Linearized AVO and poroelasticity

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ABSTRACT

This study combines the technique of amplitude variations with offset, or AVO, analysis with the theory of poroelasticity to derive a linearized AVO approximation that provides the basis for the estimation of fluid, rigidity and density parameters from the weighted stacking of pre-stack seismic amplitudes. The method proposed is a generalization of the two AVO approximations introduced by Gray et al. (1999) using the formulation introduced by Russell et al. (2003). After a review of linearized AVO theory, we present the theory of our approach. We then apply our method to both model and real datasets.

INTRODUCTION

When an incident *P*-wave wave strikes a boundary between two elastic media at an angle greater than zero, a phenomenon called mode conversion occurs, in which reflected and transmitted *P* and *S*-waves are created on both sides of the boundary, as shown in Figure 1.

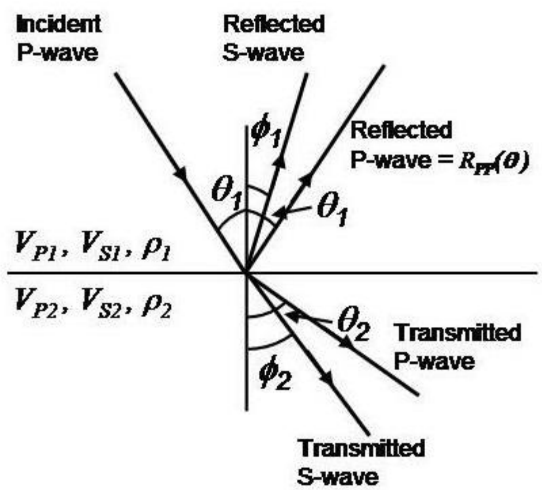

FIG. 1. Mode conversion of an incident P-wave.

The amplitudes of the reflected and transmitted waves can be derived by solving the following matrix equation (Zoeppritz, 1919):

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$$\begin{bmatrix} R_{PP} \\ R_{PS} \\ T_{PP} \\ T_{PS} \end{bmatrix} = \begin{bmatrix} -\sin\theta_1 & -\cos\phi_1 & \sin\theta_2 & \cos\phi_2 \\ \cos\theta_1 & -\sin\phi_1 & \cos\theta_2 & -\sin\phi_2 \\ \sin 2\theta_1 & \frac{V_{P1}}{V_{S1}}\cos 2\phi_1 & \frac{\rho_2 V_{S2}^2 V_{P1}}{\rho_1 V_{S1}^2 V_{P2}}\cos 2\phi_1 & \frac{\rho_2 V_{S2} V_{P1}}{\rho_1 V_{S1}^2}\cos 2\phi_2 \\ -\cos 2\phi_1 & \frac{V_{S1}}{V_{P1}}\sin 2\phi_1 & \frac{\rho_2 V_{P2}}{\rho_1 V_{P1}}\cos 2\phi_2 & -\frac{\rho_2 V_{S2}}{\rho_1 V_{P1}}\sin 2\phi_2 \end{bmatrix}^{-1} \begin{bmatrix} \sin\theta_1 \\ \cos\theta_1 \\ \sin 2\theta_1 \\ \cos 2\phi_1 \end{bmatrix} \quad (1)$$

Notice that the necessary parameters for the solution of the problem involve the individual *P*-wave velocity, *S*-wave velocity and density values on each side of the boundary, as well as the incident, reflected and transmitted angles, all of which can be derived from the incident P-wave angle using Snell's law. Although equation (1) will give precise values of the amplitudes of the reflected and transmitted waves, it does not provide an intuitive understanding of the effects of the parameter changes on the amplitudes, and is also difficult to invert (that is, given the amplitudes, what are the underlying elastic parameters which caused those amplitudes.) For these reasons, much current amplitude variation with offset (AVO) work and pre-stack inversion is based on linearized approximations to equation (1). These linearized approximations will be discussed in the next section, and we will discuss how we can re-parameterize the equations for various combinations of three physical parameters: *P*-wave velocity, *S*-wave velocity and density; *P*-wave reflectivity, *S*-wave reflectivity and density reflectivity; *P*-wave velocity, Poisson's ratio and density; the two Lamé coefficients and density; bulk modulus, shear modulus, and density.

Although the discussion of the various linearized approximations in the next section is a summary of existing work, it sets the stage for the next part of our development, in which we consider not only the velocities and densities themselves, but the effect of the fluid component of the velocity and density of the reservoir rock. In the theory developed independently by Biot (1941) and Gassmann (1951), we consider four components of the reservoir rock: its matrix, pore/fluid system, saturated state, and dry state. This is illustrated in Figure 2.

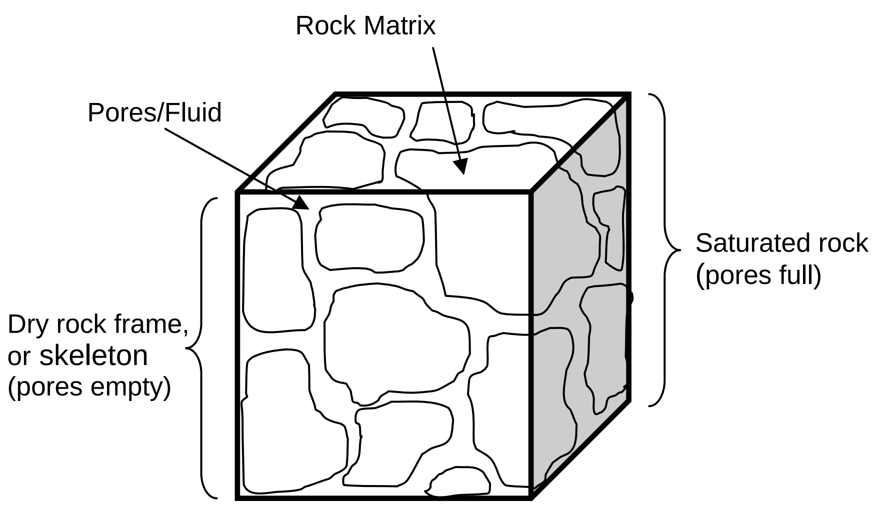


FIG. 2. In Biot-Gassmann theory, a cube of rock is characterized by four components: the rock matrix, the pore/fluid system, the dry rock frame, and the saturated frame. (Russell et al., 2003).

Based on these considerations the poroelasticity theory of Biot and Gassmann allows us to incorporate a term for the fluid component of the in-situ reservoir rock into the expression for the *P*-wave velocity. This will be discussed in the section on poroelasticity theory. Finally, we will combine poroelasticity theory and linearized AVO in such a way that the fluid component of the in-situ reservoir rock can be estimated using standard AVO least-squares extraction techniques. We will finish with both model and real data case studies that illustrate the method.

LINEARIZED AVO APPROXIMATIONS

It has been shown (Bortfeld, 1961, Richards and Frasier, 1976, Aki and Richards, 2002) that, for small changes in the *P*-wave velocity, *S*-wave velocity and density across a boundary between two elastic media, the *P*-wave reflection coefficient for an incident *P*-wave as a function of angle can be approximated by the following linearized sum of three terms:

$$R_{PP}(\theta) = \left[\frac{1}{2\cos^2\theta}\right]\frac{\Delta V_P}{V_P} + \left[\frac{-4\sin^2\theta}{\gamma_{sat}^2}\right]\frac{\Delta V_S}{V_S} + \left[\frac{1}{2} - \frac{2\sin^2\theta}{\gamma_{sat}^2}\right]\frac{\Delta\rho}{\rho}, \qquad (2)$$

where V_P, V_S and ρ are the average velocity and density values across the boundary, ΔV_P, ΔV_S and $\Delta\rho$ are the differences of the velocity and density values across the boundary, θ is the average of the incident and refracted angles, and $\gamma_{sat} = V_P/V_S$ for the in-situ (saturated) rocks. By "small" changes, we mean that equation (1) is valid where each ratio $\Delta p/p$ (which we refer to as "reflectivity" terms) is less than approximately 0.1. If we know the relationship between offset and angle for a seismic CMP gather, equation (1) can be used to extract estimates of the three reflectivities from the gather using a weighted least-squares approach. Equation (2) has also been used to perform Bayesian inversion for velocity and density (Buland and Omre, 2003). Since equation (2) was developed independently by Bortfeld, Aki and Richards, we will refer to it as the Bortfeld-Aki-Richards (B-A-R) equation.

There are several important algebraic re-arrangements of equation (2). First, it can be transformed into the three term sum given by

$$R_{PP}(\theta) = A + B\sin^2\theta + C\tan^2\theta\sin^2\theta, \qquad (3)$$

where $A = R_{P0} = \frac{1}{2}\left[\frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}\right]$ is a linearized approximation to the zero-offset *P*-wave reflection coefficient, $B = \frac{\Delta V_P}{2V_p} - \frac{4}{\gamma_{sat}^2}\frac{\Delta V_S}{V_S} - \frac{2}{\gamma_{sat}^2}\frac{\Delta\rho}{\rho}$, and $C = \frac{\Delta V_P}{2V_p}$. This equation, which was initially derived by Wiggins et al. (1983), is the basis of much of the empirical amplitude variations with offset (AVO) work performed today and has the advantage that an estimate of γ_{sat} is not needed in the weighting coefficients used to extract the three parameters (generally called the intercept, gradient, and curvature terms).

A second re-arrangement of equation (1), by Fatti et al. (1994) (based on an earlier equation by Smith and Gidlow (1986)), is given by

$$R_{PP}(\theta) = \left[1+\tan^2\theta\right]R_{P0} + \left[\frac{-8}{\gamma_{sat}^2}\sin^2\theta\right]R_{S0} + \left[\frac{2\sin^2\theta}{\gamma_{sat}^2} - \frac{1}{2}\tan^2\theta\right]R_D \,, \tag{4}$$

where R_{P0} is equal to the A term from equation (3), $R_{S0} = \left[\frac{\Delta V_S}{V_S} + \frac{\Delta\rho}{\rho}\right]$ is a linearized approximation to the S-wave reflectivity, and $R_D = \frac{\Delta\rho}{\rho}$ is the linearized density reflectivity term from equation (2). Equation (4) has been used both to extract the reflectivity terms from a CMP gather and as the basis for impedance inversion (Simmons and Backus, 1996, Hampson et al., 2006), although it does require an estimate of γ_{sat} in the weighting coefficients. Since equation (4) was developed by Smith, Gidlow and Fatti, we will call it the Smith-Gidlow-Fatti (S-G-F) equation.

Another way of re-formulating equation (1) involves transforming to parameters which are nonlinearly related to velocity and density. This involves the use of differentials as well as algebra. For example, Shuey (1982) transformed the second term in equation (3) to from dependence on V_S and ΔV_S to dependence on Poisson's ratio $\sigma = (\gamma - 2)/(2\gamma - 2)$ and changes in Poisson's ratio ($\Delta\sigma$). Shuey's gradient term B is written

$$B = A\left[D - 2(1+D)\frac{1-2\sigma}{1-\sigma}\right] + \frac{\Delta\sigma}{(1-\sigma)^2} \,, \tag{5}$$

where $D = \frac{\Delta V_P / V_P}{\Delta V_P / V_P + \Delta\rho/\rho} = \frac{\Delta V_P / V_P}{2A}$, $\sigma = \frac{\sigma_2 + \sigma_1}{2}$, $\Delta\sigma = \sigma_2 - \sigma_1$. Since Shuey did not provide his derivation of this term, and we are not aware of its publication anywhere in the literature, the derivation is given in Appendix A.

More recently, Gray et al. (1999) re-formulated equation (1) for two sets of fundamental constants: λ, μ and ρ, and K, μ and ρ, where we recall the following relationships:

$$V_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}} \,, \tag{6}$$

and

$$V_S = \sqrt{\frac{\mu}{\rho}} \,. \tag{7}$$

As with Shuey's work, this re-formulation required the use of both algebra and differentials relating λ, μ and K to V_P, V_S and ρ. Gray et al.'s two formulations are given as

$$R_{PP}(\theta) = \left(\frac{1}{4} - \frac{1}{2\gamma_{sat}^2}\right)\sec^2\theta\frac{\Delta\lambda}{\lambda} + \frac{1}{\gamma_{sat}^2}\left(\frac{1}{2}\sec^2\theta - 2\sin^2\theta\right)\frac{\Delta\mu}{\mu} + \left(\frac{1}{2} - \frac{1}{4}\sec^2\theta\right)\frac{\Delta\rho}{\rho}, \quad (8)$$

and

$$R_{PP}(\theta) = \left(\frac{1}{4} - \frac{1}{3\gamma_{sat}^2}\right)\sec^2\theta\frac{\Delta K}{K} + \frac{1}{\gamma_{sat}^2}\left(\frac{1}{3}\sec^2\theta - 2\sin^2\theta\right)\frac{\Delta\mu}{\mu} + \left(\frac{1}{2} - \frac{1}{4}\sec^2\theta\right)\frac{\Delta\rho}{\rho}. \quad (9)$$

The similarity between equations (8) and (9) can be easily noted. The only differences are that the 1/2 factor in the first and second terms in equation (8) changes to 1/3 in the first and second terms in equation (9). To understand the significance of this observation, we will first review the elements of poroelasticity theory as presented by Russell et al. (2003).

POROELASTICITY THEORY

The purpose of the present study is to show how the two formulations from Gray et al. can be generalized using the work of Russell et al. (2003). In that study, the authors used poroelasticity theory (Biot, 1941, and Gassmann, 1951) to equate the λ, μ, ρ and K, μ, ρ sets of parameters using the model shown in Figure 2. Biot (1941) used the Lamé parameters and showed that (Krief et al, 1990)

$$\lambda_{sat} = \lambda_{dry} + \beta^2 M, \quad (10)$$

where λ_{sat} is the 1st Lamé parameter for the saturated rock, λ_{dry} is the 1st Lamé parameter for the dry frame, β is the Biot coefficient, or the ratio of the volume change in the fluid to the volume change in the formation when hydraulic pressure is constant, and M is the modulus, or the pressure needed to force water into the formation without changing the volume. Conversely, Gassmann started with the bulk and shear moduli, and derived the following relationship (Krief et al, 1990):

$$K_{sat} = K_{dry} + \beta^2 M, \quad (11)$$

where K_{sat} is the bulk modulus of the saturated rock, K_{dry} is the bulk modulus of the dry rock, and β and M are the same as in equation (10). By equating equations (10) and (11), and using equation (6) to derive the relationships among K, λ and μ, the following result can be derived:

$$\mu_{sat} = \mu_{dry}. \quad (12)$$

That is, the shear modulus is unaffected by the pore fluid. This theoretical result has a strong intuitive basis, since we know that fluids do not support shear stresses, only compressive stresses.

Gassmann further showed that

$$\beta = 1 - \frac{K_{dry}}{K_m} \quad , \tag{13}$$

and

$$\frac{1}{M} = \frac{\beta - \phi}{K_m} + \frac{\phi}{K_{fl}} \quad , \tag{14}$$

where K_m is the bulk modulus of the matrix material and K_{fl} is the bulk modulus of thefluid. The advantage of using the Gassmann formulation given in equations (12) through (14) is that we can model our particular gas sand using these parameters, although it is often difficult to obtain reliable estimates for K_{dry} unless an in-situ S-wave log has been measured. It should also be noted that the K_{fl} term can be derived from knowledge of the water and hydrocarbon components by the equation

$$\frac{1}{K_{fl}} = \frac{1 - S_W}{K_{hc}} + \frac{S_W}{K_w} \, , \tag{15}$$

where S_W is the water saturation and K_{hc} and K_w are the hydrocarbon and water bulk modulii, respectively. These equations will be used in a later section to perform modeling.

If equations (13) and (14) are substituted into equation (11) the result is the expression often seen in rock-physics textbooks (e.g. Mavko et al 1998). However, we have chosen to retain the use of the term $\beta^2 M$ for the difference between the dry and saturated cases to emphasize its independence from the first term. Using $\beta^2 M$, we can rewrite the equation for P-wave velocity (equation (6)) for the saturated case with lambda and mu as

$$V_P = \sqrt{\frac{\lambda_{dry} + 2\mu + \beta^2 M}{\rho_{sat}}} \quad , \tag{16a}$$

or with the bulk and shear modulii as

$$V_P = \sqrt{\frac{K_{dry} + \frac{4}{3}\mu + \beta^2 M}{\rho_{sat}}} \quad . \tag{16b}$$

Both equations (16a) and (16b) can be written more succinctly as:

$$V_P = \sqrt{\frac{f + s}{\rho_{sat}}} \quad , \tag{17}$$

where f is a fluid/porosity term equal to $\beta^2 M$, and s is a dry-skeleton term which can be written either as $K_{dry} + \frac{4}{3}\mu$ or $\lambda_{dry} + 2\mu$. Note that in equations (16) and (17) we have assumed that $\mu = \mu_{sat} = \mu_{dry}$. In Russell et al. (2003) this formulation was applied to inverted seismic data, where we estimated the P and S-wave impedances, Z_P and Z_S, rather than velocities V_P and V_S. However, in this study, we will assume that the velocities are of prime importance. Therefore, note that we can extract the terms f and s by re-arranging equations (7) and (17) to get

$$f = \left(\rho V_P^2\right)_{sat} - \gamma_{dry}^2 (\rho V_S^2)_{sat}, \tag{18}$$

and

$$s = \gamma_{dry}^2 \mu, \tag{19}$$

where $\gamma_{dry}^2 = \left[\frac{V_P}{V_S}\right]_{dry}^2 = \frac{\lambda_{dry}}{\mu} + 2 = \frac{K_{dry}}{\mu} + \frac{4}{3}$. (Note that the term γ_{dry}^2 was labeled c in Russell et al., but we have chosen to emphasize its physical significance in this study) Finally, notice that by dividing f through by μ and realizing that $\mu = (\rho V_S^2)_{sat}$, we get $f / \mu = \gamma_{sat}^2 - \gamma_{dry}^2$.

There are several approaches to estimating γ_{dry}^2. The first is to estimate the dry-rock Poisson's ratio, σ_{dry}, noting that this is given by

$$\sigma_{dry} = \frac{\gamma_{dry}^2 - 2}{2\gamma_{dry}^2 - 2}. \tag{20}$$

Generally, the accepted value of σ_{dry} is in the order of 0.1, which corresponds to a V_P/V_S ratio of 1.5, or a γ_{dry}^2 value of 2.25.

A second approach is to perform laboratory measurements. Murphy et al (1993) measured the K_{dry}/μ ratio for clean quartz sandstones over a range of porosities and found that this value was, on average, equal to 0.9. This corresponds to a c value of 2.233. If the K_{dry}/μ value is rounded to 1.0, this implies a σ_{dry} of 0.125, and a corresponding γ_{dry}^2 value of 2.333.

Thus, there are a range of values of γ_{dry}^2 that depend on the particular reservoir being studied. Table 1 shows a range of these values and the range of their equivalent elastic constant ratios. The value of γ_{dry}^2 in this table ranges from a high of 4, meaning that λ_{dry}/μ is equal to 2, to a low of 1 1/3, meaning that K_{dry}/μ is equal to 0.

Table 1. A table of various values of the dry rock V_P/V_S ratio squared and their relationship to other elastic constants.

γdry^2	$(Vp/Vs)dry$	σdry	$Kdry/\mu$	$\lambda dry/\mu$
4.000	2.000	0.333	2.667	2.000
3.333	1.826	0.286	2.000	1.333
3.000	1.732	0.250	1.667	1.000
2.500	1.581	0.167	1.167	0.500
2.333	1.528	0.125	1.000	0.333
2.250	1.500	0.100	0.917	0.250
2.233	1.494	0.095	0.900	0.233
2.000	1.414	0.000	0.667	0.000
1.333	1.155	-1.000	0.000	-0.667

THE GENERALIZED ELASTIC CONSTANT AVO EQUATION

As shown in Appendix B, if we start with the Aki-Richards formulation given in equation (2) and use the differential given by

$$\Delta f = \frac{\partial f}{\partial V_P}\Delta V_P + \frac{\partial f}{\partial V_S}\Delta V_S + \frac{\partial f}{\partial \rho}\Delta \rho \quad , \tag{21}$$

we can re-parameterize equation (2) using the parameters $\gamma_{dry}{}^2$ and $\gamma_{sat}{}^2$. The final equation is written

$$R_{PP}(\theta) = \left[\left(1 - \frac{\gamma_{dry}^2}{\gamma_{sat}^2}\right)\frac{\sec^2\theta}{4}\right]\frac{\Delta f}{f} + \left[\frac{\gamma_{dry}^2}{4\gamma_{sat}^2}\sec^2\theta - \frac{2}{\gamma_{sat}^2}\sin^2\theta\right]\frac{\Delta\mu}{\mu} + \left[\frac{1}{2} - \frac{\sec^2\theta}{4}\right]\frac{\Delta\rho}{\rho} \tag{22}$$

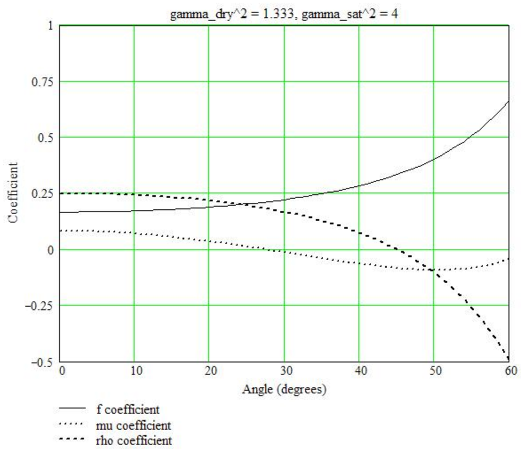
Equation (22) will be referred to as the *f-m-r* (fluid-mu-rho) equation since it gives us new physical insight into the relationship between linearized AVO and poroelasticity and is a generalization of the equations of Gray et al. (1999). The first thing to note about this equation is that the scaling parameter in front of the fluid term $\Delta f/f$ proportional to one minus the ratio of the saturated and dry V_S/V_P ratios. If $\gamma_{sat}^2 = \gamma_{dry}^2$ this term goes to zero, implying that there is no fluid component to the reservoir (i.e. we are dealing with a perfectly “dry” rock). Also, since we can never have a situation in which $\gamma_{sat}/\gamma_{dry} > 1$ (since, as seen from equations (10) and (11), the saturated values for K or λ will always be greater or equal to the dry values), the scaling coefficient for the fluid term will always be positive or zero. Secondly, if we let $\gamma_{dry}^2 = 2$, equation (22) reduces to the λ, μ, ρ formulation as given in equation (8). Finally, if we let $\gamma_{dry}^2 = 4/3$, it reduces to the K, μ, ρ formulation given in equation (9).

But do these values of 2 and 4/3 make physical sense? As discussed by Russell et al. (2003), these values are not appropriate for typical saturated rocks since, if we refer back to Table 1, a value of 4/3 implies a dry rock Poisson's ratio of -1 and a value of 2 implies a dry rock Poisson's ratio of 0, neither of which is physically realistic. A value of 2.333, which implies from Table 1 that $(K/\mu)_{\mathrm{dry}} = 1$ and the dry rock Poisson's ratio is 0.125, is more appropriate for rocks such as sandstones. In fact, Dillon et al. (2003) measured γ_{dry}^2 values as high as 3 for unconsolidated sandstones in Brazil.

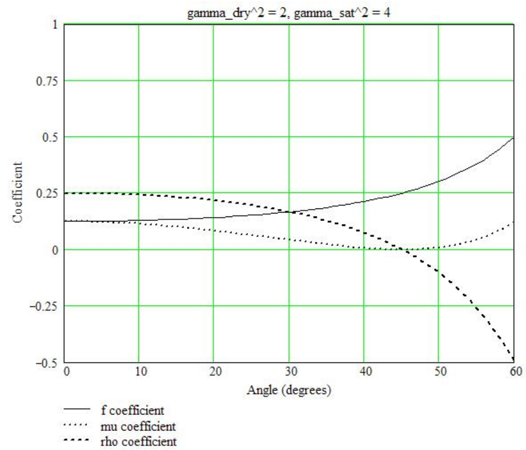
Next, note that the scaling term for $\Delta\mu/\mu$ is also dependent on both γ_{dry}^2 and γ_{sat}^2. However, the $1/\gamma_{sat}^2$ term can be factored out of both terms in the brackets and can be thought of as an overall scaling factor, leaving a first term dependent on γ_{dry}^2 minus a second term that is independent of either velocity ratio. Thus as the γ_{dry}^2 value goes up, this scaling coefficient increases.

Lastly, the density term is independent of γ_{dry}^2 and γ_{sat}^2, and is only a function of $\sec^2\theta$. Thus, the density scaling will always have the same values as a function of angle, regardless of γ_{dry}^2 and γ_{sat}^2, and will always change from positive to negative at $\theta = 45^{\circ}$ (where $\cos^2\theta = ½$). Physically, this makes sense since both γ_{dry}^2 and γ_{sat}^2 are functions of a velocity ratio, in which the density term cancels. Note, however, that this is not the same as saying that the extracted density term $\Delta\rho/\rho$ is independent of fluid, since its value will depend on the actual amplitudes of the seismic data.

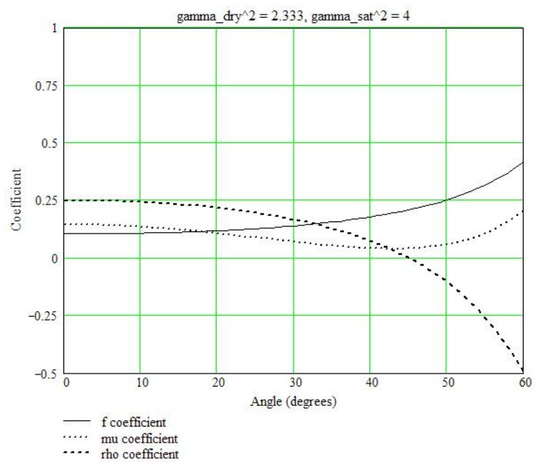
The computed curves for the various cases are shown in Figures 3 and 4. In Figure 3, the coefficients for the three terms are shown for $\gamma_{sat}^2 = 4$ and three different values of γ_{dry}^2 (1.333, 2.0 and 2.333). As discussed in the previous paragraph, the density term does not change, and so in Figure 3 can be used as a reference for the other two curves.



(a)



(b)

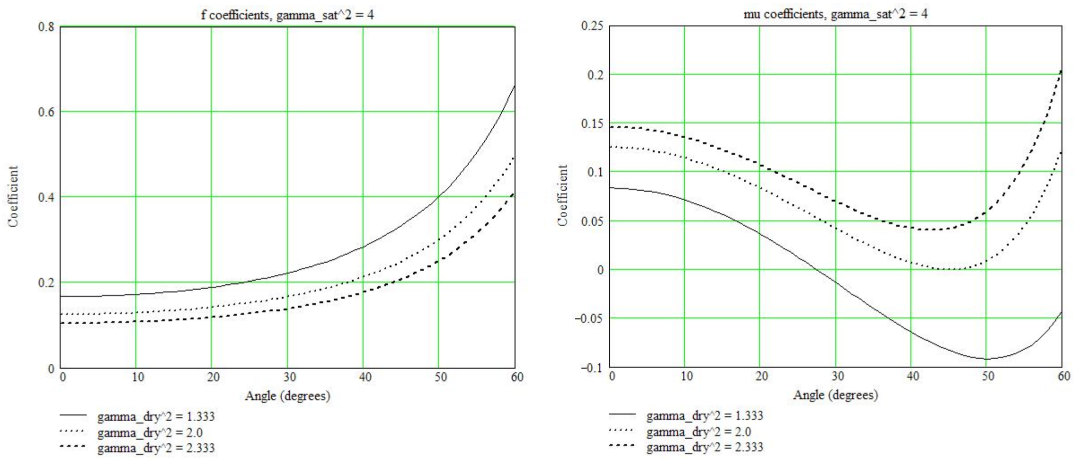


(c)

FIG. 3. Weighting coefficients for $\Delta f/f$, $\Delta\mu/\mu$, and $\Delta\rho/\rho$ as a function of angle, with $\gamma_{sat}^2 = 4$ in all cases and for (a) $\gamma_{dry}^2 = 1.333$, (b) $\gamma_{dry}^2 = 2.0$, and (c) $\gamma_{dry}^2 = 2.333$.

We can make several general observations based on the three separate plots shown in Figure 3. First, the weighting of the fluid term increases as we go out to higher angles, but the value of this term goes down as the $\gamma_{dry}^2 / \gamma_{sat}^2$ ratio increases, as was mentioned earlier. Second, the weighting on the rigidity term decreases out to about 50 degrees but then starts to increase. Also, the overall weighting on this term increases as the $\gamma_{dry}^2 / \gamma_{sat}^2$ term increases. Finally, the weighting on the density term decreases as a function of angle and eventually becomes negative at 45 degrees, as predicted.

In Figure 4, the weighting coefficients for $\Delta f/f$ and $\Delta\mu/\mu$ are shown separately, as a function of the three values of γ_{dry}^2, again with a constant value of γ_{sat}^2. This figure makes it clearer than Figure 3 that the weighting for $\Delta f/f$ goes down as γ_{dry}^2 increases, but the weighting for $\Delta\mu/\mu$ goes up as γ_{dry}^2 increases. This makes physical sense if we recall that γ_{dry}^2 represents the square of the dry rock V_P/V_S velocity ratio and γ_{sat}^2 represents the square of the saturated rock V_P/V_S velocity ratio. Thus, as γ_{dry}^2 increases for a fixed value of γ_{sat}^2, their product decreases, reducing the effect on $\Delta f/f$, but increasing the effect on the $\Delta\mu/\mu$ term.



(a) (b)

FIG. 4. Weighting coefficients for (a) $\Delta f/f$, and (b) $\Delta\mu/\mu$ as a function of angle with γ_{sat}^2 = 4 in all cases, and for γ_{dry}^2 = 1.333, 2.0 and 2.333.

WEIGHTED PARAMETER EXTRACTION

It should also be noted that equation (22) is similar to equations (2), (3), (4), (8) and (9) in that all of these three term linearized AVO expressions can be expressed as

$$R_{PP}(\theta) = ap_1 + bp_2 + cp_3, \tag{23}$$

where $a, b,$ and c are functions of θ and V_P^2/V_S^2 (dry or wet), and $p_1,\ p_2,\ and\ \ p_3$ are functions of $V_P, V_S, \rho, \sigma, f$, or μ. The only differences among the equations are the parameters we wish to compute and the values needed to compute the constants *a*, *b* and *c*. Table 2 summarizes these various values, where B-A-R stands for the Bortfeld-Aki-Richards equation, and S-G-F stands for the Smith-Gidlow-Fatti equation.

Note that the equations in Table 2 have been ranked based on the complexity of what we need to know in order to compute the constants, where for the first two equations (Wiggins and Shuey) we only need to know the angle of incidence, for the next two equations (B-A-R and S-G-F) we need to know angle and the saturated V_P/V_S ratio, and for the generalized *f-m-r* equation discussed in this section, we need to know angle and the saturated and dry V_P/V_S ratios. Thus, although the advantage of equation (22) is that we can extract the fluid component directly, the disadvantage of this equation is that we now need to estimate both γ_{dry}^2 and γ_{sat}^2 in the weighting coefficients. To determine γ_{dry}^2, more research needs to be done on rocks that don't fit the standard Biot-Gassmann model, such as shales and fractured carbonates.

Table 2. The parameters needed to estimate the various terms in the 3-term linearized AVO expressions considered in this paper.

Method	Need to know			Able to compute		
	a	b	c	p_1	p_2	p_3
Wiggins	θ	θ	θ	R_{P0}	$G(V_P, V_S, \rho)$	$\frac{\Delta V_P}{V_P}$
Shuey	θ	θ	θ	R_{P0}	$G(V_P, \sigma, \rho)$	$\frac{\Delta V_P}{V_P}$
B-A-R	θ	θ, γ_{sat}^2	θ, γ_{sat}^2	$\frac{\Delta V_P}{V_P}$	$\frac{\Delta V_S}{V_S}$	$\frac{\Delta \rho}{\rho}$
S-G-F	θ	θ, γ_{sat}^2	θ, γ_{sat}^2	R_{P0}	R_{S0}	$\frac{\Delta \rho}{\rho}$
f-μ-ρ	$\theta, \gamma_{sat}^2, \gamma_{dry}^2$	$\theta, \gamma_{sat}^2, \gamma_{dry}^2$	θ	$\frac{\Delta f}{f}$	$\frac{\Delta \mu}{\mu}$	$\frac{\Delta \rho}{\rho}$

Appendix C explains the mathematics involved in actually extracting the three parameters from a seismic gather using the least-squares approach. Let us now look at model and real data examples of the implementation of equation (22).

MODEL EXAMPLE

A model was next created consisting of two sands, both of which had the same physical parameters except for fluid content. The top sand was modeled as water-wet, with K_{fl} = 1.0, and the second sand was modeled as a gas sand with K_{fl} = 0.1. For each sand K_{dry} = 3 MPa and μ = 3 MPa, which meant that K_{dry}/μ = 1.0 in both sands. The mineral bulk modulus, K_m, of each sand was set to a value of 40 MPa, the generally accepted value for sandstone. In the wet sand, the density was set to 2.0 g/cc and in the second sand to 1.8 g/cc. Using equations (11) through (16), the velocities of the two sands could then be computed. The overlying wet sand velocities were V_P = 2259 m/s and V_S = 1225 m/s. The underlying gas sand velocities were V_P = 1977 m/s and V_S = 1291 m/s. As expected by Biot-Gassmann theory, the P-wave velocity drops and the S-wave velocity increases across the boundary. Utilizing equations (2) and (22) we can now compute the AVO curves at the elastic boundary interface for the Bortfeld-Aki-Richards and *f-m-r* approaches, respectively. These curves are shown in Figure 5.

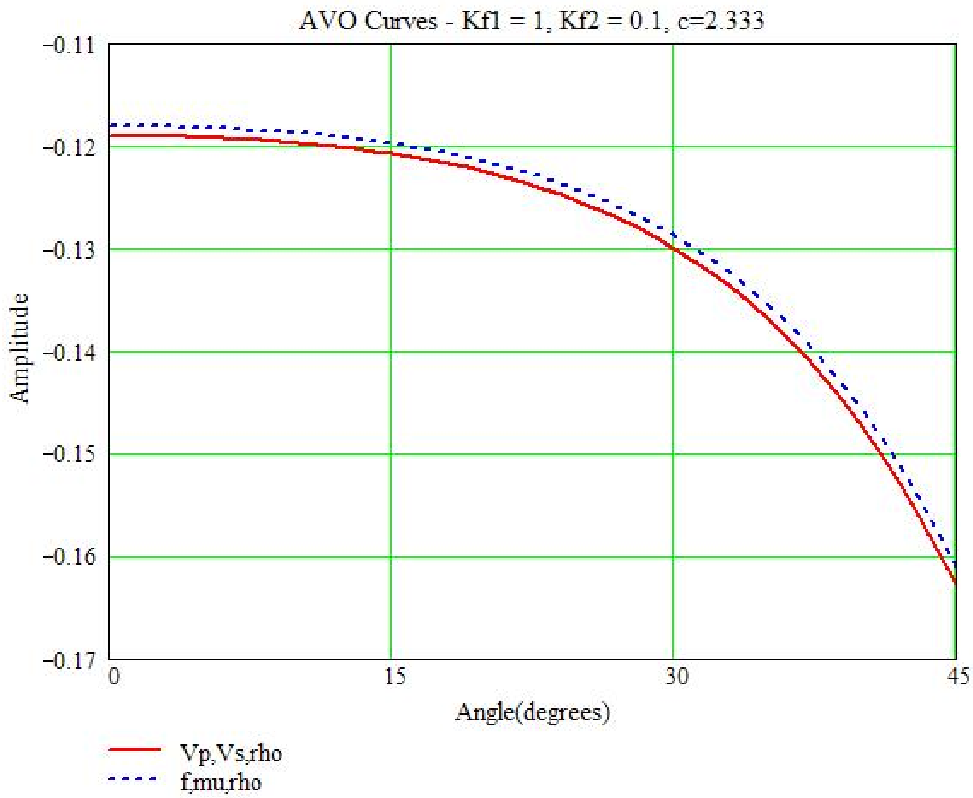


FIG. 5: The fit between curves derived from equations (2) and (21), where we modeled a wet sand over a gas sand for which K_{fluid} drops from 1.0 to 0.1 and $(K/\mu)_{dry} = 1$ in both sands.

Notice that although the two curves are not exact, they are very close. Thus, we can feel confident that if whether we extract the terms using the *f-m-r* method of one of the standard Aki-Richards reformulations, the reconstructed amplitudes will match our seismic observations.

In computing the curves in Figure 5 there is one very important point that should be made (and can serve at a large source of error if not observed). The term γ_{sat}^2 that appears in both equations (2) and (22) must be computed differently for each equation. That is, in equation (2) γ_{sat}^2 can be computed using its normal definition of $(V_P/V_S)^2_{sat}$, where V_P and V_S are the average values of the velocities across the boundaries. However, in equation (22) the terms γ_{sat}^2 and γ_{dry}^2 must be re-parameterized using the coefficients f, K_{dry} and μ, which are the averaged fluid term, dry rock bulk modulu and shear modulus across the boundary. Utilizing equation (16) through (19), the new expressions are written

$$\gamma_{dry}^2 = \frac{K_{dry}}{\mu} + \frac{4}{3}, \qquad (24)$$

and

$$\gamma_{sat}^2 = \frac{f}{\mu} + \frac{K_{dry}}{\mu} + \frac{4}{3}. \tag{25}$$

Since K_{dry} and μ don't change between layers, we can re-write equation (25) as

$$\gamma_{sat}^2 = \frac{f}{\mu} + \gamma_{dry}^2, \tag{26}$$

which is identical to a formulation we derived earlier after equation (19). To show how crucial this step is, if we use the velocity averages we get a value of γ_{sat}^2 =2.835 for example shown in Figure (5), but if we use elastic parameter averages, we get a value of γ_{sat}^2 =2.873.

REAL DATA EXAMPLE

Let us finish by looking at an actual example of the *f-m-r* approach encompassed in equation (21) using a shallow gas sand example from Alberta. Figure 6 is a display of a seismic stack which exhibits a "bright-spot" anomaly and structural high at 630 ms in the centre of the line. A successful gas well was drilled at CDP 330 on the line, and the sonic log from this gas well has been splice into the section. Notice the low velocity associated with the gas sand.

It is well known that neither the structural high nor the "bright-spot" shown on the section in Figure 6 is unambiguous when it comes to predicting gas sands. In fact, similar anomalies encountered on lines close the one shown here have false "bright-spots" caused by hard carbonate streaks and coals which lead to the drilling of unsuccessful wells. However, the use of the AVO method will help us to more accurately predict the presence of gas (although the AVO method is not totally unambiguous, and is insensitive to the actual hydrocarbon percentage in the reservoir).

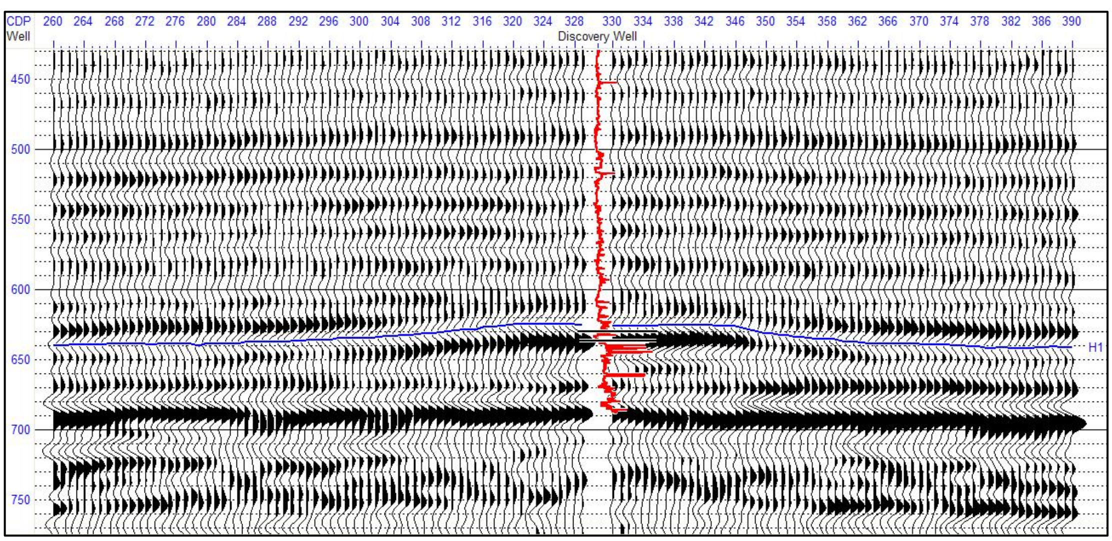


FIG. 6. The stack of line from Alberta showing a shallow "bright-spot" anomaly at 630 ms which is due to a gas sand.

Figure 7 shows some of the gathers from the line shown in Figure 6. Note that the gas sand zone has a pronounced AVO increase with offset, usually indicative of a Class 3 anomaly (in which the anomalous sand is of lower acoustic impedance than the surrounding sediments).

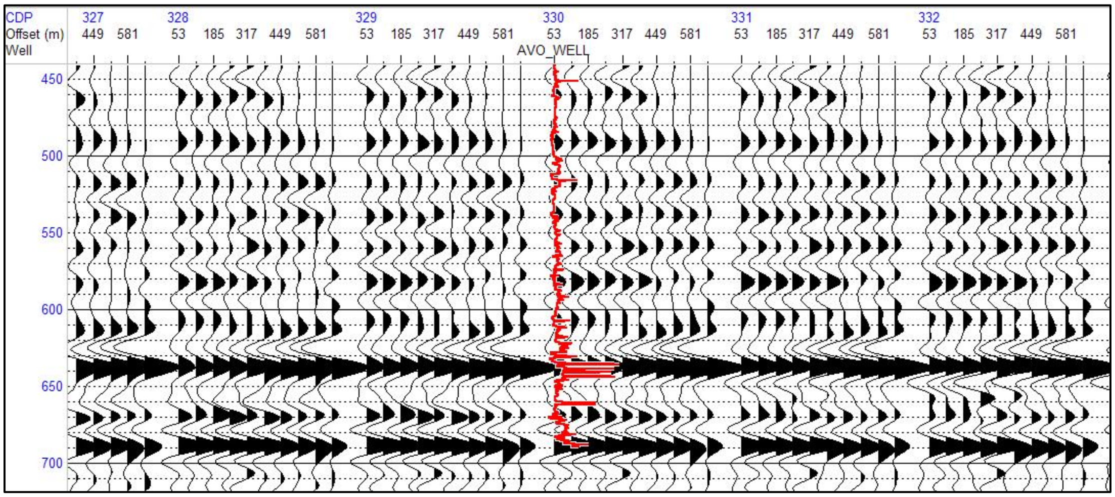


FIG. 7. The input gathers used to extract *f-m-r* parameters.

There are many approaches to interpreting the AVO anomaly shown on the gathers of Figure 7, such as intercept/gradient analysis, *P* and *S*-impedance inversion, and so on. All will do a reasonable job of delineating the gas sand. However, let us now apply the *f-m-r* analysis to this line.

In our analysis, we used a time-varying γ_{sat}^2 that was derived from the measured sonic log values and the S-wave values derived from this log using the mudrock equation $V_P = 1.16V_S + 1360$ m/s, and a constant γ_{dry}^2 value of 2.333. Figure 8 shows the extracted $\Delta f/f$ section, where red indicates a negative change and blue a positive change. On the $\Delta f/f$ section, notice the decrease in the fluid term as the gas sand is encountered and the increase as the underlying shale is encountered. Both of these observations make physical sense, since the gas sand should show a drop in its fluid effect as it is encountered on the section. Also, note how well the gas sand is delineated, giving a clear indication of both its lateral and vertical extent.

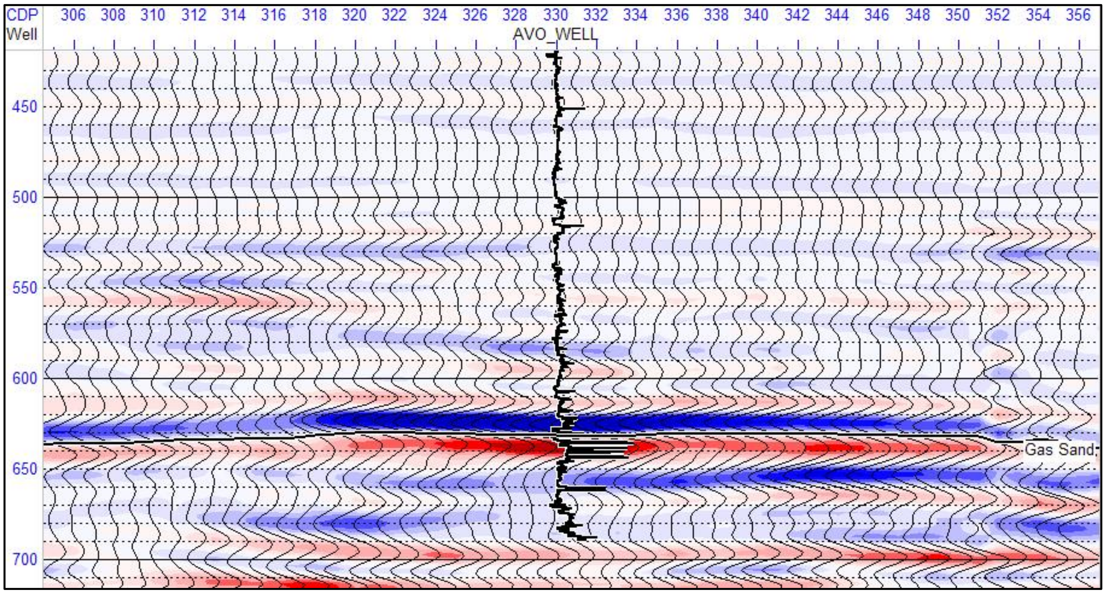


FIG. 8. The $\Delta f/f$ fluid modulus extraction for the data shown in Figures 6 and 7.

Next, Figure 9 shows the extracted$\Delta\mu/\mu$ rigidity section, where red again indicates a negative change and blue a positive change.

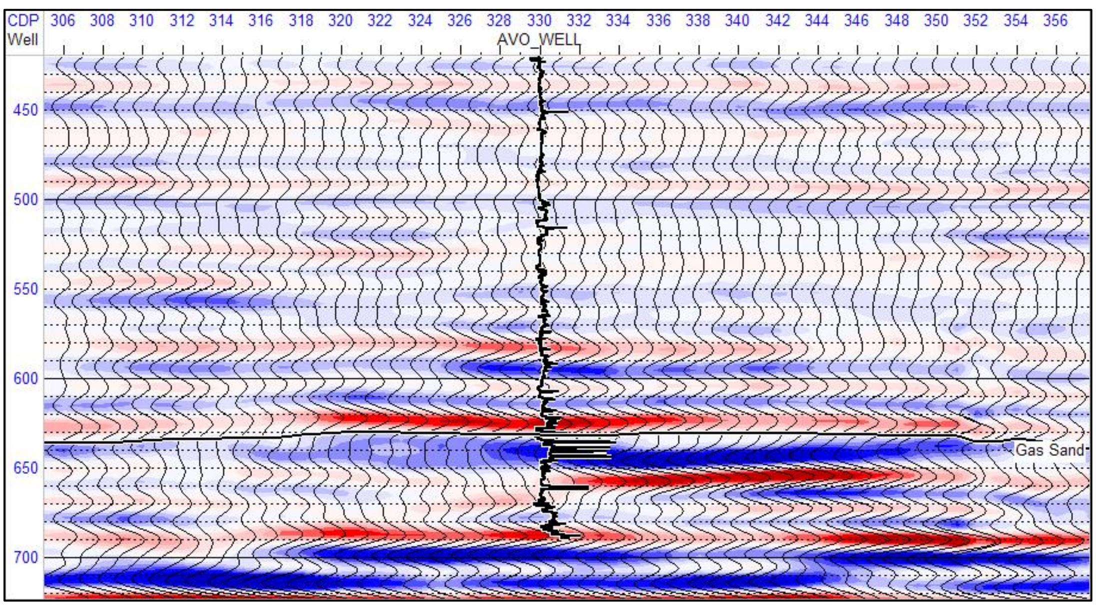


FIG. 9. The $\Delta\mu/\mu$ rigidity modulus extraction for the data shown in Figures 6 and 7.

On the section shown in Figure 9, notice the increase in the rigidity as the gas sand is encountered and the decrease as the underlying shale is encountered. Again, both of these observations make physical sense, since the rigidity term should be an indicator of the sandstone matrix, which is greater than the rigidity of the surrounding shales.

Thus, both the fluid and rigidity terms have proven to be excellent indicators of the makeup of the reservoir which has been delineated by this line. On the other hand, the

$\Delta\rho/\rho$ section which was extracted on this line was felt to be not very meaningful, because of the very short offsets, which limited the angular aperture to less than 30 degrees. On datasets in which we have an angular aperture out to 45 degrees or more, it is felt that the density section would be more reliable.

CONCLUSIONS

In this study, we combined the technique of amplitude variations with offset (AVO) analysis with the theory of poroelasticity to derive a linearized AVO approximation that provides the basis for the estimation of fluid, rigidity and density parameters from the weighted stacking of pre-stack seismic amplitudes. We showed that, by using the poroelasticity formulation discussed by Russell et al. (2003) and developed initially by Biot (1941) and Gassmann (1951), the proposed method is a generalization of the two AVO approximations introduced by Gray et al. (1999).

To fill in the background for our new method, we first presented an extensive review of linearized AVO theory, discussing the various re-parameterizations of the Bortfeld-Aki-Richards equation. We then discussed poroelasticity theory and followed this with the derivation of the fluid-mu-rho (*f-m-r*) formulation. The key parameter that was introduced into the AVO weighting coefficients was γ_{dry}^2 , the square of the dry rock V_P to V_S ratio. It was shown that the $\lambda\text{–}\mu\text{–}\rho$ formulation proposed by Gray et al. (1999) corresponded to γ_{dry}^2 = 4/3, and the $K\text{–}\mu\text{–}\rho$ formulation proposed by Gray et al. (1999) corresponded to γ_{dry}^2 = 2.0. However, a more realistic value for sandstone reservoirs is given by γ_{dry}^2 = 2.0.

We then applied our method to both model and real datasets. In our model study, we modelled a wet sand over a gas sand, and showed that we could accurately model the AVO effect using both the Bortfeld-Aki-Richards equation and the *f-m-r* equation. Finally, we applied the method to a real data example over a known gas sand. By extracting the fluid and rigidity components for this dataset, we were able to delineate the extent of the gas sand both spatially and temporally from an analysis of both sections.

It should be pointed out that a disadvantage of this approach is that we now need to estimate both γ_{dry}^2 and γ_{sat}^2 in the weighting coefficients. To determine γ_{dry}^2 , more research need to be done on rocks that don't fit the standard Biot-Gassmann model, such as shales and fractured carbonates.

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APPENDIX A

Derivation of Shuey's Equation

Shuey (1885) started with the Wiggins et al. (1983) rearrangement of the Aki-Richards equation, given by

$$R_{PP}(\theta) = A + B\sin^2\theta + C\tan^2\theta\sin^2\theta \quad , \tag{A1}$$

where $A = R_{P0} = \frac{1}{2}\left[\frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}\right]$ is a linearized approximation to the zero-offset *P*-wave reflection coefficient, $B = \frac{\Delta V_P}{2V_p} - \frac{4}{\gamma_{sat}^2}\frac{\Delta V_S}{V_S} - \frac{2}{\gamma_{sat}^2}\frac{\Delta\rho}{\rho}$, $\gamma_{sat} = V_P/V_S$ and $C = \frac{\Delta V_P}{2V_p}$. He then sought to re-parameterize this equation as a function of V_P, σ (Poisson's ratio) and ρ, rather than V_P, V_S, and ρ. Notice that the terms *A* and *C* are independent of V_S, so will remain unchanged in the re-parameterization. Thus, we only need to work with the *B*, or gradient, term.

To transform to the new set of parameters, Shuey used a differential form that relates V_S to V_P and σ, and can be written:

$$\Delta V_S = \frac{\partial V_S}{\partial V_P}\Delta V_P + \frac{\partial V_S}{\partial\sigma}\Delta\sigma \quad . \tag{A2}$$

First, we recall that, by definition, s is given by:

$$\sigma = \frac{\gamma^2 - 2}{2\gamma^2 - 2} \quad , \tag{A3}$$

where $\gamma = V_P / V_S$. Equation (A3) can be inverted to give

$$\gamma^2 = \left(\frac{V_P}{V_S}\right)^2 = \frac{2(1-\sigma)}{1-2\sigma} \Rightarrow \left(\frac{V_S}{V_P}\right)^2 = \frac{1-2\sigma}{2(1-\sigma)} \Rightarrow V_S = V_P\sqrt{\frac{1-2\sigma}{2(1-\sigma)}} \quad . \tag{A4}$$

The various equivalent relationships given in equation (A4) will come in handy when we compute the differentials in equation (A2).

Next, we note that if $\Delta\sigma = 0$, we can rewrite equation (A2) as

$$\Delta V_S = \frac{\partial V_S}{\partial V_P}\Delta V_P \Rightarrow \frac{\partial V_S}{\partial V_P} = \frac{\Delta V_S}{\Delta V_P} = \frac{V_S}{V_P} \quad . \tag{A5}$$

That is, if there is no change in the Poisson's ratio, there is no change in the V_P/V_S ratio. However, if there is a change in Poisson's ratio between layers, as is normal, we can write for the second term in equation (A2):

$$\frac{\partial V_S}{\partial \sigma} = -\frac{V_P^2}{4V_S}\left(\frac{1}{(1-\sigma)^2}\right) \Rightarrow \frac{\partial V_S}{\partial \sigma}\Delta\sigma = -\frac{V_P^2}{4V_S}\left(\frac{\Delta\sigma}{(1-\sigma)^2}\right). \tag{A6}$$

Substituting equations (A5) and (A6) into equation (A2) and dividing both sides through by V_S, we get:

$$\frac{\Delta V_S}{V_S} = \frac{\Delta V_P}{V_P} - \frac{\gamma_{sat}^2}{4}\left(\frac{\Delta\sigma}{(1-\sigma)^2}\right), \tag{A7}$$

where $\gamma_{sat}^2 = \left(\frac{V_S}{V_S}\right)_{sat}^2$. Substituting equation (A7) back into the gradient term B in equation (A1), we get

$$\begin{aligned} B &= \frac{\Delta V_P}{2V_p} - \frac{4}{\gamma_{sat}^2}\left[\frac{\Delta V_P}{V_P} - \frac{\gamma_{sat}^2}{4}\left(\frac{\Delta\sigma}{(1-\sigma)^2}\right)\right] - \frac{2}{\gamma_{sat}^2}\frac{\Delta\rho}{\rho} \\ &= \frac{\Delta V_P}{2V_p} - \frac{4}{\gamma_{sat}^2}\frac{\Delta V_P}{V_P} + \frac{\Delta\sigma}{(1-\sigma)^2} - \frac{2}{\gamma_{sat}^2}\frac{\Delta\rho}{\rho} \end{aligned} \tag{A8}$$

To complete the derivation, we still need to re-express the velocity ratio in terms of V_P and σ. This is done in the following way, using equation (A4):

$$\begin{aligned} B &= \frac{\Delta V_P}{2V_p} - 2\left(\frac{1-2\sigma}{1-\sigma}\right)\frac{\Delta V_P}{V_P} - \left(\frac{1-2\sigma}{1-\sigma}\right)\frac{\Delta\rho}{\rho} + \frac{\Delta\sigma}{(1-\sigma)^2} \\ &= \frac{\Delta V_P}{2V_p} - \left(\frac{1-2\sigma}{1-\sigma}\right)\frac{\Delta V_P}{V_P} - \left(\frac{1-2\sigma}{1-\sigma}\right)\left(\frac{\Delta\rho}{\rho} + \frac{\Delta V_P}{V_P}\right) + \frac{\Delta\sigma}{(1-\sigma)^2} \\ &= A\left[\frac{\Delta V_P/V_p}{2A} - 2\left(1 + \frac{\Delta V_P/V_p}{2A}\right)\left(\frac{1-2\sigma}{1-\sigma}\right)\right] + \frac{\Delta\sigma}{(1-\sigma)^2} \end{aligned} \tag{A9}$$

where $A = \frac{1}{2}\left(\frac{\Delta\rho}{\rho} + \frac{\Delta V_P}{V_P}\right)$. This can be re-expressed as

$$B = A\left[D - 2(1+D)\left(\frac{1-2\sigma}{1-\sigma}\right)\right] + \frac{\Delta\sigma}{(1-\sigma)^2} \tag{A10}$$

where $D = \frac{\Delta V_P/V_p}{2A}$. This is the expression for B found in Shuey (1985).

APPENDIX B

Derivation of the *f-m-r* equation

We start by re-writing equation (2) with the common denominator $\rho V_P{}^2$:

$$R_{PP}(\theta) = \frac{\frac{1}{2}\Delta\rho V_P^2 + \frac{1}{2}\rho V_P \Delta V_P \sec^2\theta - 2\left(\Delta\rho V_S^2 + 2\rho V_S \Delta V_S\right)\sin^2\theta}{\rho V_P^2} . \qquad \text{(B1)}$$

Keep in mind that the V_P and V_S values are for the saturated rock. Next, for convenience we will re-write equation (12), which was given as

$$f = \rho V_P^2 - \gamma_{dry}^2 \rho V_S^2 . \qquad \text{(B2)}$$

Recall the chain rule of multi-variable calculus, which can be written for $\Delta f(V_P,V_S,\rho)$ as

$$\Delta f = \frac{\partial f}{\partial V_P}\Delta V_P + \frac{\partial f}{\partial V_S}\Delta V_S + \frac{\partial f}{\partial \rho}\Delta\rho . \qquad \text{(B3)}$$

Applying equation (B3) to equation (B2) gives

$$\Delta f = 2\rho V_P \Delta V_P - 2c\rho V_S \Delta V_S + \left(V_P^2 - \gamma_{dry}^2 V_S^2\right)\Delta\rho . \qquad \text{(B4)}$$

Re-arranging equation (B4) gives

$$\Delta\rho V_S^2 + 2\rho V_S \Delta V_S = \frac{1}{\gamma_{dry}^2}\left(\Delta\rho V_P^2 + 2\rho V_P \Delta V_P - \Delta f\right) . \qquad \text{(B5)}$$

Equation (B5) can then be substituted into equation (B1) to give

$$R_P(\theta) = \frac{\frac{1}{2}\Delta\rho V_P^2 + \frac{1}{2}\rho V_P \Delta V_P \sec^2\theta - \frac{2}{\gamma_{dry}^2}\left(\Delta\rho V_P^2 + 2\rho V_P \Delta V_P - \Delta f\right)\sin^2\theta}{\rho V_P^2} , \qquad \text{(B6)}$$

which can be re-arranged to give

$$R_P(\theta) = \frac{V_P^2 \Delta\rho\left(\frac{1}{2} - \frac{2}{c}\sin^2\theta\right) + \rho V_P \Delta V_P\left(\frac{1}{2}\sec^2\theta - \frac{4}{\gamma_{dry}^2}\sin^2\theta\right) + \Delta f\left(\frac{2}{\gamma_{dry}^2}\sin^2\theta\right)}{\rho V_P^2} \qquad \text{(B7)}$$

To find the dependence on μ, note that equation (B2) can also be written

$$f = \rho V_P^2 - \gamma_{dry}^2 \mu \quad . \tag{B8}$$

The chain rule for $\Delta f(V_P,\mu,\rho)$ can then be written

$$\Delta f = \frac{\partial f}{\partial V_P}\Delta V_P + \frac{\partial f}{\partial \mu}\Delta\mu + \frac{\partial f}{\partial \rho}\Delta\rho \quad . \tag{B9}$$

Applying equation (B9) to equation (B8) gives

$$\Delta f = 2\rho V_P \Delta V_P - \gamma_{dry}^2 \Delta\mu + V_P^2 \Delta\rho \quad , \tag{B10}$$

Re-arranging equation (B10) gives

$$\rho V_P \Delta V_P = \frac{\gamma_{dry}^2}{2}\Delta\mu - \frac{1}{2}V_P^2\Delta\rho + \frac{1}{2}\Delta f \quad . \tag{B11}$$

Let us now evaluate the second term in the numerator on the right hand side of equation (B7) after substituting equation (B11). This gives

$$\begin{aligned}\rho V_P \Delta V_P\left(\frac{1}{2}\sec^2\theta - \frac{4}{\gamma_{dry}^2}\sin^2\theta\right) &= \left(\frac{\gamma_{dry}^2}{2}\Delta\mu - \frac{1}{2}V_P^2\Delta\rho + \frac{1}{2}\Delta f\right)\left(\frac{1}{2}\sec^2\theta - \frac{4}{\gamma_{dry}^2}\sin^2\theta\right) \\ &= \Delta\mu\left(\frac{\gamma_{dry}^2}{4}\sec^2\theta - 2\sin^2\theta\right) + V_P^2\Delta\rho\left(\frac{2}{\gamma_{dry}^2}\sin^2\theta - \frac{1}{4}\sec^2\theta\right) + \Delta f\left(\frac{1}{4}\sec^2\theta\right)\end{aligned} \quad .$$

Substituting equation (B12) into equation (B7) we note that several terms cancel, giving:

$$R_P(\theta) = \frac{V_P^2\Delta\rho\left(\frac{1}{2} - \frac{1}{4}\sec^2\theta\right) + \Delta\mu\left(\frac{\gamma_{dry}^2}{4}\sec^2\theta - 2\sin^2\theta\right) + \Delta f\left(\frac{1}{2}\sec^2\theta\right)}{\rho V_P^2} \quad . \tag{B13}$$

Next, we can simplify equation (A13) by dividing through by ρV_P^2 to get

$$R_P(\theta) = \left(\frac{1}{2}\sec^2\theta\right)\frac{\Delta f}{\rho V_P^2} + \left(\frac{\gamma_{dry}^2}{4}\sec^2\theta - 2\sin^2\theta\right)\frac{\Delta\mu}{\rho V_P^2} + \left(\frac{1}{2} - \frac{1}{4}\sec^2\theta\right)\frac{\Delta\rho}{\rho} \quad , \tag{B14}$$

where we have also re-arranged the terms. Equation (B14) is close to our final form, but we would like to eliminate the ρV_P^2 term and end up with the terms $\Delta f / f$ and $\Delta\mu / \mu$.

To do this for the first term on the right hand side of equation (B14), note that we can write

$$\frac{f}{\rho V_P^2} = \frac{\rho V_P^2 - \gamma_{dry}^2 \rho V_S^2}{\rho V_P^2} = 1 - \gamma_{dry}^2 \left(\frac{V_S}{V_P}\right)_{sat}^2 = 1 - \frac{\gamma_{dry}^2}{\gamma_{sat}^2}, \qquad \text{(B15)}$$

where we have made use of the γ notation introduced earlier. This implies that

$$\frac{1}{\rho V_P^2} = \frac{1 - (\gamma_{dry}^2 / \gamma_{sat}^2)}{f}. \qquad \text{(B16)}$$

For the second term in equation (B14), note that

$$\frac{\mu}{\rho V_P^2} = \frac{\rho V_S^2}{\rho V_P^2} = \left(\frac{V_S}{V_P}\right)_{sat}^2 = \frac{1}{\gamma_{sat}^2}, \qquad \text{(B17)}$$

or

$$\frac{1}{\rho V_P^2} = \frac{1}{\mu \gamma_{sat}^2}. \qquad \text{(B18)}$$

Substituting equations (B16) and (B18) into equation (B14) leads to the final expression:

$$R_{PP}(\theta) = \left[\left(\frac{1}{4} - \frac{\gamma_{dry}^2}{4\gamma_{sat}^2}\right)\sec^2\theta\right]\frac{\Delta f}{f} + \left[\frac{\gamma_{dry}^2}{4\gamma_{sat}^2}\sec^2\theta - \frac{2}{\gamma_{sat}^2}\sin^2\theta\right]\frac{\Delta\mu}{\mu} + \left[\frac{1}{2} - \frac{\sec^2\theta}{4}\right]\frac{\Delta\rho}{\rho}. \qquad \text{(B19)}$$

APPENDIX C

Three Parameter AVO Parameter Extraction

It should be pointed out that all the three term AVO expressions that we have written in this paper can be expressed by the general equation

$$R_{PP}(\theta) = ap_1 + bp_2 + cp_3, \qquad \text{(C1)}$$

where a, b, and c are functions of θ and V_P^2/V_S^2 (dry or wet), and $p_1,\ p_2,\ and\ p_3$ are functions of $V_P, V_S, \rho, \sigma, f$, or μ. For N traces, where we know the angles, we can write the following set of N linear equations with three unknowns:

$$\begin{array}{lllll}
R_{PP}(\theta_1) = R_{P1} & = a_1 p_1 & + b_1 p_2 & + c_1 p_3 \\
R_{PP}(\theta_2) = R_{P2} & = a_2 p_1 & + b_2 p_2 & + c_2 p_3 \\
\vdots \qquad \vdots & \vdots & \vdots & \vdots \\
R_{PP}(\theta_N) = R_{PN} & = a_N p_1 & + b_N p_2 & + c_N p_3
\end{array},$$

which can be written in matrix form as

$$\begin{bmatrix} R_{PP}(\theta_1) \\ R_{PP}(\theta_2) \\ \vdots \\ R_{PP}(\theta_N) \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \vdots & \vdots & \vdots \\ a_N & b_N & c_N \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_2 \end{bmatrix}, \tag{C2}$$

or, more succinctly, as

$$R = MP. \tag{C3}$$

This can be solved using the least-squares inverse given by:

$$P = (M^T M + \lambda\, I)^{-1} M^T R \, , \tag{C4}$$

where λ is a pre-whitening term and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

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