Various topics in pseudo-differential operator theory applied to scalar *qP* wave propagation in a transversely isotropic medium

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ABSTRACT

Scalar wave equation approximations for quasi-compressional (qP) propagation in a transversely isotropic medium are developed and solution options presented. The initial eikonals are obtained from linearization of the exact eikonal, or more correctly the linearized phase velocity, as well as from other approximations of the exact eikonal. The linearized approximate phase velocities or eikonals are used to construct partial differential equation of order four in spatial derivatives and order two in time using pseudo-differential operator theory. The assumption that some or all of the 2D model space is rotated at some angle with respect to a Cartesian model coordinate system is examined in a cursory manner. That the medium is weakly anelliptic is understood. Also, as the anisotropic parameters are usually spatially dependent this fact is taken explicitly into consideration when constructing the partial differential equation. The degenerate, or elliptical case, is also investigated as it is much simpler with results that should at least grossly approach the full scalar *qP* wave equation.

INTRODUCTION

There are a number of directions in which to proceed if a scalar wave approximation for quasi-compressional (qP) propagation in a transversely isotropic medium, or a related expression if the medium is assumed to be rotated at some spatially varying angle with respect to model coordinates, is sought (Igel et al, 1995, Alkhalifah, 1998a and 1998b, Zhang et al., 2003, Zhang et al., 2004, Zhang et al., 2004). Clearly some approximations must be made, most often that the media are weakly anelliptic. Under this assumption, a linearized formulation of the phase velocity (Backus, 1965) and corresponding eikonal would be a reasonable starting point as would a weakly anelliptic approximation of the exact eikonal equation. From these a number of procedural methods may be taken for similar scalar wave equations describing *qP* wave propagation in a weakly anelliptic *TI* medium. What has appeared to be neglected in the literature on this topic is the likelihood that the anisotropic parameters are spatially dependent. This may seem a trivial matter after all the approximations that have been made to the point where a scalar *qP* wave equation is to be reconstructed from a linearized eikonal equation, usually by utilizing pseudo-differential operator theory. However, it is thought that this matter should be given at least minimal attention. Writing the scalar approximation in a symmetric form adds some mathematical rigor to the theoretical and hence numerical development. As the eikonal is obtained from a symmetric operator, it would seem appropriate that any scalar equation that approximates the original equation should also be composed of symmetric operators. A comparison of several methods used to obtain the approximate scalar *qP* wave equation in a *TI* medium are developed.

THEORETICAL DEVELOPMENT

In this section three similar scalar wave equations for qP wave propagation in a *TI* medium are developed with the focus being on the approximations required to obtain each. The equation developed by Alkhalifah (1998a, 1998b) is used as a reference and is presented in Appendix C. As two different notation schemes are (necessarily) used, the relationships between the two notation schemes are given in Table 1 and the similarities of the various scalar wave equations are shown. The assumptions that are required to be made suggest that the eikonal equation presented by Podvin and Lecomte (1991) and Lecomte (1992, 1993) – initially assuming a suitable eikonal based on intuition, derived from experience in the field of anisotropic wave propagation methods, could have been used as a starting point. As a consequence much mathematical gymnastics done away with and the determination of the anelliptic term by empirical methods, usually involving travel times, could have been immediately pursued.

Approximation 1:

A linearized quasi-compressional (qP) eikonal in a transversely isotropic medium may be obtained from the linearized qP phase velocity, $v_{qP}(x_k, n_k)$, (Backus, 1965) given by

$$v_{qP}(x_k, n_k) = A_{11}n_1^2 + A_{33}n_3^2 + E_{13}n_1^2n_3^2 \tag{1}$$

where the unit phase velocity vector is $\mathbf{n} = (n_1, n_3)$ $(|\mathbf{n}| = 1)$ and the x_k dependence indicates that the A_{ii} and E_{13} may be functions of position. Divide equation (1) to obtain a pseudo-eikonal with, $p_k = n_k / v_{qP}(x_k, n_k)$ being the slowness vector components, to obtain

$$G_{qP}(x_k, n_k, p_k) = A_{11}p_1^2 + A_{33}p_3^2 + E_{13}p_1p_3n_1n_3 = 1. \tag{2}$$

To remove the dependence of (2) on the n_k, the identities $v_{qP}(x_k, n_k)/v_{qP}(x_k, n_k) = 1$ and $n_1^2 + n_3^2 = 1$ are introduced to produce the linearized eikonal

$$G_{qP}(x_k, p_k) = A_{11}p_1^2 + A_{33}p_3^2 + \frac{E_{13}p_1^2p_3^2}{p_1^2 + p_3^2} = 1. \tag{3}$$

The slowness vector is defined as $\mathbf{p} = (p_1, p_3)$, where the components, p_k, have been defined above. The density normalized anisotropic parameters, A_{ij}, have the dimensions of velocity squared and as previously mentioned the A_{ii} and E_{13} may be spatially dependent. The term defining the deviation from the elliptical (the linearized anelliptical term) is

$$E_{13} = 2\left(A_{13} + 2A_{55}\right) - \left(A_{11} + A_{33}\right). \tag{4}$$

In an alternate notation, $E_{13} = 2A_{33}\left(\delta^{(\ell)} - \varepsilon\right)$ where $\delta^{(\ell)}$ is the linearized form of δ and ε is given by the standard definition, $\varepsilon = \left(A_{11} - A_{33}\right)/2A_{33}$, the following expression for $\delta^{(\ell)}$ results

$$\delta^{(\ell)} = \frac{\left(A_{13} + 2A_{55}\right) - A_{33}}{A_{33}} \tag{5}$$

It is not unreasonable to assume that a solution for some amplitude potential is of the form

$$\textit{Amplitude potential} \propto \exp\left[-i\omega t + ik_1x_1 + ik_3x_3\right]. \tag{6}$$

The associated pseudo – differential operators with the exponential terms are defined as $ik_1 = i\omega p_1$ and $ik_3 = i\omega p_3$, and given in temporal and spatial operator notation as

$$-i\omega \to \partial_t \qquad ik_1 \to \partial_1 \qquad ik_3 \to \partial_3 \tag{7}$$

Equation (3) may be rewritten in a symmetric form, for some operation "$\circ$", on some potential function ϕ as

$$\left\{p_1^2A_{11}p_1^2 + p_3^2A_{33}p_3^2 + p_1p_3\tilde{E}_{13}p_1p_3 - \left(p_1A_{00}p_1 + p_3A_{00}p_3\right)\right\} \circ \phi = 0 \tag{8}$$

where now

$$\tilde{E}_{13} = 2\left(A_{13} + 2A_{55}\right). \tag{9}$$

has been obtained as follows

$$\begin{aligned} p_1p_3\tilde{E}_{13}p_1p_3 &= p_1p_3E_{13}p_1p_3 + p_1p_3\left(A_{11} + A_{33}\right)p_1p_3 \\ &= p_1p_3\left[2\left(A_{13} + 2A_{55}\right)\right]p_1p_3 \end{aligned} \tag{10}$$

For convenience, the quantity $A_{00}\left(x_k\right) \equiv 1 \ \forall\left[x_k : \left(-\infty < x_k < \infty\right)\right]$ has temporarily been introduced.

The partial derivative operations defined in equation (7) are introduced by first pre-multiplying equation (8) by $\left(-i\omega\right)^4$. It is assumed that (8) operates on some function (potential) $\phi\left(x,z,t\right)$ such that a force is defined by $\mathbf{u} = \nabla\phi$ and as a result pressure is given by $P = \nabla\cdot\nabla\phi = \nabla\cdot\mathbf{u} = \nabla^2\phi$ to yield

$$\partial_1^2\left(A_{11}\partial_1^2\phi\right)+\partial_3^2\left(A_{33}\partial_3^2\phi\right)+\partial_1\partial_3\left(\tilde{E}_{13}\partial_1\partial_3\phi\right)-\partial_t^2\left(\partial_1^2+\partial_3^2\right)\phi=0 \tag{11}$$

The above equation incorporates the most general case of a linearized quasi-compressional scalar equation, as it assumes that the anisotropic parameters A_{ij} are spatially dependent, i.e., $A_{ij}\equiv A_{ij}\left(x_1,x_3\right)$. It is often convenient to write a part of equation (11) in terms of pressure so that with

$$\partial_t^2\left(\partial_1^2+\partial_3^2\right)\phi=\partial_t^2\left(\nabla^2\phi\right)=\partial_t^2\left(\nabla\cdot\mathbf{u}\right)=\partial_t^2 P \tag{12}$$

equation (11) becomes

$$\partial_1^2\left(A_{11}\partial_1^2\phi\right)+\partial_3^2\left(A_{33}\partial_3^2\phi\right)+\partial_1\partial_3\left(\tilde{E}_{13}\partial_1\partial_3\phi\right)-\partial_t^2 P=0 \tag{13}$$

or in a more convenient notation

$$\Re\left(0\right)\left[\phi\left(x_1,x_3,t\right)\right]-\partial_t^2 P=0 \tag{14}$$

where the function $\Re\left(\chi\right)$ may be inferred from equation (13) with the parameter, χ, of the function, $\Re\left(\chi\right)$, indicating its orientation to the global model Cartesian coordinate system. Applying an orthonormal rotation of the type discussed in Appendix B has the property that what has been designated above as pressure $\left(P\right)$ is invariant with respect to this type of rotation, which is what would be expected.

It should be noted at this point that in a number of papers on this topic the assumption that the relationship between the potential, ϕ, and the pressure. P, is given by $P=\partial_t^2\phi$, where here the derivation leads to the relationship that $P=\nabla^2\phi$. This requires that to recover ϕ, assuming that P is known at the current time step, requires ϕ must be obtained as the solution of banded matrix problem, that is, a pentadiagonal matrix problem, which adds a significant amount of complexity to the solution of this problem. An implicit, alternating direction (*ADI*) algorithm to accomplish this is described in Appendix A. The explanation for $P=\partial_t^2\phi$ replacing $P=\nabla^2\phi$ in other approximations in the literature is dealt with in a subsequent section.

Continuing with the problem at hand, equation (13) is solved with zero initial values, i.e.,

$$\phi\big|_{t=0}=\partial_t\phi\big|_{t=0}=0 \tag{15}$$

and

$$P\big|_{t=0} = \partial_t P\big|_{t=0} = 0 \tag{16}$$

A point source term, $F\left(x_1, x_3, t\right)$, may be introduced in equation (13) for $\phi\left(x_1, x_3, t\right)$ at $\mathbf{x}^0 = \left(x_1^0, x_3^0\right)$ (Mikhailenko, 1980)

$$\mathbf{u}\left(x_1, x_3, t\right) = \nabla F\left(x_1, x_3\right) f\left(t\right) = \nabla\left[\delta\left(x_1 - x_1^0\right)\delta\left(x_3 - x_3^0\right)\right] f\left(t\right) \tag{17}$$

so that

$$F\left(x_1, x_3, t\right) = \delta\left(x_1 - x_1^0\right)\delta\left(x_3 - x_3^0\right) f\left(t\right) \tag{18}$$

In the above $\delta(\zeta)$ is the Dirac delata function and $f\left(t\right)$ is some (usually band limited) source wavelet.

Approximation 2:

The derivation of this form of the scalar *qP* wave equation in a *TI* medium will use equation (2) as a starting point, viz.,

$$G_{qP}\left(x_k, n_k, p_k\right) = A_{11}p_1^2 + A_{33}p_3^2 + E_{13}p_1p_3n_1n_3 = 1 \tag{19}$$

In this instance, rather than introducing the two equalities that precede equation (3), the approximation $V^2 \big/ v_{qP}^2\left(p_k\right) = 1$, for some V, which may be spatially dependent, to be determined is introduced. In light of the eikonal employed in the first section in this report this assumption may be considered somewhat problematic. It would not be unreasonable to assume that V^2 is, as $A_{33} \le v_{qP}^2\left(p_k\right) \le A_{11}$, such that the behavior of V^2 be similar, i.e., $A_{33} \le V^2 \le A_{11}$. Proceeding leads to the eikonal

$$G_{qP}\left(x_k, p_k\right) = A_{11}p_1^2 + A_{33}p_3^2 + V^2E_{13}p_1^2p_3^2 = 1 \tag{20}$$

Lecomte (1992) employed a similar equation, in migration algorithms,

$$A_{11}p_1^2 + A_{33}p_3^2 + A_\delta p_1^2p_3^2 = 1 \tag{21}$$

where the spatially dependent quantity A_δ is required to be obtained empirically.

Introducing the pseudo-differential operators as in the preceding section results in

$$\left\{(i\omega)^2 A_{11}\left(-ik_1\right)^2 + (i\omega)^2 A_{33}\left(-ik_3\right)^2 + V^2E_{13}\left(-ik_1\right)^2\left(-ik_3\right)^2 - (i\omega)^4\right\}\circ\phi = 0 \tag{22}$$

for some potential, ϕ and subsequently to the partial differential equation

$$A_{11}\partial_t^2\partial_1^2\phi + A_{33}\partial_t^2\partial_1^2\phi + V^2E_{13}\partial_1^2\partial_3^2\phi - \partial_t^4\phi = 0 \tag{23}$$

Letting $P = \partial_t^2\phi$, yields

$$A_{11}\partial_1^2 P + A_{33}\partial_1^2 P + V^2 E_{13}\partial_1^2\partial_3^2\phi - \partial_t^2 P = 0 \tag{24}$$

To determine the value of $V^2 E_{13}$, recall that

$$E_{13} = 2\left(A_{13} + 2A_{55}\right) - \left(A_{11} + A_{33}\right). \tag{25}$$

Using equation (T.7) from Table 1 has

$$\left(1+2\eta\right)V_{NMO}^2 = \left(1+2\varepsilon\right)A_{33} = A_{11} \tag{26}$$

and the linearized term $V^2 E_{13}$ becomes

$$V^2 E_{13} = A_{11}\left[2\left(A_{13} + 2A_{55}\right) - \left(A_{11} + A_{33}\right)\right] \tag{27}$$

which for $A_{55} = 0$ reduces to

$$V^2 E_{13} = A_{11}\left[A_{13} - \left(A_{11} + A_{33}\right)\right] \tag{28}$$

Comparing the above result to that obtained by the derivation by Alkhalifah (1998b), (Appendix C) requires that the following quantities satisfy the relation

$$\left(A_{13}^2 - A_{11}A_{33}\right) \approx A_{11}\left[A_{13} - \left(A_{11} + A_{33}\right)\right] \tag{29}$$

under the assumption of weak anellipticity.

Approximation 3:

From the exact eikonal for *qP* wave propagation in a *TI* medium (Gassmann, 1964), after the approximation for moderate anellipticity has been introduced, a scalar wave equation may be constructed in a manner similar to that used in Approximation 1. The approximate eikonal in this case is given by

$$G_{qP}\left(p_1, p_3\right) = A_{11}p_1^2 + A_{33}p_3^2 + \left(A_{13}^2 - A_{11}A_{33}\right)p_1^2 p_2^2 = 1 \tag{30}$$

where the condition $A_{55} = 0$ has been introduced. In contrast to the previous case, some velocity V^2 $\left(A_{33} \le V^2 \le A_{11}\right)$ is not introduced to obtain (30). Rather, it is required that the *qP* phase velocity is replaced by the related degenerate elliptical phase velocity

$$v_{qP}^2\left(\theta\right) \approx \left[v_{qP}^{(ellip.)}\left(\theta\right)\right]^2 = \left[A_{11}\sin^2\theta + A_{33}\cos^2\theta\right] \tag{31}$$

to obtain equation (30). It should be noted at this point that it is this form of the scalar wave equation for *qP* wave propagation in a *TI* medium that was derived by Alkhalifah (1998b) and discussed in more detail in Appendix C. The assumption that $A_{55} = 0$ was introduced earlier here, than in that work, as a necessity to avoid dealing with a more complex initial approximate equation.

Some function (potential) $\phi(x,z,t)$ is introduced such that a resulting pseudo-pressure is given by $P = \partial_t^2 \phi$ to yield

$$\partial_t^2 P(x,z,t) = A_{11}\partial_x^2 P(x,z,t) + A_{33}\partial_z^2 P(x,z,t) + \left(A_{13}^2 - A_{11}A_{33}\right)\left(\partial_{xz}^2\right)^2 \phi(x,z,t) \quad (32)$$

where

$$A_D\big|_{A_{55}=0} = \left(A_{13}+A_{55}\right)^2 - \left(A_{11}-A_{55}\right)\left(A_{33}-A_{55}\right)\Big|_{A_{55}=0} = A_{13}^2 - A_{11}A_{33} \quad (33)$$

Rewriting in symmetric operator notation has

$$\partial_t^2 P(x,z,t) = \partial_x\left[A_{11}\partial_x P(x,z,t)\right] + \partial_z\left[A_{33}\partial_z P(x,z,t)\right] + \partial_{xz}^2\left[\left(A_{13}^2 - A_{11}A_{33}\right)\partial_{xz}^2 \phi(x,z,t)\right] \quad (34)$$

This problem is also solved, with zero initial conditions, and some point source excitation as in the previous case. Often the method of solution involved is to return equation (32) to Fourier operator notation and solve the resultant problem in a numerical fashion, computing the spatial derivatives using a discrete Fourier transform (*DFT*). A second order finite difference approach is then employed to obtain the temporal derivatives, i.e.

$$\partial_t^2 P(t) = \frac{P(t+\Delta t) - 2P(t) + P(t-\Delta t)}{(\Delta t)^2} \quad (35)$$

with

$$P(t) = \frac{\phi(t+\Delta t) - 2\phi(t) + \phi(t-\Delta t)}{(\Delta t)^2} \quad (36)$$

The solution of (32), using finite difference methods exclusively, is given in Rector et al. (2002).

THE DEGENERATE (ELLIPTICAL) CASE

It may be useful to look at the elliptical case of this problem as it can produce results similar to the general formulation in a much less complex manner. For this problem, the anelliptic term is zero, i.e., E_{13}, so that equations (3) and (7) become

$$G_{qP}(x_k, p_k) = A_{11}p_1^2 + A_{33}p_3^2 = 1 \quad (37)$$

and

$$\{p_1 A_{11} p_1 + p_1 A_{33} p_1 - A_{00}\} \circ \phi = 0 \tag{38}$$

respectively, where all quantities have been previously defined. The resulting unrotated problem to be solved by finite difference methods may be written in terms of the temporal and spatial partial differential operators, ∂_t, ∂_1 and ∂_3 in the following form

$$\partial_1 \left(A_{11}\partial_1\phi\right) + \partial_3 \left(A_{33}\partial_3\phi\right) - \partial_t^2\phi = F\left(x_1, x_3, t\right) \quad , \tag{39}$$

for some source term, $F\left(x_1, x_3, t\right)$, and again subject to the initial conditions

$$\phi\big|_{t=0} = \partial_t\phi\big|_{t=0} = 0 \tag{40}$$

In a rotated coordinate system transformation, discussed in Appendix B this problem becomes

$$\begin{aligned}
&\cos^2\chi\,\partial_x\left(A_{11}\partial_x\phi\right) + \sin^2\chi\,\partial_z\left(A_{11}\partial_z\phi\right) + \\
&\sin^2\chi\,\partial_x\left(A_{33}\partial_x\phi\right) + \cos^2\chi\,\partial_z\left(A_{33}\partial_z\phi\right) - \\
&\cos\chi\sin\chi\left[\partial_x\left(A_{11}\partial_z\phi\right) + \partial_z\left(A_{11}\partial_x\phi\right) - \partial_x\left(A_{33}\partial_z\phi\right) - \partial_x\left(A_{33}\partial_z\phi\right)\right] \\
&-\partial_t^2\phi = 0
\end{aligned} \tag{41}$$

which as previously stated is much simpler than the general case discussed as it is dependent only on the potential, $\phi\left(x_1, x_3, t\right)$. For this simple case the results are comparable, at least in general behavior, to the case where an anelliptic term is introduced. The approximations used in introducing the anelliptic term in subsections (2), (3) and Appendix C suggest that the use of the degenerate *qP* wave equation might produce acceptable results in the early stages of the analysis of field data.

NUMERICAL RESULTS

The model used to produce numerical results for the degenerate elliplitical qP scalar wave equation is shown in Figure (1). The anisotropic parameters and densities are defined in the figure. In the elliptical case $E_{13} \equiv 0.0 m^2/s^2$. The algorithm for this case, described in the preceding section, is computed for rotation angles of ± 30 degrees, as well as 0 degrees, in the second (elliplitical) layer. The results are shown in Figures (2) with descriptions given in the associated figure caption. Rather than a single point source of *P* – waves, a sequence of identical point sources are placed at each surface node of the finite difference grid to approximate a stacked section. In all of the trace panels, the direct arrivals have been removed from the receivers located at the surface to enhance the display of the reflected *PP* events.

It is evident in the figures that, as expected, the position of the diffracting corner between layers 3 and 4 is misplaced. The correct position of the diffracting corner, again as would be expected, lies at the midpoint of the positions indicate in the positive and negative synthetics. This can be verified by using a rotation angle of 0 degrees in the anisotropic layer.

A further numerical experiment that will be carried out here is a comparison of the homogeneous and inhomogenous scalar wave equations, which in an isotropic medium would read as

$$c^2(\mathbf{x})\nabla^2\phi(\mathbf{x},t) - \partial_t^2\phi(\mathbf{x},t) = \delta(\mathbf{x}-\mathbf{x}_0)\, f(t) \tag{42}$$

$$\nabla\cdot\left[c^2(\mathbf{x})\nabla\phi(\mathbf{x},t)\right] - \partial_t^2\phi(\mathbf{x},t) = \delta(\mathbf{x}-\mathbf{x}_0)\, f(t) \tag{43}$$

By inspection, it would appear that they would not differ much from one another, given the same velocity model. However, for the model shown in Figure (1), and the panels in Figure (3) it is clear that the do not produce the same results. The four panels include the solution obtained using a finite difference solution of the elliptical equivalents of equations (42) and (43) together with their sums and differences.

CONCLUSIONS

A linearized qP eikonal for a transversely isotropic (TI) medium is arranged into a symmetric formulation to produce, after the replacement of wave number vector components by its differential operator equivalents. The symmetric operator formulation forces differentiation of the anisotropic parameters that define the medium This is a variation of what has been presented in the literature, where these parameters have been assumed spatially independent, in similar scalar wave approximations. For generality, the possibility of an arbitrary coordinate rotation, with respect to the model system is introduced. If the problem is to be solved by finite difference methods on a rectangular grid, this angle may vary in an arbitrary manner as may the anisotropic parameters. As a reference the elliptical (degenerate) problem is briefly addressed, as it produces, for weak anelliptical models, a reference synthetic using a much less complicated finite difference algorithm. This may be useful in economically checking the results obtained by the general method.

Further other formulations of a scalar type wave equation for $P-\text{waves}$ in a *TI* medium are investigated and compared. It is determined in this process what assumptions and approximations are required to be made to arrive at an equation that is not as complicated as the first approximation presented, yet stil model the wave type in question.

As a final note, a comparison of homogeneous and inhomogeneous scalar wave equations for the *TI* elliptical scalar $P-\text{wave}$ equations. It would be thought that these would be quite similar. However, a numerical display of the results shows a marked difference, indicating that some further research is indicated in this area.

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APPENDIX A: A PENTADIAGONAL *ADI* SYSTEM SOLUTION SEPARATED INTO TWO IMPLICIT TRIDIAGONAL SYSTEMS

Consider the equation $\nabla^2\phi = P$ in a two spatial dimension system, where at the time step $n+1$, the value of $P_{i,j}^{n+1}$ is known at all $(i,j) \rightarrow (i(\Delta x), j(\Delta z))$. The operator $\nabla^2\phi$

$$\frac{\phi_{i+1,j}^{n+1} + \phi_{i-1,j}^{n+1} + \phi_{i,j+1}^{n+1} + \phi_{i,j-1}^{n+1} - 4\phi_{i,j}^{n+1}}{h^2} = P_{i,j}^{n+1} \qquad \text{(A.1)}$$

is first separated into its two constituent parts, $\partial_x^2\phi$ and $\partial_z^2\phi$ in the following manner

$$\frac{\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}}{h^2} + \frac{\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}}{h^2} = P_{i,j}^{n+1} \tag{A.2}$$

However, the quantities in the two terms on the *LHS* of equation (A.2) are just the original pentadiagonal system rewritten in another form. The equation (A.2) may be recast in either of the two following manners, with either operation assumed to be known at the time step *n*, so that either

$$\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1} = h^2 P_{i,j}^{n+1} - \phi_{i,j+1}^{n} - 2\phi_{i,j}^{n} + \phi_{i,j-1}^{n} \tag{A.3}$$

or

$$\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1} = h^2 P_{i,j}^{n+1} - \phi_{i+1,j}^{n} - 2\phi_{i,j}^{n} + \phi_{i-1,j}^{n} \tag{A.4}$$

Consider equation (A.3) and write it at the half time step $n+1/2$, for the x spatial direction as

$$\begin{aligned}\left[\phi_{i+1,j}^{n+1/2} - 2\phi_{i,j}^{n+1/2} + \phi_{i-1,j}^{n+1/2}\right] &= 2h^2 P_{i,j}^{n+1} - \\ &2\left[\phi_{i,j+1}^{n} - 2\phi_{i,j}^{n} + \phi_{i,j-1}^{n}\right] - \left[\phi_{i+1,j}^{n} - 2\phi_{i,j}^{n} + \phi_{i-1,j}^{n}\right]\end{aligned} \tag{A.5}$$

which may be followed by

$$\begin{aligned}\left[\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}\right] &= 2h^2 P_{i,j}^{n+1} - \\ &2\left[\phi_{i,j+1}^{n} - 2\phi_{i,j}^{n} + \phi_{i,j-1}^{n}\right] - \left[\phi_{i+1,j}^{n+1/2} - 2\phi_{i,j}^{n+1/2} + \phi_{i-1,j}^{n+1/2}\right]\end{aligned} \tag{A.6}$$

to obtain the values of ϕ at the $(n+1)^{th}$ time step. The alternating direction aspect of this set of equations results from using the *z* spatial direction at the next half time step and full time step to evaluate ϕ of what would now be the $(n+2)^{th}$ time step. (The time superscripts will not be incremented in what follows to reflect this.

The finite difference analogue for equation (A.4) at the half time step $n+1/2$, for the *z* spatial directions is given by

$$\begin{aligned}\left[\phi_{i,j+1}^{n+1/2} - 2\phi_{i,j}^{n+1/2} + \phi_{i,j-1}^{n+1/2}\right] &= 2h^2 P_{i,j}^{n+1} - \\ &2\left[\phi_{i+1,j}^{n} - 2\phi_{i,j}^{n} + \phi_{i-1,j}^{n}\right] - \left[\phi_{i,j+1}^{n} - 2\phi_{i,j}^{n} + \phi_{i,j-1}^{n}\right]\end{aligned} \tag{A.7}$$

which is followed by the full time step analogue for incrementing the time step count by one using the *z* spatial direction

$$\left[\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}\right] = 2h^2 P_{i,j}^{n+1} - 2\left[\phi_{i+1,j}^{n} - 2\phi_{i,j}^{n} + \phi_{i-1,j}^{n}\right] - \left[\phi_{i,j+1}^{n+1/2} - 2\phi_{i,j}^{n+1/2} + \phi_{i,j-1}^{n+1/2}\right] \tag{A.8}$$

It is to be remembered that the solutions of equations (A.5) – (A.8) over the (x, z) plane must be done using a tridiagonal procedure, such as that of Thomas (Conte and deBoor, 1972).

APPENDIX B: ROTATION THEORY FOR INITIAL PROBLEM CONSIDERED

In a rotated (primed) system, with respect to the reference or model coordinate system, the orthonormal (length preserving) transform through some angle χ of the $(ik_j,\ j = 1,3.)$ must be made and implemented, most easily in equation (8), through the relations

$$\begin{bmatrix} ik_x \\ ik_z \end{bmatrix} = \begin{bmatrix} \cos\chi & +\sin\chi \\ -\sin\chi & \cos\chi \end{bmatrix} \begin{bmatrix} ik_1 \\ ik_3 \end{bmatrix} \tag{B.1}$$

where the sign choice is dictated by how the rotated coordinates are defined with respect to the model coordinates, in other words, the manner of specifying χ.

As the rotation transformation is orthonormal, length is preserved, so that

$$(ik_1)^2 + (ik_3)^2 = (ik_x)^2 + (ik_z)^2, \tag{B.2}$$

or equivalently

$$\partial_1^2 + \partial_3^2 = \partial_x^2 + \partial_z^2. \tag{B.3}$$

This is consistent with *P*, (pressure) being a scalar quantity. Taking the inverse of equation (B.1) has

$$\begin{bmatrix} ik_1 \\ ik_3 \end{bmatrix} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix} \begin{bmatrix} ik_x \\ ik_x \end{bmatrix} \tag{B.4}$$

Implementing this rotation transformation results in equation (14) having the form

$$\Re(\chi)\left[\phi(x_1, x_3, t)\right] - \partial_t^2 P = 0 \tag{B.5}$$

which together with equation (13) may be rearranged to yield

$$\Big\{\left(\cos^2\chi\partial_x^2 - 2\cos\chi\sin\chi\partial_x\partial_z + \sin^2\chi\partial_z^2\right)A_{11}\times$$
$$\left(\cos^2\chi\partial_x^2 - 2\cos\chi\sin\chi\partial_x\partial_z + \sin^2\chi\partial_z^2\right)+$$

$$\left(\sin^2\chi\partial_x^2 + 2\cos\chi\sin\chi\partial_x\partial_z + \cos^2\chi\partial_z^2\right)A_{33}\times$$
$$\left(\sin^2\chi\partial_x^2 + 2\cos\chi\sin\chi\partial_x\partial_z + \cos^2\chi\partial_z^2\right)+ \tag{B.6}$$

$$\left(\cos\chi\sin\chi\left[\partial_x^2 - \partial_z^2\right] + \left[\cos^2\chi - \sin^2\chi\right]\partial_x\partial_z\right)\tilde{E}_{13}\times$$
$$\left(\cos\chi\sin\chi\left[\partial_x^2 - \partial_z^2\right] + \left[\cos^2\chi - \sin^2\chi\right]\partial_x\partial_z\right)-$$

$$\partial_t^2\left(\partial_x^2 + \partial_z^2\right)\Big\}\phi = 0 = \Re(\chi)[\phi] - \partial_t^2 P$$

Further rearrangement produces an extremely long and complicated equation which is not used here and as a consequence will not be pursued at this time.

APPENDIX C: AN ALTERNATE *qP* SCALAR WAVE EQUATION IN A *TI* MEDIUM

A scalar wave equation for *qP* waves in a *TI* medium, equivalent to that derived in an earlier section of the work is given by Alkhalifah (1998a). The (three dimensional) migration dispersion relation for *qP* wave propagation may be written for the case of a 2D *TI* medium as

$$p_3^2 = \frac{V^2}{V_{P_0}^2}\left[\frac{1}{v^2} - \frac{p_1^2}{1 - 2V^2\eta p_1^2}\right] \tag{C.1}$$

which in the wavenumber domain has the form

$$k_3^2 = \frac{V^2}{V_{P_0}^2}\left[\frac{\omega^2}{v^2} - \frac{\omega^2 k_1^2}{\omega^2 - 2V^2\eta k_1^2}\right] \tag{C.2}$$

for some vertical velocity $V_{P_0}^2 = A_{33}$,some undetermined velocity V^2, and the definition of the quantity η given below, is used as a starting point. Introducing the operator notation for k_1, k_3 and ω as specified in equation (6) the following partial differential equation is obtained

$$\partial_t^2 P = (1+2\eta)V^2\partial_1^2 P + V_{P_0}^2\partial_3^2 P - 2\eta V^2 V_{P_0}^2\partial_x^2\partial_z^2\phi\,. \tag{C.3}$$

where

$$P(x,z,t) = \partial_t^2 \phi(x,z,t) \tag{C.4}$$

$P(x,z,t)$ being specified as pressure.

This equation evolves in subsequent works Alkhalifah (1998b), using the *qP* eikonal in the form presented by Tsvankin (2001), to

$$\partial_t^2 P = (1+2\eta) V_{NMO}^2 \partial_1^2 P + V_{P_0}^2 \partial_3^2 P - 2\eta V_{NMO}^2 V_{P_0}^2 \partial_1^2 \partial_3^2 \phi \tag{C.5}$$

where the previously unspecified velocity V^2 has been replaced by V_{NMO}^2, the normal moveout (*NMO*) velocity. It should be mentioned at this point that the authors cited in this section indicate that for $\eta = 0$, equation (C.5) reduces to the isotropic scalar wave equation. However, $\eta = 0$ corresponds to $\varepsilon = \delta$, which is the condition for the degenerate form of *qP* wave propagation, specifically, the elliptical problem. Such may be shown to be the case if expansion of η into its constituent parameters is done before coming to the former conclusion.

Replacing the parameters in the above equation by the notation, using the previously defined, together with the sequence of definitions given in Table 1, the following partial differential equation is obtained

$$\partial_t^2 P = A_{11} \partial_1^2 P + A_{33} \partial_3^2 P + A_D A_{33}^2 \big/ \left[A_{33} \left(A_{33} - A_{55} \right) \right] \partial_1^2 \partial_3^2 \phi \tag{C.6}$$

which is very similar to equation (34) obtained earlier if written in the symmetric form

$$\partial_t^2 P = \partial_1 \left(A_{11} \partial_1 P \right) + \partial_3 \left(A_{33} \partial_3 P \right) + \partial_1 \partial_3 \left[A_D A_{33}^2 \big/ \left[A_{33} \left(A_{33} - A_{55} \right) \right] \partial_1 \partial_3 \phi \right] \tag{C.7}$$

In the papers referenced in this work, the equations referenced are usually solved with the condition $V_{S_0}^2 = A_{55} = 0$, which results in equation (C.7) having the form

$$\partial_t^2 P = \partial_1 \left(A_{11} \partial_1 P \right) + \partial_3 \left(A_{33} \partial_3 P \right) + \partial_1 \partial_3 \left[\left(A_{13}^2 - A_{11} A_{33} \right) \partial_1 \partial_3 \phi \right] \tag{C.8}$$

Additionally it follows that this problem solved with zero initial conditions

$$P\big|_{t=0} = \partial_t P\big|_{t=0} = \phi\big|_{t=0} = \partial_t \phi\big|_{t=0} = 0 \tag{C.9}$$

with the option of introducing a point source with a band limited source wavelet

$$\phi(x_s, z_s, t) = \delta(x - x_s)\,\delta(z - z_s)\, f(t), \left(0 \le t \le n_s \Delta t\right) \tag{C.10}$$

A second order finite difference approach is then employed to obtain the temporal derivatives, i.e.

$$\partial_t^2 P(t) = \frac{P(t+\Delta t) - 2P(t) + P(t-\Delta t)}{(\Delta t)^2} \tag{C.11}$$

with

$$P(t) = \frac{\phi(t+\Delta t) - 2\phi(t) + \phi(t-\Delta t)}{(\Delta t)^2} \tag{C.12}$$

As previously mentioned, the solution of (C.6), using finite difference methods exclusively, is given in Rector et al. (2002).

Table 1. Relationships between different methods of specifying parameters describing a *TI* medium.

$$V_{NMO}^2 = V_{P_0}^2 (1+\delta) = A_{33}(1+\delta) \qquad \text{(T.1)}$$

$$\eta = \frac{\varepsilon - \delta}{1+2\delta} \qquad \text{(T.2)}$$

$$\varepsilon = \frac{(A_{11} - A_{33})}{2A_{33}} \qquad \text{(T.3)}$$

$$\delta = \frac{(A_{13}+A_{55})^2 - (A_{33}-A_{55})^2}{2A_{33}(A_{33}-A_{55})} \qquad \text{(T.4)}$$

$$\varepsilon - \delta = -\left[\frac{(A_{13}+A_{55})^2 - (A_{11}-A_{55})(A_{33}-A_{55})}{2A_{33}(A_{33}-A_{55})}\right] = -\frac{A_D}{2A_{33}(A_{33}-A_{55})} \qquad \text{(T.5)}$$

$$A_D = (A_{13}+A_{55})^2 - (A_{11}-A_{55})(A_{33}-A_{55}) \qquad \text{(T.6)}$$

$$(1+2\eta)V_{NMO}^2 = (1+2\varepsilon)A_{33} = A_{11} \qquad \text{(T.7)}$$

$$\eta = \frac{\varepsilon-\delta}{1+2\delta} = \frac{-A_D}{2\left((A_{13}+A_{55})^2 + A_{55}(A_{33}-A_{55})\right)} \qquad \text{(T.8)}$$

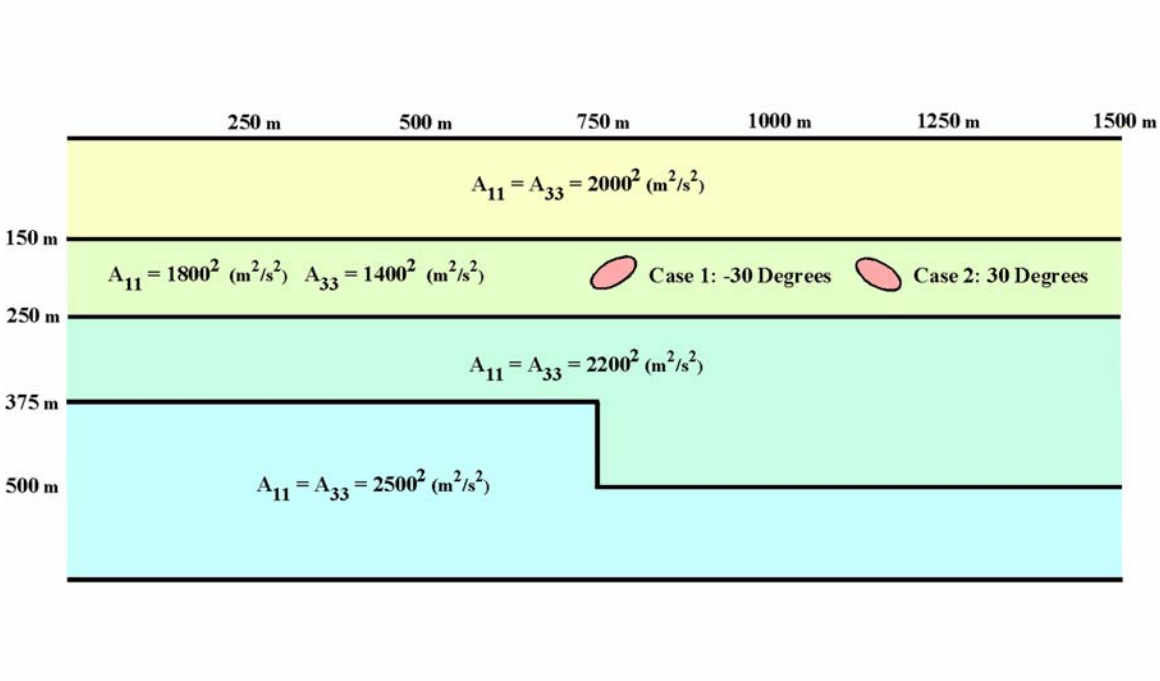

FIG. 1. Model used in computing synthetics in Figures 2 and 3.

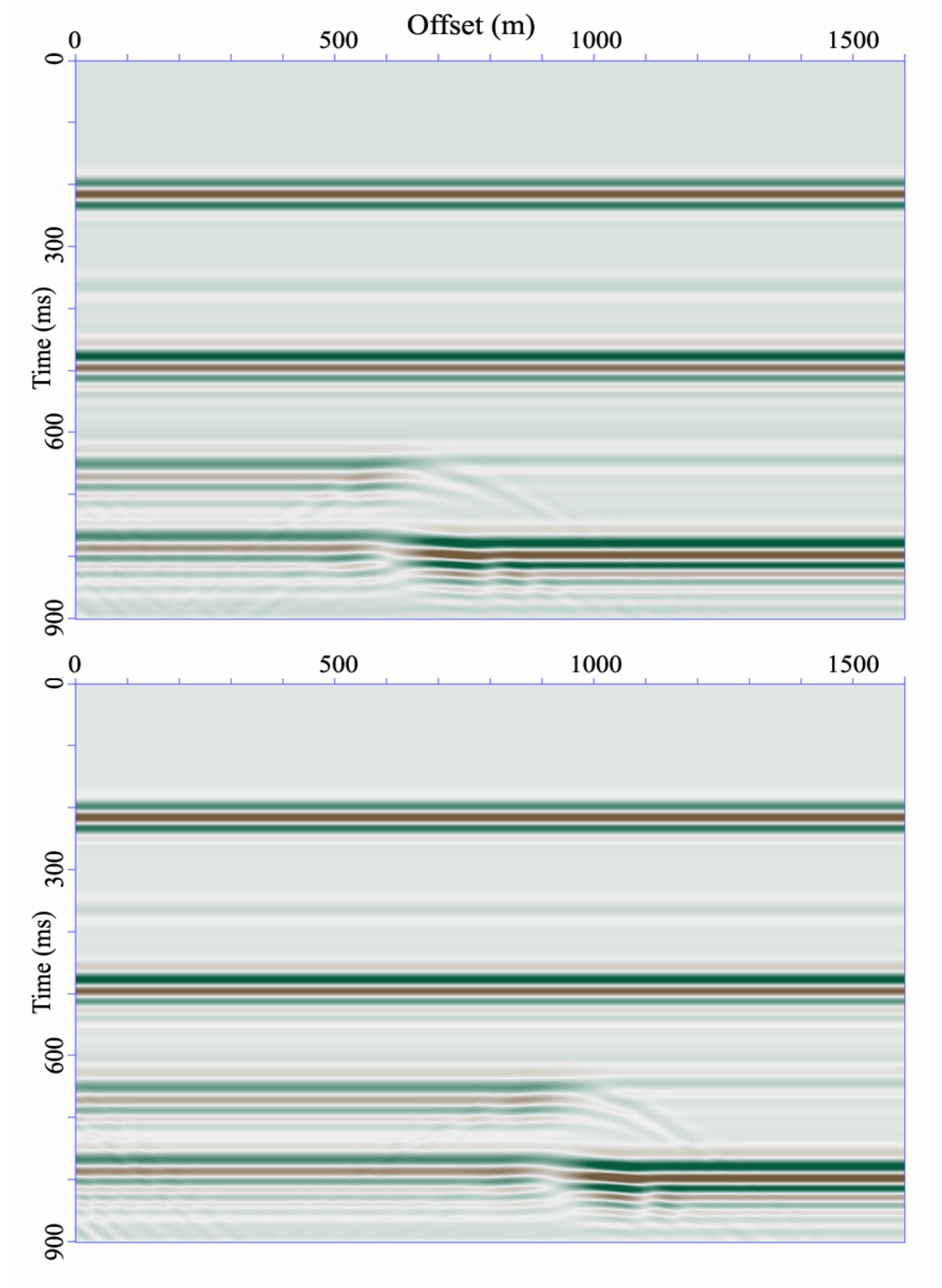

FIG. 2. Positive (upper) and negative (lower) anisotropic axes rotation as shown in Figure 1.

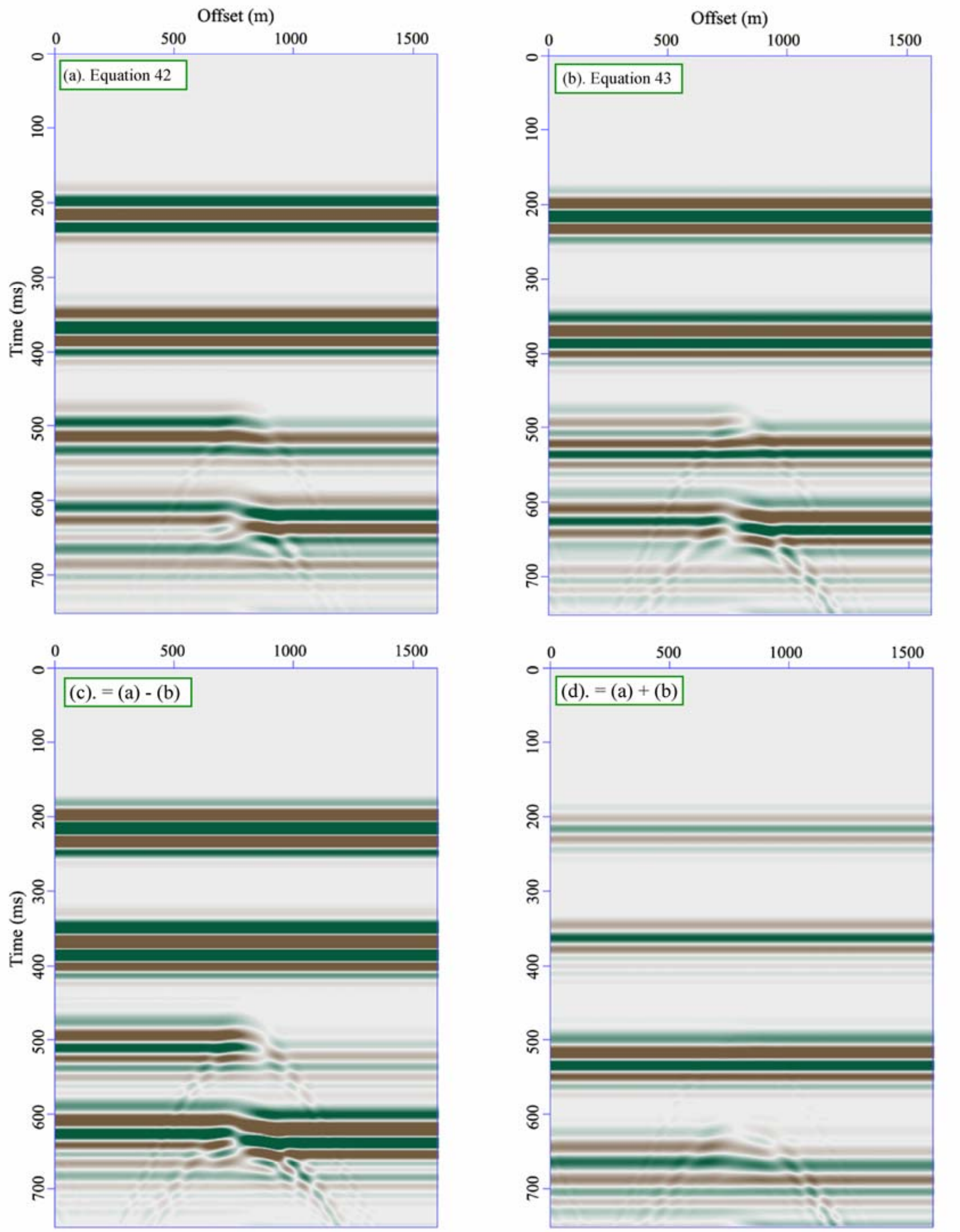

FIG. 3. Comparison of synthetics for Model 1 with no rotation in the second layer. Equations (42) and (43) are used here, panels (a) and (b). The difference of panels (a) and (b) are shown in panels (c) and (d).