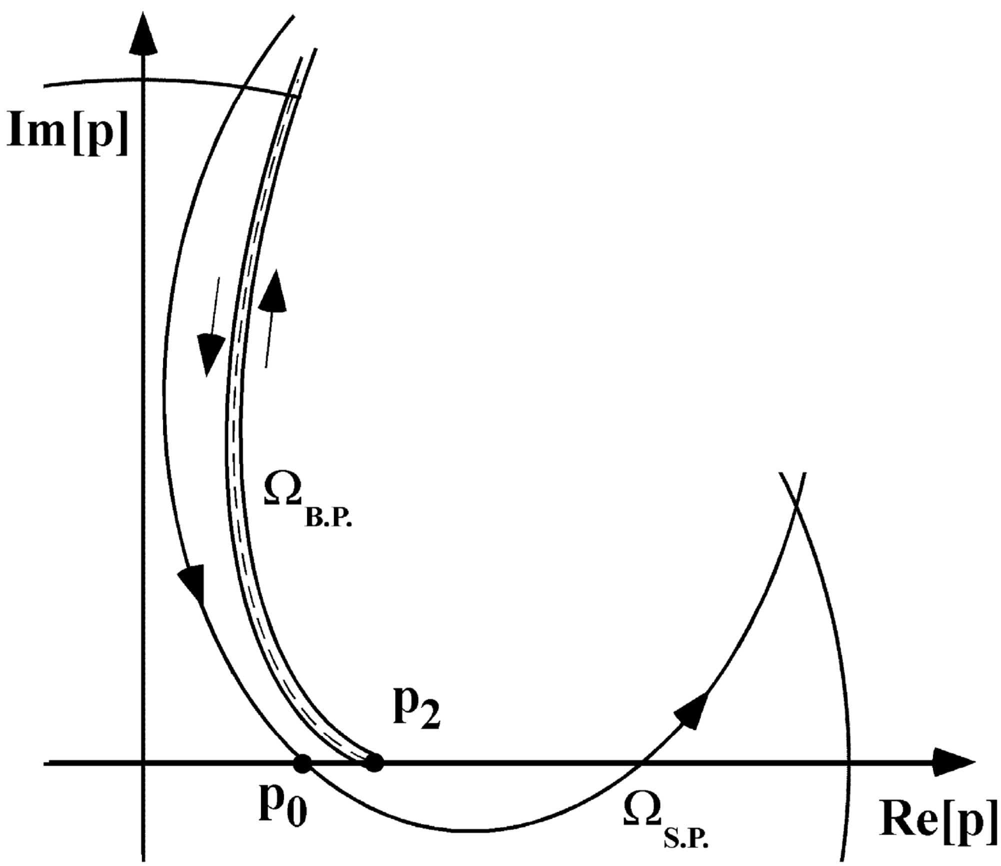


Figure 2. The saddle point and branch point contours corresponding to the reflected $P_1S_1$ reflected ray $(p_0)$ and the critically refracted $P_1P_2S_1$ ray. The parameterizations of the contours are given by equations (13) and (32). The parts of the total contour are designated as $\Omega_{S.P.}$ and $\Omega_{B.P.}$, respectively.

## HEAD WAVE $P_1P_2S_1$

The $P_1P_2S_1$ critically refracted (head) wave which exists if the $P$ – wave velocity $(\alpha_1)$ and hence the $S_V$ – wave velocity $(\beta_1)$ in medium 1 (upper medium) are less than the $P$ – wave velocity $(\alpha_2)$ in medium 2 (lower medium). The velocity distribution assumed is $\alpha_2 > \alpha_1 > \beta_2 > \beta_1$. If this distribution is such that $\alpha_2 > \beta_2 > \alpha_1 > \beta_1$ there is the possibility of two critically refracted waves, $P_1P_2S_1$ and $P_1S_2S_1$. There are geometrical rays paths associated with the critical refracted arrival(s) in both distributions.

The contribution to the generalized reflected wave field due to the reflection of a $P$ – wave from the layer – half space boundary is obtained from the integration around the branch cut corresponding to the component of the horizontal slowness vector $p$ being equal to $p = p_2 = \alpha_2^{-1}$. The problem is stated in equation (10) of the preceding section for the horizontal component of particle displacement, if the saddle point contour is replaced by the branch cut contour, i.e.

$$u(r,z,\omega) = \omega^{3/2}\left(\frac{\beta_1}{\alpha_1}\right)\frac{A(\omega)}{\sqrt{2\pi r}}e^{-i\pi/4}\int_{b.c.} R_{P1S1}(p)\,\exp\left[i\omega(rp+\eta_1 h+\xi_1 z)\right]\frac{p^{1/2}\,\xi_1\,dp}{\eta_1} \quad (31)$$

with "*b.c.*" indicating *branch cut*. In this case the change of variable used to produce the branch cut integral contribution is

$$\left(p_1^2 - p^2\right)^{1/2} = \left(p_1^2 - p_2^2\right)^{1/2} - ye^{-i\pi/4}, \quad y-\text{real}, \quad 0 < y < \infty \quad (32)$$

The radical $\eta_2$ is again the quantity of concern as it changes sign (phase) by a factor of $\pi$ from one side of the branch cut to the other. Thus for this integral to have any finite value the integral around the branch cut must be such that it is twice the integral of $y$ from $0$ to $\infty$. It should also be noted that the major contribution to the branch cut integral occurs in the vicinity of $p = p_2$ $(y=0)$. In the vicinity of $p = p_2$, the radical $\eta_2$ may be approximated, using the variable change defined in equation (32), in the following manner

$$\eta_2 \approx \sqrt{2}\left[-ye^{-i\pi/4}\right]^{1/2}\left(p_1^2 - p_2^2\right)^{1/4} = \sqrt{2}e^{3i\pi/8}y^{1/2}\left(p_1^2 - p_2^2\right)^{1/4} \quad (33)$$

Quantities that will be required in the computations that follow are

$$\frac{dp}{\eta_1} = dye^{-i\pi/4} \quad \rightarrow \quad \frac{dp}{dy} = \eta_1 e^{-i\pi/4} \quad \text{and} \quad \frac{d^2p}{dy^2} = -\frac{p_1^2 e^{-i\pi/2}}{p^3} \quad .$$

The reflection coefficient in the proximity of the branch point is approximated as (Equation A.22)

$$R_{P1S1}(p) \approx C_1(p_2) + C_2(p_2)\eta_2(p) \tag{34}$$

The second term is the only one that makes a contribution to the branch cut integral as $\eta_2(p)$ differs by $\pi$ on either side of the branch cut. The first term does not, so that the integration along one side of the branch cut cancels the integration along the other.

The Taylor series expansion of the exponential term differs somewhat in this problem when compared to the saddle point case as here, $df(p)/dp \neq 0$, so that with

$$f(p) = rp + \eta_1 h + \xi_1 z \tag{35}$$

$$f(p) \approx f(p_2) + \left[\frac{df}{dp}\frac{dp}{dy}\right]_{y=0} y + \left[\frac{d^2 f}{dp^2}\left(\frac{dp}{dy}\right)^2 + \frac{df}{dp}\frac{d^2 p}{dy^2}\right]_{y=0} y^2 + \cdots \tag{36}$$

where the evaluation of terms in the series at $y = 0$ is the same as taking their value at $p = p_2$. Thus

$$i\omega f(p) \approx i\omega\tau_{PPS} + \left[\omega\ell\hat{\eta}_1 e^{i\pi/4}\right] y - \omega a^2 y^2 + \cdots \tag{37}$$

so that equation (31) may be written more compactly as

$$u(r,z,\omega) = 2\omega\left(\frac{\beta_1}{\alpha_1}\right) A(\omega)\, e^{-i\pi/8}\left[\frac{\omega^2 p_2\left(p_1^2 - p_2^2\right)\hat{\xi}_1^4}{\pi^2 r^2}\right]^{1/4} e^{i\omega\tau_{PPS}}\, C_2(p_2) \times \int_0^\infty y^{1/2} \exp\left[-\omega a^2 y^2 - \omega\ell\hat{\eta}_1 e^{-3i\pi/4} y\right] dy \tag{38}$$

where the quantities $a^2$ and $\ell$ are defined in the following manner

$$a^2 = \left(\frac{h}{\alpha_1^2\hat{\eta}_1^3} + \frac{z}{\beta_1^2\hat{\xi}_1^3}\right)\hat{\eta}_1^2 + \frac{\ell p_1^2}{p_2^3} \tag{39}$$

$$\ell = r - p_2\left(\frac{h}{\hat{\eta}_1} + \frac{z}{\hat{\xi}_1}\right) \quad \text{or} \quad \ell = r - \frac{h\sin\theta_c^P}{\cos\theta_c^P} - \frac{z\sin\theta_c^{S_V}}{\cos\theta_c^{S_V}} = r - r_c\,. \tag{40}$$

As $r$ is the source receiver offset, then $r_c$ is the offset at which the critically refracted arrival first appears and which defines the angles $\theta_c^P$ and $\theta_c^{S_V}$. Now, consider the integral

$$I_H = \int_0^\infty y^{1/2} \exp\left[-\omega a^2 y^2 - \omega \ell \hat{\eta}_1 e^{-3i\pi/4} y\right] dy \tag{41}$$

From Gradshteyn and Ryzik (1980), the following solution may be obtained in terms of the Parabolic Cylinder Function of order $-3/2$ as

$$I_H = \frac{\Gamma(3/2)}{\left(2\omega a^2\right)^{3/4}} \exp\left[\frac{i\omega e^{-i\pi/2} \ell^2 \hat{\eta}_1^2}{8a^2}\right] D_{-3/2}\left[\sqrt{\frac{\omega}{2}} \frac{\ell \hat{\eta}_1 e^{-3i\pi/4}}{a}\right] \tag{42}$$

so that the horizontal component of the critically refracted wave has the form

$$\begin{aligned} u(r,z,\omega) = \left(\frac{\beta_1}{\alpha_1}\right) A(\omega) e^{-i\pi/8} \left[\frac{\omega^3 p_2 \left(p_1^2 - p_2^2\right) \hat{\xi}_1^4}{2^3 a^6 r^2}\right]^{1/4} e^{i\omega\tau_{PPS}} C_2(p_2) \times \\ \exp\left[\frac{i\omega e^{-i\pi/2} \ell^2 \hat{\eta}_1^2}{8a^2}\right] D_{-3/2}\left[\sqrt{\frac{\omega}{2}} \frac{\ell \hat{\eta}_1 e^{-3i\pi/4}}{a}\right] \end{aligned} \tag{43}$$

Introducing the high frequency approximation for $D_{-3/2}(z)$ $\left[D_{-3/2}(z) \sim e^{-z^2/4} / z^{3/2}\right]$ yields

$$u(r,z,\omega) = -\left(\frac{\beta_1}{\alpha_1}\right) A(\omega) \frac{\hat{\xi}_1 p_2^{1/4} \left(p_1^2 - p_2^2\right)^{1/4}}{r^{1/2} \ell^{3/2} \hat{\eta}_1^{3/2}} e^{i\omega\tau_{PPS}} C_2(p_2) \tag{44}$$

This form of the expression for the horizontal component of the critically refracted wave is in agreement with formulae obtained employing less complex methods of solution. In the region of $p = p_2$ this form of the solution cannot be used as $\ell = 0$ here, yielding an infinite value for the displacement component.

## CONCLUSIONS

The zero order high frequency or geometrical optics approximation for the both the $P_1 P_1$ and $P_1 S_1$ reflected arrivals at an interface between two elastic media is questionable in the vicinity of a branch point. This refers not only to post critical reflections but also to pre-critical reflected arrivals. The reason for this is that the basic saddle point or stationary phase type solution assumes in its solution method that the only quantities that are rapidly varying are those in the exponential terms. For a saddle point in the vicinity of a branch point, say at $p = p_k$, the associated radical, $\zeta_k = \left(p_k^2 - p^2\right)^{1/2}$, also varies rapidly. As a consequence, it must be isolated in the integrand and its effects included in the solution. This was the main objective of this report in that a correction to the zero order high frequency approximation in terms of the Parabolic Cylinder Function of order $1/2$ was derived. This additional term resulted from the rewriting of the $R_{P1S1}$ reflection coefficient, presented in detail in Appendix A, and the equation obtained employed to get the modified solution for the horizontal component of the reflected $P_1S_1$ particle component displacement. The vertical component follows in a straightforward manner from this initial analysis. For completeness the critically refracted (head) wave arrival corresponding to the branch point was also considered. A higher order approximation in terms of the Parabolic Cylinder Function of order $-3/2$ was presented as well as the large argument formula.

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## APPENDIX A: DISPLACEMENT REFLECTION COEFFICIENT $\mathbf{R_{P1S1}}$

The plane wave particle displacement $P_1S_1$ reflection coefficient at a plane interface between two elastic media has the form (Aki and Richards, 1980)

$$R_{P1S1}\left(p\right)=\frac{-2\eta_1 p\alpha_1\left(ab+cd\eta_2\xi_2\right)}{\beta_1 D} \tag{A.1}$$

where $p$ is the horizontal component of the slowness vector. For completeness, all of the steps in the rewriting of the reflection coefficient $R_{P1S1}\left(p\right)$ will be detailed here. The preliminary quantities require definition

$$D = EF + GHp^2 \tag{A.2}$$

$$\eta_i = \left(\alpha_i^{-2} - p^2\right)^{1/2},\ i = 1, 2. \tag{A.3}$$

$$\xi_i = \left(\beta_i^{-2} - p^2\right)^{1/2},\ i = 1, 2. \tag{A.4}$$

$$E = b\eta_1 + c\eta_2 \tag{A.5}$$

$$F = b\xi_1 + c\xi_2 \tag{A.6}$$

$$G = a - d\eta_1\xi_2 \tag{A.7}$$

$$H = a - d\eta_2\xi_1 \tag{A.8}$$

with

$$a = \rho_2\left(1 - 2\beta_2^2 p^2\right) - \rho_1\left(1 - 2\beta_1^2 p^2\right) \tag{A.9}$$

$$b = \rho_2\left(1 - 2\beta_2^2 p^2\right) + 2\rho_1\beta_1^2 p^2 \tag{A.10}$$

$$c = \rho_1\left(1 - 2\beta_1^2 p^2\right) + 2\rho_2\beta_2^2 p^2 \tag{A.11}$$

$$d = 2\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right) \tag{A.12}$$

Near the branch point $p = \alpha_2^{-1}$, the radical $\eta_2$ is the quantity which varies most rapidly. For this reason any approximation to $R_{P1S1}\left(p\right)$ should reflect this. A standard manner in accomplishing this is rewrite equation (A.1) as

$$R_{P1P1}(p)=\frac{A_1(p)+A_2(p)\eta_2}{B_1(p)+B_2(p)\eta_2} \tag{A.13}$$

$$R_{P1S1}(p)=\frac{-\left[2\eta_1 p(\alpha_1/\beta_1)ab\right]-\left[2\eta_1 p(\alpha_1/\beta_1)cd\xi_2\right]\eta_2}{D} \tag{A.14}$$

$$\begin{aligned} D &= EF+GHp^2 \\ &=(b\eta_1+c\eta_2)F+G(a-d\eta_2\xi_1)p^2 \\ &=\left(b\eta_1 F+aGp^2\right)+\left(cF-dG\xi_1 p^2\right)\eta_2 \end{aligned} \tag{A.15}$$

$$R_{P1S1}(p)=\frac{\left[A_1(p)+A_2(p)\eta_2\right]}{\left[B_1(p)+B_2(p)\eta_2\right]} \tag{A.16}$$

where $A_i(p)$ and $B_i(p)\,(i=1,2)$ are determined from equation (A.1) and subsequent definitions, and are given explicitly below. The approximation to $R_{P1S1}(p)$ begins by multiplying equation (A.16) by unity, viz.,

$$R_{P1S1}(p)=\frac{A_1(p)B_1(p)-\left[A_1(p)B_2(p)-A_2(p)B_1(p)\right]\eta_2}{\left[B_1^2(p)-B_2^2(p)\eta_2^2\right]} \tag{A.17}$$

$$\begin{aligned} A_1(p)&=-2\eta_1 p(\alpha_1/\beta_1)ab \\ A_2(p)&=-2\eta_1 p(\alpha_1/\beta_1)cd\xi_2 \\ B_1(p)&=b\eta_1 F+aGp^2 \\ B_2(p)&=cF-dG\xi_1 p^2 \end{aligned} \tag{A.18}$$

which results in an equation of the form

$$R_{P1S1}(p)=C_1(p)+C_2(p)\eta_2 \tag{A.19}$$

with

$$C_1(p)=\frac{A_1(p)}{B_1(p)} \tag{A.20}$$

$$C_2(p)=\frac{-\left[A_1(p)B_2(p)-A_2(p)B_1(p)\right]}{\left[B_1(p)\right]^2} \tag{A.21}$$

where $C_i(p)\,(i=1,2)$ has been written in terms of $A_i(p)$ and $B_i(p)\,(i=1,2)$.

In the vicinity of $p=\alpha_2^{-1}=p_2$ it is not unreasonable, as $C_1(p)$ and $C_2(p)$ may be shown to be slowly varying functions of $p$, to make the following approximation:

$$R_{P1S1}(p) \approx C_1(p_2) + C_2(p_2)\eta_2(p) \quad . \tag{A.22}$$

However, if the problem of a saddle point in the vicinity of the branch point at $p = \alpha_2^{-1} = p_2$ is being considered, it is advisable to evaluate $C_1(p)$ and $C_2(p)$ at the saddle point, such that

$$R_{P1S1}(p) \approx C_1(p_0) + C_2(p_0)\eta_2(p) \tag{A.23}$$

Even in the isotropic homogeneous acoustic case, the problem of $P_1S_1$ reflection at the interface between two acoustic media results in a caustic, providing the phase of the reflection coefficient is taken into consideration. This is a consequence of the rapid variation of $\eta_2$ in this region.

In the region $p \approx \alpha_2^{-1} = p_2$ and $p > \alpha_2^{-1} = p_2$ the reflected wave and the critically refracted wave due to the branch point at $p = \alpha_2^{-1} = p_2$ produce what is termed the interference wave.

Approximation to $\eta_2$ near $p = p_2$ yields:

$$\begin{aligned} \eta_2 &= \left(p_2^2 - p^2\right)^{1/2} = \left(p_2 + p\right)^{1/2}\left(p_2 - p\right)^{1/2} \\ &\approx \left(2p_2\right)^{1/2}\left(p_2 - p\right)^{1/2}. \end{aligned} \tag{A.24}$$

For the case of a saddle point in the vicinity of a branch point the integration contour often used is of the form

$$\left(p_1^2 - p^2\right)^{1/2} = \left(p_1^2 - p_0^2\right)^{1/2} - ye^{-i\pi/4}, \quad y - \text{real}, \quad -\infty > y > \infty \tag{A.25}$$

It is required during the course of computations to obtain a useable expression for $\eta_2 = \left(p_2^2 - p^2\right)^{1/2}$ other than that given in equation (A.24).

$$\begin{aligned} \eta_2 &= \left(p_2^2 - p^2\right)^{1/2} \\ &= \left[\left(p_1^2 - p^2\right)^{1/2} - \left(p_1^2 - p_2^2\right)^{1/2}\right]^{1/2}\left[\left(p_1^2 - p^2\right)^{1/2} + \left(p_1^2 - p_2^2\right)^{1/2}\right]^{1/2} \end{aligned} \tag{A.26}$$

Introducing equation (A.25) into (A.26) yields

$$\eta_2 = \left[\left(p_1^2 - p_0^2\right)^{1/2} - \left(p_1^2 - p_2^2\right)^{1/2} - ye^{-i\pi/4}\right]^{1/2} \times$$
$$\left[\left(p_1^2 - p_0^2\right)^{1/2} + \left(p_1^2 - p_2^2\right)^{1/2} - ye^{-i\pi/4}\right]^{1/2} \qquad \text{(A.27)}$$

In the high frequency limit for a branch point near the saddle point at $p = p_1$ this may be approximated as

$$\eta_2 = \left[\left(p_1^2 - p_0^2\right)^{1/2} + \left(p_1^2 - p_2^2\right)^{1/2}\right]^{1/2} \left[\left(p_1^2 - p_0^2\right)^{1/2} - \left(p_1^2 - p_2^2\right)^{1/2} - ye^{-i\pi/4}\right]^{1/2} \qquad \text{(A.28)}$$

for $p_0 < p_2$, and

$$\eta_2 = -i\left[\left(p_1^2 - p_0^2\right)^{1/2} + \left(p_1^2 - p_2^2\right)^{1/2}\right]^{1/2} \left[\left(p_1^2 - p_2^2\right)^{1/2} - \left(p_1^2 - p_0^2\right)^{1/2} + ye^{-i\pi/4}\right]^{1/2} \qquad \text{(A.29)}$$

for $p_0 > p_2$ as a branch point has been passed. Using (A.28) for $p_0 < p_2$ requires that

$$\arg\left[\left(p_1^2 - p_2^2\right)^{1/2} - \left(p_1^2 - p_0^2\right)^{1/2} + ye^{-i\pi/4}\right]^{1/2} = \pi/2 \qquad \text{(A.30)}$$

In reduced notation, equation (A.29) has the form

$$\eta_2 = -i\varepsilon_+ \left[\varepsilon_-^2 + ye^{-i\pi/4}\right]^{1/2} \qquad \text{(A.31)}$$

where the definitions of $\varepsilon_\pm$ may be obtained from (A.29).

## APPENDIX B: PARABOLIC CYLINDER FUNCTION

In the problem under consideration here, the Parabolic Cylinder Function (PCF) of complex argument, $z$, of a special type, specifically, $z = (1+i)y$, where $y$ is a real quantity, is considered. The PCF is required for all combinations of positive and negative $z$ and its complex conjugate. The problem is simplified somewhat in that if a solution can be obtained for $D_{-3/2}\left[(1+i)y\right]$, $y \geq 0$, all other required order types, specifically $\nu = 1/2$, may be obtained using the following relations:

$$D_p\left(z^*\right) = D_p^*(z) \quad \text{and} \quad D_p(-z) = (-1)^p D_p(z) \qquad \text{(B.1)}$$

together with the functional relation

$$D_p(z) = \frac{\Gamma(p+1)}{(2\pi)^{1/2}}\left[e^{i\pi p/2} D_{-p-1}(iz) + e^{i\pi p/2} D_{-p-1}(-iz)\right] \qquad \text{(B.2)}$$

with "*" indicating complex conjugate (Gradshteyn and Rhyzik, 1980).

The PCF may be written in the form of an initial value problem of a second order ordinary differential equation. For an arbitrary order $p$ and argument $z$ the following equation is valid

$$\frac{d^2 D_p}{dz^2} + \left(p + 1/2 - z^2/4\right) D_p = 0, \tag{B.3}$$

subject to the initial conditions

$$D_p(0) = \frac{2^{p/2}\pi^{1/2}}{\Gamma\left[(1-p)/2\right]} \tag{B.4}$$

and

$$\frac{dD_p(0)}{dz} = -\frac{2^{p/2}(2\pi)^{1/2}}{\Gamma\left[-p/2\right]} \tag{B.5}$$

Differential equation problems of this special formulation may be simplified for solution purposes if the following variable changes are made

$$D_p(z) = u(z) + i\,v(z)\,, \tag{B.6}$$

$u$ and $v$ are real functions of $y$, and

$$z = (1+i)\,y\,, \tag{B.7}$$

$y$ is a real variable. The above changes of variables yield the following initial value problem involving a system of first order ordinary differential equations

$$\frac{dr}{dy} = 2\,(p + 1/2)\,v - y^2 u \tag{B.8}$$

$$\frac{ds}{dy} = -2\,(p + 1/2)\,u - y^2 v \tag{B.9}$$

$$\frac{du}{dy} = r \tag{B.10}$$

$$\frac{dv}{dy} = s \tag{B.11}$$

The initial conditions for this problem at $y = 0$ are

$$u(0) = \frac{2^{p/2}\pi^{1/2}}{\Gamma\left[(1-p)/2\right]} \tag{B.12}$$

$$v(0) = 0 \tag{B.13}$$

$$\frac{du(0)}{dy} = \frac{dv(0)}{dy} = -\frac{2^{p/2}(2\pi)^{1/2}}{\Gamma\left[-p/2\right]} \tag{B.14}$$

After an evaluation of numerical ODE solvers, Gear's method (Gear, 1971) for the numerical solution of systems of ordinary differential equations (initial value problems) was chosen as it provides acceptable results, partially as a result of the fact that the Jacobian used in the solution of equation (9) may be obtained analytically. As an adaptive finite difference grid method is used, a reasonably accurate solution in the problematic area, $y \approx 0$, can be expected. There are a number of routines in various mathematical computing libraries, which are based on the above method, which produce little variation in the results and require about equal amounts of computational time. The algorithm used here is the IMSL routine DGEAR.

This approach of solving an initial value problem comprised of first order ordinary differential equations (ODEs) produces accurate results. Rather than compute the solution once for a range of values, tabulate the results and write to an external device for later recall, it is more efficient to compute the function values as required and enhancing computational speed by using the restart option contained in most of the algorithms. Apart from an independent check of the accuracy of the ODE solver, alternative methods are sought, if not in the full range of interest of the independent variable, then at least in part, and the ODE solution used as required in all other areas.

It has been found that by retaining the first three terms in the asymptotic expansion of $D_{-3/2}\left[(1+i)\,y\right]$ for $y \geq 0$, and using a 64 bit word in the calculations for both the ODE method and the asymptotic expansion, 10 to 13 floating point digits of accuracy are obtained. The asymptotic expansion, valid in the first quadrant of the complex $z$-plane for $|z| >> 1$, $|z| >> p$, is (Gradshteyn and Rhyzik, 1980)

$$D_p(z) \approx e^{-z^2/4} z^p \left(1 - \frac{p(p+1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4 z^4} - \cdots \right) \quad . \tag{B.15}$$

The ODE solution to the parabolic cylinder function was sought at increments of $\Delta y = 0.01$ in the range $0.0 < y \leq 8.0$. The user supplied (estimated) $\Delta y_{\min}$ was set to $1.0\times10^{-04}$ which the routine DGEAR modified to $1.0\times10^{-07}$ in the vicinity of $y \approx 0.0$. Over the range, $0 < y \leq 8.0$, $\Delta y$ never became larger than $5.0\times10^{-03}$. This is partially, but not totally, due to the value of the user specified relative tolerance, initially set to $1.0\times10^{-12}$. After experimentation it was reset to $1.0\times10^{-10}$, producing the same results as the previous tolerance.