Finite difference – finite integral transform hybrid techniques: the coupled P-S_V problem in a radially symmetric vertically inhomogeneous medium

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ABSTRACT

The hybrid finite difference – finite integral transform method is developed for the $P-S_V$ wave propagation problem in a radially symmetric vertically inhomogeneous medium. Apart from the development of the finite difference analogues of the transformed equations of motion, a number of numerical considerations are addressed. As in most problems where numerical methods are employed in the solution, there are several areas that are given special attention to indicate how to improve run times and accuracy. Often this method of problem solving is referred to (erroneously) as the *pseudo-spectral method*. The solution approach described here is more general in that uniformly sampled grids of any spatial dimension are not required. It may be correct to say that the pseudo-spectral method is a subset of what is discussed here.

Presentation of the theory, with consideration given to finite Hankel transform theory, the development of finite difference analogues, stability analysis and numerical considerations to exploit the highly parallel nature of the problem are included. Numerical results for a range of geological models, for both amplitude versus offset (AVO) and vertical seismic profile (VSP) applications are presented.

INTRODUCTION

This method is often referred to as the pseudo-spectral method, but due to the extensive work done in the specific area considered here by B.G. Mikhailenko and A.S. Alekseev it is sometimes referred to, in seismic applications as the Alekseev-Mikhailenko Method (AMM). However, much of their work is relatively physically inaccessible and a considerable amount of the more significant contributions are in Russian. To maintain a citation data base of reasonable size and relevance, only a few noteworthy references will be given as any further pursuit of the literature may be obtained from the following cited works within the context of that part of the total problem that a reader may require more information regarding. This shortened list includes: Gazdag (1973, 1981), Kosloff and Baysal (1982), Alekseev and Mikhailenko (1980), Mikhailenko and Korneev (1984) and Mikhailenko (1985). The last of these is possibly the most general, dealing with numerous types of seismic wave propagation.

The medium type that is considered here is a radially symmetric, vertically inhomogeneous medium, which precludes the presence of lateral inhomogeneities, for AVO and VSP applications. The elastodynamic equations relevant to this geometry are investigated for the coupled $P-S_V$ particle displacement. Removal of the radial coordinate by a Hankel transform greatly reduces the physical resources (memory) requirements. Computation time is of the order of a 2D finite difference problem.

However, 3D geometrical spreading is integral to the method so that this problem would be referred to as a 2.5D numerical experimentation.

THEORETICAL DEVELOPMENT

General Theory

Consider the problem of coupled $P-S_V$ wave propagation in a radially symmetric (no lateral inhomogeneities), vertically inhomogeneous half space. The equations of motion are defined by the elastodynamic equation (Aki and Richards, 1980)

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+(\nabla\lambda)(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}+(\nabla\mu\cdot\nabla)\mathbf{u}+\nabla(\nabla\mu\cdot\mathbf{u})- \rho\,\partial^2\mathbf{u}/\partial t^2=\mathbf{F}(r,z,t) \tag{1}$$

where

$$\mathbf{u}\equiv\mathbf{u}(r,z,t)=(u_r,u_z) \tag{2}$$

is the vector particle displacement, r the radial coordinate, z the vertical coordinate, t is time, λ, μ are the elastic parameters of the medium and ρ is the density, all three of which may be dependent on z. There is no dependence on the coordinate θ in a cylindrical, (r,θ,z), coordinate system.

The problem is solved subject to the initial conditions

$$\mathbf{u}\big|_{t=0}=\left.\frac{\partial\mathbf{u}}{\partial t}\right|_{t=0}=0 \tag{3}$$

and the free surface boundary conditions that are required to be satisfied are

$$\sigma_{zz}\big|_{z=0}=0 \qquad \text{and} \qquad \sigma_{rz}\big|_{z=0}=0\,, \tag{4}$$

that is, the normal stress and shear stress are zero at the free surface.

The two typical types of sources, $\mathbf{F}(r,z,t)$, used in seismic applications are (B.G. Mikhailenko, 1980):

1. Vertical point force :

$$\mathbf{F}(r,z,t)=\frac{\delta(r)}{2\pi r}\delta(z-z_s)\,f(t)\,\mathbf{n_z}\,. \tag{5}$$

where $\mathbf{n_z}$ is a unit vector in the z (vertical downwards) direction.

2. Explosive point source of *P* waves:

$$\mathbf{F}(r,z,t)=\nabla\left(\frac{\delta(r)}{2\pi r}\delta(z-z_s)\right)f(t). \tag{6}$$

In the above, $\delta(\xi)$ is the Dirac delta function and $f(t)$ is some band limited source wavelet, about which more will be said later.

Assuming an explosive point source of *P* waves the radial, $u_r(r,z,t)$, and vertical, $u_z(r,z,t)$, components of particle displacement for the problem specified are given by

$$\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial u_z}{\partial z}\right]+\frac{\partial}{\partial z}\left[\frac{(\lambda+2\mu)}{r}\frac{\partial(ru_r)}{\partial r}\right]-\frac{\mu}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u_r}{\partial z}\right]- \\ 2\frac{\partial\mu}{\partial z}\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{\mu}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u_z}{\partial r}\right]-\rho\frac{\partial^2 u_z}{\partial t^2}=\frac{\delta(r)}{2\pi r}\frac{\partial}{\partial z}\left[\delta(z-z_s)\right]f(t) \tag{7}$$

and

$$\frac{\partial}{\partial z}\left[\mu\frac{\partial u_r}{\partial z}\right]+\frac{\partial}{\partial r}\left[(\lambda+2\mu)\frac{\partial u_z}{\partial z}\right]-\frac{\partial}{\partial z}\left[\mu\frac{\partial u_z}{\partial r}\right]+ \\ 2\frac{\partial\mu}{\partial z}\frac{\partial u_z}{\partial r}+(\lambda+2\mu)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right]-\rho\frac{\partial^2 u_r}{\partial t^2}=\frac{\partial}{\partial r}\left[\frac{\delta(r)}{2\pi r}\right]\delta(z-z_s)f(t) \tag{8}$$

where $\lambda=\lambda(z)$, $\mu=\mu(z)$ are the depth dependent elastic parameters and $\rho=\rho(z)$ is the density.

In terms of $u_r(r,z,t)$ and $u_z(r,z,t)$, the expressions for the normal stress and shear stresses, which are zero at the free surface have the form

$$\sigma_{zz}=\lambda\left[\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right]+(\lambda+2\mu)\frac{\partial u_z}{\partial z} \tag{9}$$

$$\sigma_{rz}=\mu\left[\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right] \tag{10}$$

Introducing the finite Hankel integral transforms and the transformed vector designation $\mathbf{G}(\tilde{k}_i,k_i,z,t)=\left(S(\tilde{k}_i,z,t),R(k_i,z,t)\right)$ has

$$R\left(k_i, z, t\right) = \int_0^a u_z\left(r, z, t\right) J_0\left(\tilde{k}_i r\right) r\, dr \tag{11}$$

$$S\left(\tilde{k}_i, z, t\right) = \int_0^a u_r\left(r, z, t\right) J_1\left(k_i r\right) r\, dr \tag{12}$$

where k_i and $\tilde{k}_i$ are the roots of the transcendental equations

$$J_0\left(\tilde{k}_i r\right) = 0 \tag{13}$$

and

$$J_1\left(k_i r\right) = 0, \tag{14}$$

respectively. Using the two formulations of the of Hankel transforms discussed in Appendix A, it may be shown that both of the inverse series summations may be accomplished using only the roots of one of the Bessel function transcendental equation, $J_1\left(k_i r\right) = 0$, so that

$$u_z\left(r, z, t\right) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{R\left(k_i, z, t\right) J_0\left(k_i r\right)}{\left[J_0\left(k_i a\right)\right]^2} \tag{15}$$

$$u_r\left(r, z, t\right) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{S\left(k_i, z, t\right) J_1\left(k_i r\right)}{\left[J_0\left(k_i a\right)\right]^2} \tag{16}$$

Thus, both inverse series summations may be taken over the roots of one rather than two transcendental equations and as a consequence, $\mathbf{G}\left(k_i, z, t\right) = \left(S\left(k_i, z, t\right), R\left(k_i, z, t\right)\right)$. The matter of what, numerically, constitutes an infinite number of terms in the inverse series summations will be addressed in due course. It will be shown that an earlier assumption that the source wavelet be band limited is significant in this determination. As the only spatial direction in which a finite difference is used is the *z* direction, an economical manner to account for spurious reflections from the boundary at $z = a$ is to introduce damping conditions in the vicinity of the lower *z* boundary of the form $\gamma\left(z\right)\partial R/\partial t$ and $\gamma\left(z\right)\partial S/\partial t$. A safe estimate for the length of this zone is of the order of *2* wavelengths *(WL)*.(B.G. Mikhailenko, 1980, and equation (34)). Alternatively, the explicit exponential damping discussed in Reynolds (1978) and Cerjan et al. (1985) may be used in the vicinity of the boundary at $z = z_{\max}$, where $z_{\max}$ is the finite maximum depth of the model, with similar results. The well known boundary damping methods discussed in Clayton and Enquist (1977) may also be considered.

Applying the appropriate Hankel transforms to equations (7) and (8) results in:

$$\frac{\partial}{\partial z}\left[\mu\frac{\partial S}{\partial z}\right]-(\lambda+2\mu)k_j\frac{\partial R}{\partial z}+k_j\frac{\partial[\mu R]}{\partial z}- \\ 2k_j\frac{\partial\mu}{\partial z}R-(\lambda+2\mu)k_j^2S-\rho\frac{\partial^2 S}{\partial t^2}-\gamma\frac{\partial S}{\partial t}=\frac{1}{4\pi}\frac{\partial}{\partial z}\left(\delta(z-z_s)\right)f(t) \tag{17}$$

$$\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial R}{\partial z}\right]+k_i\frac{\partial}{\partial z}\left[(\lambda+2\mu)S\right]-\mu k_i\frac{\partial S}{\partial z}-2k_i\frac{\partial\mu}{\partial z}S- \\ \mu k_i^2R-\rho\frac{\partial^2 R}{\partial t^2}-\gamma\frac{\partial R}{\partial t}=-\frac{k_i}{4\pi}\delta(z-z_s)f(t) \tag{18}$$

The development of a finite difference analogue for terms of the form

$$\frac{\partial}{\partial z}\left[\zeta(z)\frac{\partial\psi(k_i,z,t)}{\partial z}\right] \tag{19}$$

is given in Appendix B

If it is assumed that the elastic parameters have no spatial dependence, that is they are homogeneous throughout the model or some part thereof, the Hankel transformed equations (17) and (18) take the simplified forms given below. For convenience, it is assumed that the first two grid points in z, at the free surface are of this form.

$$v_s^2\frac{\partial^2 S}{\partial z^2}-k_i\left(v_p^2-v_s^2\right)\frac{\partial R}{\partial z}-k_i^2v_p^2S-\frac{\partial^2 S}{\partial t^2}=0 \tag{20}$$

$$v_p^2\frac{\partial^2 R}{\partial z^2}+k_i\left(v_p^2-v_s^2\right)\frac{\partial S}{\partial z}-k_i^2v_s^2R-\frac{\partial^2 R}{\partial t^2}=0 \tag{21}$$

The Hankel transformed shear and normal stresses required at the free surface as boundary conditions are

Normal stress:

$$\sigma_{zz}\big|_{z=0}=\rho\left[k_i\left(v_p^2-2v_s^2\right)S+v_p^2\frac{\partial R}{\partial z}\right]_{z=0}=0 \tag{22}$$

or

$$\left.\frac{\partial R}{\partial z}\right|_{z=0}=\left.\frac{k_i\left(v_p^2-2v_s^2\right)S}{v_p^2}\right|_{z=0} \tag{23}$$

and

Shear stress:

$$\sigma_{rz}\big|_{z=0} = \mu\left(\frac{\partial S}{\partial z} - k_i R\right)\bigg|_{z=0} = 0 \tag{24}$$

or

$$\frac{\partial S}{\partial z}\bigg|_{z=0} = k_i R\big|_{z=0}. \tag{25}$$

The finite difference analogues for equations (23) and (25) may be written as

$$R_{-1}^m = R_1^m - \frac{2\Delta z\, k_i \left(v_p^2 - 2v_s^2\right)_0 S_0^m}{\left(v_p^2\right)_0} = \frac{\left(v_p^2\right)_0 R_1^m - 2\Delta z\, k_i \left(v_p^2 - 2v_s^2\right)_0 S_0^m}{\left(v_p^2\right)_0} \tag{26}$$

and

$$S_{-1}^m = S_1^m - 2\Delta z\, k_i R_0^m \tag{27}$$

Free Surface Finite Difference Analogues

Using equations (26) and (27), equations (20) and (21) for the horizontal and vertical components of particle displacement at the free surface $(z = 0)$ may be determined. In the following, the spatial and temporal increments are Δz and Δt.

The finite difference analogue for the horizontal component of particle displacement at the free surface may be written as

$$\begin{aligned} S_0^{m+1} &= \frac{2(\Delta t)^2 \left(v_s^2\right)_0}{(\Delta z)^2} S_1^m - \frac{2\, k_i (\Delta t)^2 \left(v_s^2\right)_0}{(\Delta z)} R_0^m - S_0^{m-1} \\ &+ \left[2 - \frac{2(\Delta t)^2\left(v_s^2\right)_0}{(\Delta z)^2} + k_i^2 (\Delta t)^2 \frac{\left(v_p^2 - v_s^2\right)_0 \left(v_p^2 - 2v_s^2\right)_0 - \left(v_p^2\right)_0^2}{\left(v_p^2\right)_0}\right] S_0^m \end{aligned} \tag{28}$$

Continuing with the vertical component of particle displacement yields, at the free surface, the finite difference analogue

$$\begin{aligned} R_0^{m+1} &= \frac{2(\Delta t)^2\left(v_p^2\right)_0 R_1^m}{(\Delta z)^2} - \frac{2k_i (\Delta t)^2 \left(v_p^2 - 2v_s^2\right)_0 S_0^m}{(\Delta z)} \\ &+ \left[2 + k_i^2 (\Delta t)^2 \left(v_p^2 - 2v_s^2\right)_0 - \frac{2(\Delta t)^2 \left(v_p^2\right)_0}{(\Delta z)^2}\right] R_0^m - R_0^{m-1} \end{aligned} \tag{29}$$

General Point Finite Difference Analogues

Horizontal component:

$$
\begin{aligned}
S_n^{m+1} = \Bigg\{ & \frac{(\Delta t)^2}{\rho_n (\Delta z)^2}\left(b_{n+1}S_{n+1}^m - (b_{n+1}+b_n)S_n^m + b_n S_{n-1}^m\right) - \\
& \frac{k_j(\Delta t)^2}{2\rho_n(\Delta z)}(\lambda+\mu)_n\left(R_{n+1}^m - R_{n-1}^m\right) - \\
& \frac{k_j(\Delta t)^2}{2\rho_n(\Delta z)}(\mu_{n+1}-\mu_{n-1})R_n^m - \frac{k_j^2(\Delta t)^2}{\rho_n}(\lambda+2\mu)_n S_n^m + 2S_n^m - \\
& \left(1-\frac{\gamma_n(\Delta t)}{2\rho_n}\right)S_n^{m-1}\Bigg\} \Bigg/ \left(1+\frac{\gamma_n(\Delta t)}{2\rho_n}\right)
\end{aligned}
\tag{30}
$$

Vertical component:

$$
\begin{aligned}
R_n^{m+1} = \Bigg\{ & \frac{(\Delta t)^2}{(\Delta z)^2\rho_n}\left(a_{n+1}R_{n+1}^m - (a_{n+1}+a_n)R_n^m + a_n R_{n-1}^m\right) + \\
& \frac{(\Delta t)^2 k_j}{2(\Delta z)\rho_n}\left((\lambda+2\mu)_{n+1} - (\lambda+2\mu)_{n-1}\right)S_n^m + \frac{(\Delta t)^2 k_j}{2(\Delta z)\rho_n}(\lambda+\mu)_n\left(S_{n+1}^m - S_{n-1}^m\right) \\
& - \frac{(\Delta t)^2}{(\Delta z)\rho_n}k_j(\mu_{n+1}-\mu_{n-1})S_n^m - \frac{k_i^2}{\rho_n}\mu_n R_n^m + 2R_n^m - \\
& \left(1-\frac{\gamma_n(\Delta t)}{2\rho_n}\right)R_n^{m-1}\Bigg\} \Bigg/ \left(1+\frac{\gamma_n(\Delta t)}{2\rho_n}\right)
\end{aligned}
\tag{31}
$$

Various numerical considerations

There are numerous minor questions that also need to be discussed. A few of the more significant ones are briefly addressed below.

- Numerical implementation of the operation "$\nabla\left(\delta\left(z-z_0\right)\right)$". The source term $F\left(z-z_j\right)=\nabla\left(\delta\left(z-z_j\right)\right)$ may be approximated by

$$
\frac{\partial\left[\delta\left(z-z_j\right)\right]}{\partial z} \approx \frac{\delta\left(z_{j+1}\right)-\delta\left(z_{j-1}\right)}{2\Delta z}. \tag{32}
$$

Possibly a more numerically correct way of accomplishing this is to consider the approximate relation

$$F\left(z-z_j\right)=\delta\left(z-z_j\right)\approx\sqrt{\frac{m_0}{\pi}}\exp\left[-n_0\left(z-z_j\right)^2\right]\Bigg|_{n_o\to\infty} \tag{33}$$

and proceed using this formulation.

- Number of points per wavelength (WL). As there are several ways in which a WL has been defined, this topic will clarified first. Given that $v_{\min}$ is the minimum velocity encountered in the z – direction, where both the v_P and v_S velocity values are considered in this determination, and $f_{\max}$ is that frequency beyond which the spectrum of the band limited source is zero (numerically), the definition of a WL used here is

$$WL=\frac{v_{\min}}{f_{\max}}. \tag{34}$$

As the spatial finite difference analogues used here are of second order accuracy, at least $10\ \text{points}/WL$ would be required to keep grid dispersion to a minimum. In the problem considered here, where the vertical and horizontal components of particle displacement as well as all required elastic parameters need only to be specified at a sequence of depth points (one spatial dimension), the use of two to four times this minimum number does not cause any major problems in allocating space for the arrays required during the computation process. When compared to schemes employing finite difference methods, this ability to increase the number of points per WL in this manner allows for a simple manner to perform an analysis of grid dispersion versus points per WL.

CONCLUSIONS

The theory and development of finite difference analogues for $P-S_V$ wave propagation in a plane parallel layered model has been presented. The radial coordinate was removed using a finite Hankel transform prior to implementation of finite difference process. What results are a coupled system of finite difference equations in only depth and time. The radial component is recovered by applying an inverse Hankel transform summation, which although infinite, may be truncated if a band limited source wavelet is used. The synthetic traces produced using this method have 3D spreading and the amount of computer resources is reduced considerably as the vertical and horizontal components of particle displacement as well as all required elastic parameters need only to be specified at a sequence of depth points – one spatial dimension as opposed to two.

The finite difference analogues given are accurate to second order in both time and space (depth). The analogues for a surface point as well as general points within the medium are given. Provisions for either a vertical or explosive point source of P – waves are included in the derivations. A number of points regarding this seismic modeling process, especially where some mathematical rigor is required are dealt with in a series of Appendices.

Using the formulae presented here it should be possible write a hybrid finite difference – finite integral transform programs for a variety of source – receiver configurations including AVO and VSP.

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APPENDIX A: FINITE HANKEL TRANSFORM

Although the two following finite Hankel transform methods may be found in the literature (Sneddon, 1972, for example), it was felt that for completeness they should be included here, at least in an abbreviated theorem formulation. The finite Hankel transform of the first kind is a direct application of the following theorem.

Theorem I: If $f\left(x\right)$ satisfies Dirichlet's conditions in the interval $\left(0,a\right)$ and if its Hankel transform in that range is defined to be

$$H_{\mu}^{(1)}\left[f\left(x\right)\right]\equiv f_{\mathrm{J}}\left(\xi_{j}\right)=\int_{0}^{a}x\,f\left(x\right)J_{\mu}\left(\xi_{j}x\right)dx \qquad \text{(A.1)}$$

where ξ_j is a root of the transcendental equation

$$J_{\mu}\left(\xi_{j}a\right)=0 \qquad \text{(A.2)}$$

then, at any point in the interval $\left(0,a\right)$ at which the function $f\left(x\right)$ is continuous ,

$$f\left(x\right)=\frac{2}{a^{2}}\sum_{j=1}^{\infty}f_{\mathrm{J}}\left(\xi_{j}\right)\frac{J_{\mu}\left(\xi_{j}x\right)}{\left[J_{\mu+1}\left(\xi_{j}x\right)\right]^{2}} \qquad \text{(A.3)}$$

where the sum is taken over all the positive roots of equation (A.2).

The finite Hankel transform and inverse of the second kind used in the text are given as follows:

Theorem II: If $f\left(x\right)$ satisfies Dirichlet's conditions in the interval $\left(0,a\right)$ and if its Hankel transform in that range is defined to be

$$H_{\mu}^{(1)}\left[f\left(x\right)\right]\equiv f_{\mathrm{J}}\left(\xi_{j}\right)=\int_{0}^{a}x\,f\left(x\right)J_{\mu}\left(\xi_{j}x\right)dx \qquad \text{(A.4)}$$

in which ξ_j is a root of the transcendental equation

$$\xi_{j}J_{\mu}'\left(\xi_{j}a\right)+h\,J_{\mu}\left(\xi_{j}a\right)=0 \qquad \text{(A.5)}$$

then, at each point in the interval $\left(0,a\right)$ at which the function $f\left(x\right)$ is continuous ,

$$f\left(x\right)=\frac{2}{a^{2}}\sum_{j=1}^{\infty}\frac{\xi_{j}^{2}f_{\mathrm{J}}\left(\xi_{j}\right)}{h^{2}+\left(\xi_{j}^{2}-\mu^{2}/a^{2}\right)}\frac{J_{\mu}\left(\xi_{j}x\right)}{\left[J_{\mu}\left(\xi_{j}x\right)\right]^{2}} \qquad \text{(A.6)}$$

where the sum is taken over all the positive roots of (A.5) and *h* is determined from a boundary operator **N** at $x=a$ defined as

$$\mathbf{N}[f] = \frac{df(a)}{dx} + h\, f(a) = 0\,. \tag{A.7}$$

APPENDIX B: FINITE DIFFERENCE ANALOGUE

For determining the finite difference analogue in the case of an operation of the type

$$\frac{\partial}{\partial z}\left[\zeta(z)\frac{\partial B(z)}{\partial z}\right] \tag{B.1}$$

let

$$w(z) = \zeta(z)\frac{\partial B(z)}{\partial z} \tag{B.2}$$

or equivalently

$$w(z)\frac{\partial z}{\zeta(z)} = \partial B(z) \tag{B.3}$$

which may be written as

$$w_{k-1/2}\int_{z_{k-1}}^{z_k}\frac{dz}{\zeta(z)} = B_k - B_{k-1} \tag{B.4}$$

or

$$w_{k-1/2} = \left(B_k - B_{k-1}\right)\left[\int_{z_{k-1}}^{z_k}\frac{dz}{\zeta(z)}\right]^{-1} \tag{B.5}$$

In a similar manner

$$w_{k+1/2}\int_{z_k}^{z_{k+1}}\frac{dz}{\zeta(z)} = B_{k+1} - B_k \tag{B.6}$$

or

$$w_{k+1/2} = \left(B_{k+1} - B_k\right)\left[\int_{z_k}^{z_{k+1}}\frac{dz}{\zeta(z)}\right]^{-1} \tag{B.7}$$

so that

$$\frac{\partial w(z)}{\partial z} = \frac{\partial}{\partial z}\left[\zeta(z)\frac{\partial B(z)}{\partial z}\right] \tag{B.8}$$

whose finite difference analogue is of the form

$$\frac{\partial w(z)}{\partial z} \approx \frac{\left(w_{k+1/2} - w_{k-1/2}\right)}{\Delta z} \tag{B.9}$$

which in terms of $\gamma(z)$ and B_j may be written as

$$\frac{\partial}{\partial z}\left[\zeta(z)\frac{\partial B(z)}{\partial z}\right] \approx \frac{\left(B_{k+1} - B_k\right)}{(\Delta z)^2}\left[\frac{1}{\Delta z}\int_{z_k}^{z_{k+1}}\frac{dz}{\zeta(z)}\right]^{-1} - \frac{\left(B_k - B_{k-1}\right)}{(\Delta z)^2}\left[\frac{1}{\Delta z}\int_{z_{k-1}}^{z_k}\frac{dz}{\zeta(z)}\right]^{-1} \tag{B.10}$$

from which it follows that

$$\frac{\partial}{\partial z}\left[\zeta(z)\frac{\partial B(z)}{\partial z}\right] \approx \frac{\chi_{k+1}B_{k+1} - \left(\chi_{k+1} + \chi_k\right)B_k + \chi_k B_{k-1}}{(\Delta z)^2} \tag{B.11}$$

where the χ_k are obtained as

$$\chi_k = \left[\frac{1}{\Delta z}\int_{z_{k-1}}^{z_k}\frac{dz}{\zeta(z)}\right]^{-1} = \frac{2\zeta_k\zeta_{k-1}}{\zeta_k + \zeta_{k-1}} \tag{B.12}$$

using the trapezoidal numerical integration scheme.

APPENDIX C: TERMS IN INVERSE SUMMATION SERIES

The analytic Fourier transform of the Gabor wavelet

$$f(t) = \cos(\omega_0 t)\exp\left[-\left(\frac{\omega_0 t}{\gamma}\right)^2\right] \tag{C.1}$$

is, apart from some constant multiplicative terms,

$$F(\omega) = \frac{\pi^{1/2}}{\omega_0}\exp\left[-\frac{\gamma^2}{4}\left(1 + \omega/\omega_0\right)^2\right]\cosh\left(\frac{\omega\gamma^2}{\omega_0}\right). \tag{C.2}$$

The horizontal wave number, k, in a coordinate system with cylindrical symmetry is related to the angular frequency as

$$k = \frac{\omega}{v} \tag{C.3}$$

where v is velocity and ω is the circular frequency. It will be assumed that some upper bound, ω_u, on the band limited spectrum of the source wavelet has been determined, often through numerical integration of the spectrum and then reintegration to the value ω_u up to which about 99.99% of the initial integration. Once ω_u has been determined, the value of k_u may be obtained as

$$k_u = \frac{\omega_u}{v_{\min}} \tag{C.4}$$

with $v_{\min}$ being the minimum velocity P or S_V encountered on the spatial grid which is one dimensional. It is known from numerical experiments that a good approximation for the duration of the Gabor wavelet in the time domain is γ / f_0. For some arbitrary k_i in the inverse series,

$$k_i = \frac{\zeta_i}{a} \tag{C.5}$$

where the values of ζ_i are the roots of the transcendental equation

$$J_1(\zeta_i) = 0 \tag{C.6}$$

so that

$$k_u = \frac{\omega_u}{v_{\min}} = \frac{\zeta_u}{a} \tag{C.7}$$

or equivalently

$$\zeta_u = \frac{a\,\omega_u}{v_{\min}} \tag{C.8}$$

indicating that the number of terms which must be considered to adequately approximate the infinite series summation increases linearly with *a*. It may be seen upon examination of equation (C.2) that the spectral width of the Gabor wavelet decreases with increasing values of γ. With the value of $\gamma = 4$ used here, $\omega_u \approx 2\omega_0$, so that with the predominant wavelength defined in terms of the predominant circular frequency and the minimum velocity encountered, $\lambda_0 = f_0 / v_{\min}$ equation (C.7) becomes

$$\zeta_u = 4\pi\alpha \tag{C.9}$$

In the above equation, $\alpha = a/\lambda_0$, a dimensionless quantity relating the predominant wavelength with the pseudo – boundary introduced at $r = a$. For large values of *i*, the relation approximate relation $\zeta_i \approx \pi i$ holds (Abramowitz and Stegun, 1980). Thus the number of terms *N*, required to approximate the infinite series is, with $\zeta_u \approx \pi N$, given as

$$N \approx 4\alpha\,. \tag{C.10}$$

For comparison purposes, going through the derivation with $\gamma = 5$ results in the value of N being given as

$$N \approx 8\alpha/5 \tag{C.11}$$

which is less than that estimated for $\gamma = 4$, as would be expected.

APPENDIX D: STABILITY ANALYSIS

For a system of coupled hyperbolic equations, Mitchell (1977) and Richtmeyer and Morton (1967) both (among others) consider a detailed manner, employing a Fourier series decomposition to determine the finite difference stability criteria. For completeness, this analysis will be presented in a condensed form here as a consequence of a number of typographic errors in the literature (Pasco, 1984).

Defining $v_s^2 = \mu/\rho$ and $v_p^2 = (\lambda + 2\mu)/\rho$ the homogeneous form of the coupled transformed $P - S_V$ equations of motion, in terms of transformed vertical $\left(R\left(k_j, z, t\right)\right)$ and horizontal $\left(S\left(k_j, z, t\right)\right)$ vector components of particle displacement with the source term removed may be written as

$$\begin{aligned} S_n^{m+1} = {} & \left(\frac{\Delta t\left(v_s\right)_n}{\Delta z}\right)^2 S_{n+1}^m - \left\{2\left(\frac{\Delta t\left(v_s\right)_n}{\Delta z}\right)^2 - 2 + k_i^2\left(\Delta t\right)^2\left(v_p^2\right)_n\right\} S_n^m + \\ & \left(\frac{\Delta t\left(v_s\right)_n}{\Delta z}\right)^2 S_{n-1}^m - \frac{k_i\left(\Delta t\right)^2\left(v_p^2 - v_s^2\right)_n}{2\left(\Delta z\right)}\left(R_{n+1}^m - R_{n-1}^m\right) - P_n^m \end{aligned} \tag{D.1}$$

$$\begin{aligned} R_n^{m+1} = {} & \left(\frac{\left(v_p\right)_n \Delta t}{\Delta z}\right)^2 R_{n+1}^m - \left\{2\left(\frac{\left(v_p\right)_n \Delta t}{\Delta z}\right)^2 - 2 + k_i^2\left(\Delta t\right)^2\left(v_s^2\right)_n\right\} R_n^m + \\ & \left(\frac{\left(v_p\right)_n \Delta t}{\Delta z}\right)^2 R_{n-1}^m + \frac{k_i\left(\Delta t\right)^2\left(v_p^2 - v_s^2\right)_n}{2\Delta z}\left(S_{n+1}^m - S_{n-1}^m\right) - Q_n^m \end{aligned} \tag{D.2}$$

$$Q_n^{m+1} = R_n^m \tag{D.3}$$

$$P_n^{m+1} = S_n^m \tag{D.4}$$

In the above, Q_n^{m+1} and P_n^{m+1} have been introduced to reformulate the finite difference analogues to a two level time scheme in four unknowns.

What is required is to determine an amplification matrix whose eigenvalues, η_i, $\left(i = 1, 2, 3, 4\right)$ are such that stability, dependent on the spectral radius, is ensured at all grid points at a given time level, $t_m = \left[(m+1)\Delta t, m\Delta t, (m-1)\Delta t \right]$, where it is convenient and is generally applicable for this analysis to choose $m=1$ The implication is that the maximum of $|\eta_i|$ must be uniformly bounded, i.e., $|\eta_i| < 1$ for all t_m, $(0 \le t_m \le T = n_{\max}\Delta t)$.

It follows from Lax's Equivalence Theorem[1] (Richtmeyer and Morton, 1967) that given a finite difference approximation to a properly posed initial value problem, as is the case here, stability is a necessary and sufficient condition for convergence to the continuous problem. Consistency requires that the truncation error tends to zero as $\Delta t \to 0$ for $0 \le t \le T$, T being the time length of the finite difference computation. The amplification factors (eigenvalues), η_i, are obtained by substituting the error function E into equations (D.1) and (D.2). What results is a system of equations in R_n^{m+1}, S_n^{m+1}. P_n^{m+1} and Q_n^{m+1}. The form of the error functions (Mitchell, 1977) are

$$E = \sum_i A_i \exp\left[i\eta_i \left(k\,\Delta z \right) \right] \tag{D.5}$$

Defining the propagation vector

$$\mathbf{W}_n^m = \left[P_n^m, Q_n^m, R_n^m, S_n^m \right]^T \tag{D.6}$$

with the relation between consecutive depth points of an element of W_n^m $(W = P,Q,R,S)$ is given by

$$W_{n+1}^m = e^{i\ell\Delta z} W_n^m \tag{D.7}$$

Substitution of equation (D.7) into the system of equations (D.1) - (D.4) yields the linear equation set

$$\mathbf{W}_n^{m+1} = \bar{\mathbf{B}}\,\mathbf{W}_n^m \tag{D.8}$$

where

$$\bar{\mathbf{B}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & B_{33} & B_{34} \\ 0 & -1 & -B_{34} & B_{44} \end{bmatrix} \tag{D.9}$$

with

[1] "A consistent finite difference scheme for a partial differential equation for which the initial-value problem is well posed is convergent if and only if it is stable."

$$B_{33} = \left(\frac{v_P \Delta t}{\Delta z} \right)^2 \left[2 - 4\sin^2\left(\ell \Delta z/2\right) - k_j^2 v_S^2 \left(\Delta t\right)^2 \right] \quad \text{(D.10)}$$

$$B_{34} = \left(\frac{m k_j \left(\Delta t\right)^2}{\Delta z} \right) \left[\left(v_P^2 - v_S^2 \right) \sin\left(\ell \Delta z\right) \right] \quad \text{(D.11)}$$

$$B_{44} = \left(\frac{v_S \Delta t}{\Delta z} \right)^2 \left[2 - 4\sin^2\left(\ell \Delta z/2\right) - k_j^2 v_P^2 \left(\Delta t\right)^2 \right] \quad \text{(D.12)}$$

The amplifications factors are the eigenvalues, η_i, of the matrix $\bar{\mathbf{B}}$, obtained from

$$\det\left[\mathbf{B} - \eta \mathbf{I}\right] = 0 \quad \text{(D.13)}$$

which in general terms may be written in terms of invariant coefficients as

$$\begin{aligned} &\eta^4 - \left(\eta_1 + \eta_2 + \eta_3 + \eta_4\right)\eta^3 + \\ &\quad \left[\eta_1\eta_2 + \eta_3\eta_4 + \left(\eta_1 + \eta_2\right)\left(\eta_3 + \eta_4\right)\right]\eta^2 - \\ &\quad \left[\eta_1\eta_2\left(\eta_3 + \eta_4\right) + \eta_3\eta_4\left(\eta_1 + \eta_2\right)\right]\eta + \eta_1\eta_2\eta_3\eta_4 = 0 \end{aligned} \quad \text{(D.14)}$$

or specifically for this case as

$$\eta^4 - \left(B_{33} + B_{44}\right)\eta^3 + \left(B_{33}B_{44} + B_{34}^2\right)\eta^2 + 1 = 0 \quad \text{(D.15)}$$

so that comparing (D.14) and (D.14) yields

$$\sum_{\ell=1}^{4} \lambda_\ell = \left(B_{33} + B_{44}\right). \quad \text{(D.16)}$$

The condition for stability is $\left|\lambda_\ell\right| < 1,\ \mathit{for}\ \ell = 1, \ldots, 4.$ This leads to the conditional

$$\left| \sum_{\ell=1}^{4} \lambda_\ell \right| \le \sum_{\ell=1}^{4} \left|\lambda_\ell\right| < 4 \quad \text{(D.17)}$$

which together with (D.16) to

$$\left| 4 - 4\sin^2\left(\ell \Delta z/2\right) \left[\left(\frac{v_P \Delta t}{\Delta z} \right)^2 + \left(\frac{v_S \Delta t}{\Delta z} \right)^2 \right] - k_j^2 \left(v_P^2 + v_S^2 \right)\left(\Delta t\right)^2 \right| < 4 \quad \text{(D.18)}$$

or equivalently, as $\sin^2\left(\ell \Delta z/2\right) < 1$,

$$\left| 4 - \left[\left(\frac{v_P \Delta t}{\Delta z} \right)^2 + \left(\frac{v_S \Delta t}{\Delta z} \right)^2 \right] - k_j^2 \left(v_P^2 + v_S^2 \right)\left(\Delta t\right)^2 \right| < 4 \quad \text{(D.19)}$$

defines the requirement for stability of the finite difference scheme, The resulting two inequalities from (D.18) explicitly stating a stability condition are

$$\left[\left(\frac{v_P \Delta t}{\Delta z}\right)^2 + \left(\frac{v_S \Delta t}{\Delta z}\right)^2\right] + k_j^2 \left(v_P^2 + v_S^2\right)\left(\Delta t\right)^2 > 0 \qquad \text{(D.20)}$$

and

$$\left[\left(\frac{v_P \Delta t}{\Delta z}\right)^2 + \left(\frac{v_S \Delta t}{\Delta z}\right)^2\right] + k_j^2 \left(v_P^2 + v_S^2\right)\left(\Delta t\right)^2 < 2\,. \qquad \text{(D.21)}$$

Inequality (D.20) is trivially satisfied as all quantities are positive definite. Thus, the necessary and sufficient condition for the stability of the finite difference problem under consideration is that (D.21) be satisfied.