Wavefield extrapolation via Sturm-Liouville transforms

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ABSTRACT

We develop a procedure for building a wavefield extrapolator using a Fourier-like transform based on solutions to the Sturm-Liouville problem. This is a report on the NSERC summer research project of the second author.

INTRODUCTION

One approach to seismic imaging is to use numerical wavefield extrapolators to model the propagation of a seismic wave through a complex medium, and using the results to deduce the reflectivity events within the medium. Designing an efficient, accurate extrapolator is a challenging project, and many trade-offs are made in practice to produce software code that works well enough for the project at hand.

In this paper, we are interested in pursuing a method to find an exact extrapolator for certain interesting, but relatively simple media – in particular, the case of a transversely inhomogeneous medium. The goal is to produce models for which we have exact solutions, and use them to test various extrapolators and establish their performance by making direct comparisons with the exact solution. This paper is a preliminary report, on a mathematical method for computing exact solutions in the transversely inhomogeneous case.

The basic approach is to use separation of variables to reduce the wave equation to a Sturm-Liouville differential equation, plus a number of simple harmonic equations. We then create a transform based on the solutions to the Sturm-Liouville DE, as well as its inverse, then use these transforms to give the complete solution to the wave equation.

We first demonstrate how this works in the 1D wave equation. We give a complete result for the two-block velocity profile, to illustrate the idea, and provide both asymmetric and symmetric transforms. We then suggest how to proceed in the 2D wave equation. More work is needed to solve this case.

THE 1D WAVE EQUATION

Beginning with the 1D wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c(x)^2}\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

we separate variables by looking for solutions of the form $u(x,t) = X(x)T(t)$ to obtain two linked ordinary differential equations (ODEs)

$$X'' + \frac{\omega^2}{c(x)^2}X = 0, \qquad T'' + \omega^2 T = 0, \tag{2}$$

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where ω^2 is the separation constant. The first equation for X is a Sturm-Liouville equation, a well-studied ordinary differential equation† that can be solved explicitly for many reasonable choices of velocity profiles $c(x)$. Since this is a second order equation, there are two independent solutions for each choice of the parameter ω^2; by separating the negative and positive values of ω, we identify a parameterized family of solutions $X_\omega(x)$, indexed by real values for ω. The second equation is solved in complex exponential form $T(t) = e^{i\omega t}$.

For fixed ω, two solutions are obtained, $X_\omega(x)e^{i\omega t}$ and $X_\omega(x)e^{-i\omega t}$. By integrating linear combinations of these two solutions, over all values of ω, we obtain the general solution to the 1D wave equation in the form

$$u(x,t) = \int_{\mathbb{R}} a(\omega)X_\omega(x)e^{i\omega t} + b(\omega)X_\omega(x)e^{-i\omega t}d\omega, \tag{3}$$

where $a(\omega), b(\omega)$ are arbitrary functions.

To determine a, b, additional conditions such as the initial value equations

$$u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x) \tag{4}$$

are required.

Plugging in to the previous equation, one obtains

$$u_0(x) = \int_{\mathbb{R}} [a(\omega) + b(\omega)]X_\omega(x)d\omega, \qquad u_1(x) = \int_{\mathbb{R}} i\omega[a(\omega) - b(\omega)]X_\omega(x)d\omega. \tag{5}$$

Let's assume that u_0, u_1 can be written uniquely as a linear combination of the Sturm-Liouville functions, with coefficients $\widehat{u_0}(\omega), \widehat{u_1}(\omega)$ respectively, thus

$$u_0(x) = \int_{\mathbb{R}} \widehat{u_0}(\omega)X_\omega(x)d\omega, \qquad u_1(x) = \int_{\mathbb{R}} \widehat{u_1}(\omega)X_\omega(x)d\omega. \tag{6}$$

We obtain two simple linear equations,

$$\widehat{u_0}(\omega) = a(\omega) + b(\omega), \qquad \widehat{u_1}(\omega) = i\omega[a(\omega) - b(\omega)] \tag{7}$$

which are easily solved for the unknowns a, b.

The mathematically challenging part is to find a definition of the 1D transform $f(x) \mapsto \widehat{f}(\omega)$ so that we can write, uniquely,

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\omega)X_\omega(x)d\omega. \tag{8}$$

This will be the transform applied to both u_0 and u_1, as we saw in Eqn. 6.

†See Courant and Hilbert (1937).

THE 1D TRANSFORM: TWO-BLOCK VELOCITY PROFILE

We look for a transform of the form

$$\widehat{f}(\omega) = \int_{\mathbb{R}} \frac{f(x)}{c(x)^2} \overline{X'_\omega(x)} dx, \tag{9}$$

where $X'_\omega(x)$ is another parameterized family of solutions to the Sturm-Liouville problem. The motivation for this form comes from the mathematical properties of certain positive definite forms related to the Sturm-Liouville equation. We omit the details of the motivation, but simply observe in the following that we can successfully find a transform in this form.

To see how this works, consider a piecewise constant velocity profile of the form

$$c(x) = \begin{cases} c_1 & \text{if } x < 0 \\ c_2 & \text{if } x > 0 \end{cases} \tag{10}$$

For each parameter ω, a solution to the related Sturm-Liouville equation is given by

$$X_\omega(x) = \begin{cases} e^{i\frac{\omega}{c_1}x} & \text{if } x < 0 \\ \frac{c_1+c_2}{2c_1} e^{i\frac{\omega}{c_2}x} + \frac{c_1-c_2}{2c_1} e^{-i\frac{\omega}{c_2}x} & \text{if } x > 0 \end{cases}, \tag{11}$$

where the constants $(c_1 \pm c_2)/2c_1$ were chosen so that $X_\omega(x)$ is continuous at $x = 0$, as is its first derivative. A dual family of solutions is defined by

$$X'_\omega(x) = \begin{cases} \frac{c_2+c_1}{2c_2} e^{i\frac{\omega}{c_1}x} + \frac{c_2-c_1}{2c_2} e^{-i\frac{\omega}{c_1}x} & \text{if } x < 0 \\ e^{i\frac{\omega}{c_2}x} & \text{if } x > 0 \end{cases} \tag{12}$$

Because the $X'_\omega(x)$ are piecewise complex exponentials, the transform defined in Eqn. 9 is closely related to the Fourier transform. In fact,

$$\widehat{f}(\omega) \equiv \int_{\mathbb{R}} \frac{f(x)}{c(x)^2} \overline{X'_\omega(x)} dx \tag{13}$$

$$= \frac{c_2 + c_1}{2c_1^2 c_2} F_-(\frac{\omega}{c_1}) + \frac{c_2 - c_1}{2c_1^2 c_2} F_-(-\frac{\omega}{c_1}) + \frac{1}{c_2^2 c_2} F_+(\frac{\omega}{c_2}), \tag{14}$$

where $F_+(\omega)$ is the usual Fourier transform of function f restricted to the positive real numbers, and $F_-(\omega)$ is the Fourier transform of f restricted to the negative reals. It is routine to verify that the inverse equation 8 is satisfied, up to a constant. In fact, we have

$$f(x) = c \int_{\mathbb{R}} \widehat{f}(\omega) X_\omega(x) d\omega., \tag{15}$$

where the constant c is just

$$c = \frac{c_1 c_2}{c_1 + c_2} \frac{1}{\sqrt{2\pi}}. \tag{16}$$

SYMMETRIC TRANSFORMS

It is interesting to note that sometimes it is possible to choose the functions $X_\omega(x), X'\omega(x)$ to be the same; that is, so that the same function appears in both the forward transform

$$\widehat{f}(\omega) = \int_{\mathbb{R}} \frac{f(x)}{c(x)^2}\overline{X_\omega(x)}dx \tag{17}$$

and its inverse

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\omega)X_\omega(x)d\omega. \tag{18}$$

In the case of the two-block velocity model, we look for solutions of the form

$$X_\omega(x) = \begin{cases} Ae^{2\pi i\frac{\omega}{c_1}x} + Be^{-2\pi i\frac{\omega}{c_1}x} & \text{if } x < 0 \\ Ce^{2\pi i\frac{\omega}{c_2}x} + De^{-2\pi i\frac{\omega}{c_2}x} & \text{if } x > 0 \end{cases} \tag{19}$$

where we have rescaled the parameter ω to measure wavenumber in natural units rather than in radians. Plugging this form into Eqns. 17,18 we obtain three equations

$$A^2 + B^2 = c_1, \qquad C^2 + D^2 = c_2, \qquad AD + BC = 0 \tag{20}$$

while requiring continuity of $X_\omega(x)$ and its derivative, at $x = 0$, gives two equations

$$A + B = C + D, \qquad \frac{A-B}{c_1} = \frac{C-D}{c_2}. \tag{21}$$

This gives five equations in four unknowns, which usually is not solvable. We got lucky, though, and find the following values give a solution:

$$A = \frac{\sqrt{c_1}}{\sqrt{2}}\frac{\sqrt{c_1}+\sqrt{c_2}}{\sqrt{c_1+c_2}} \tag{22}$$

$$B = \frac{\sqrt{c_1}}{\sqrt{2}}\frac{\sqrt{c_2}-\sqrt{c_1}}{\sqrt{c_1+c_2}} \tag{23}$$

$$C = \frac{\sqrt{c_2}}{\sqrt{2}}\frac{\sqrt{c_1}+\sqrt{c_2}}{\sqrt{c_1+c_2}} \tag{24}$$

$$D = \frac{\sqrt{c_2}}{\sqrt{2}}\frac{\sqrt{c_1}-\sqrt{c_2}}{\sqrt{c_1+c_2}}. \tag{25}$$

Thus in the two block velocity model, we are able to give explicitly a symmetric transform and inverse, using a single family of solutions to the Sturm-Liouville problem.

NUMERICAL WORK

Using MATLAB, we have verified that the 1D transform defined above works as described. That is, we have an expansion of any function in terms of solutions to the Sturm-Liouville problem, with the transform defined through the dual solutions.

We also tested the symmetric solution, where the same functions $X_\omega = X'_\omega$ are used in the forward and inverse transform, and numerically verified its properties.

THE 2D WAVE EQUATION

Consider solving the 2D wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c(x)^2}\frac{\partial^2 u}{\partial t^2}, \tag{26}$$

in spatial variables x, z, where the velocity of propagation $c = c(x)$ depends only on one spatial variable. This is a special case that is of interest when doing wavefield extrapolation in the z direction; thus the velocity variations only occur in the direction transverse to the direction of extrapolation.

By separating variables, one looks for solutions of the form $u(x, z, t) = X(x)Z(z)T(t)$ and obtains three linked ordinary differential equations (ODEs),

$$X'' + (\frac{\omega^2}{c(x)^2} - \zeta^2)X = 0, \qquad Z'' + \zeta^2 Z = 0, \qquad T'' + \omega^2 T = 0, \tag{27}$$

where ζ, ω are the separation constants. The first equation for X is again a Sturm-Liouville ordinary differential equation. For any value of parameters ζ, ω, a solution $X(x) = X_{\zeta,\omega}(x)$ can be found for the Sturm-Liouvile equation. The other two ODEs are solved using complex exponentials, with $Z(z) = e^{i\zeta z}, T(t) = e^{i\omega t}$. For fixed ζ, ω, a solution to the wave equation is obtained, in the form of the linear combination

$$u(x, z, t) = aX_{\zeta,\omega}(x)e^{i\zeta z}e^{i\omega t} + bX_{\zeta,\omega}(x)e^{i\zeta z}e^{-i\omega t}. \tag{28}$$

A general solution to the wave equation is obtained by taking arbitrary sums of these solutions, thus we expect to solve in the form

$$u(x, z, t) = \int\int a(\zeta, \omega)X_{\zeta,\omega}(x)e^{i\zeta z}e^{i\omega t} + b(\zeta, \omega)Xi_{\zeta,\omega}(x)e^{i\zeta z}e^{-i\omega t}d\zeta d\omega, \tag{29}$$

where the coefficient functions $a(\zeta, \omega), b(\zeta, \omega)$ are arbitrary.

As in the 1D case, the initial conditions at $t = 0$ are enough to specify the functions a, b, as the equations

$$\begin{aligned} u(x, z, 0) &= \int\int [a(\zeta, \omega) + b(\zeta, \omega)]X_{\zeta,\omega}(x)e^{i\zeta z}d\zeta d\omega, & (30)\\ \frac{\partial u}{\partial t}(x, z, 0) &= \int\int i\omega[a(\zeta, \omega) - b(\zeta, \omega)]X_{\zeta,\omega}(x)e^{i\zeta z}d\zeta d\omega, & (31)\end{aligned}$$

are invertible provided the transforms are properly defined.

That is, we are looking to find a definition of the 2D transform $f(x, z) \mapsto \widehat{f}(\zeta, \omega)$ so that we can write, uniquely

$$f(x, z) = \int\int \widehat{f}(\zeta, \omega)X_{\zeta,\omega}(x)e^{i\zeta z}d\zeta d\omega. \tag{32}$$

This will be the transform applied to both u_0 and u_1.

THE 2D TRANSFORM

We look for a transform of the form

$$\widehat{f}(\zeta,\omega) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(x,z)}{c(x)^2}\overline{X'_{\zeta,\omega}(x)}e^{-i\zeta z}dxdz, \tag{33}$$

where $X'_{\zeta,\omega}(x)$ is another parameterized family of solutions to the Sturm-Liouville problem. That is, the transform of f is obtained by taking the inner product of $f(x,z)/c(x)^2$ with the function $X_{\zeta,\omega}(x)Z_\zeta(z)$, so

$$\widehat{f}(\zeta,\omega) = \langle f/c^2, X'_{\zeta,\omega}Z_\zeta\rangle. \tag{34}$$

The guiding rule is that we expect to find X, X' so that

$$\langle X_{\zeta,\omega}Z_\zeta/c^2, X'_{\zeta',\omega'}Z'_\zeta\rangle = \delta(\zeta-\zeta')\delta(\omega-\omega'), \tag{35}$$

where these are Dirac delta functions.

We can verify that the functions $X_{\zeta,\omega}Z_\zeta, X'_{\zeta',\omega'}Z'_\zeta$ are orthogonal when $(\zeta,\omega)\neq(\zeta',\omega')$.

Theorem 1 *Suppose $X_{\zeta,\omega}(x), X'_{\zeta,\omega}(x)$ are solutions to the Sturm-Liouville equation*

$$\frac{d^2X}{dx^2} + (\frac{\omega^2}{c(x)^2} - \zeta^2)X = 0.$$

Then the 2D functions $(x,z)\mapsto \frac{1}{c(x)}X_{\zeta,\omega}(x)e^{i\zeta z}$, $(x,z)\mapsto \frac{1}{c(x)}X'_{\zeta',\omega'}(x)e^{i\zeta' z}$ are orthogonal on $\mathbb{R}^2$ whenever $(\zeta,\omega^2)\neq(\zeta',(\omega')^2)$.

The proof proceeds by noting that the functions $e^{i\zeta z}$, $e^{i\zeta' z}$ are orthogonal in 1D whenever $\zeta\neq\zeta'$ (this is the usual result of the Fourier transform on the real line). So when these parameters are different, we integrate over the z variable first and get zero. In the case where $\zeta=\zeta'$, integrating over z first gives infinity. So instead, we integrate over the x variable first, and consider the 1D inner product

$$\langle \frac{1}{c}X_{\zeta,\omega}, \frac{1}{c}X'_{\zeta,\omega'}\rangle = \int \frac{1}{c(x)^2}X_{\zeta,\omega}(x)\overline{X'_{\zeta,\omega'}(x)}dx. \tag{36}$$

To see this is zero, note

$$
\begin{aligned}
(\omega)^2\langle\frac{1}{c}X_{\zeta,\omega},\frac{1}{c}X'_{\zeta,\omega'}\rangle &= \int\frac{\omega^2}{c(x)^2}X_{\zeta,\omega}(x)\overline{X'_{\zeta,\omega'}(x)}dx & (37)\\
&= \int(\frac{\omega^2}{c(x)^2}-\zeta^2)X_{\zeta,\omega}(x)\overline{X'_{\zeta,\omega'}(x)}dx+\zeta^2\langle X_{\zeta,\omega},X'_{\zeta,\omega'}\rangle & (38)\\
&= \int\left(-\frac{d^2}{dx^2}X_{\zeta,\omega}(x)\right)\overline{X'_{\zeta,\omega'}(x)}dx+\zeta^2\langle X_{\zeta,\omega},X'_{\zeta,\omega'}\rangle & (39)\\
&= \int X_{\zeta,\omega}(x)\overline{\left(-\frac{d^2}{dx^2}X'_{\zeta,\omega'}(x)\right)}dx+\zeta^2\langle X_{\zeta,\omega},X'_{\zeta,\omega'}\rangle & (40)\\
&= \int X_{\zeta,\omega}(x)\overline{(\frac{(\omega')^2}{c(x)^2}-\zeta^2)X'_{\zeta,\omega'}(x)}dx+\zeta^2\langle X_{\zeta,\omega},X'_{\zeta,\omega'}\rangle & (41)\\
&= \int X_{\zeta,\omega}(x)\overline{\frac{(\omega')^2}{c(x)^2})X'_{\zeta,\omega'}(x)}dx & (42)\\
&= (\omega')^2\langle\frac{1}{c}X_{\zeta,\omega},\frac{1}{c}X'_{\zeta,\omega'}\rangle. & (43)
\end{aligned}
$$

That is,

$$((\omega)^2-(\omega')^2)\langle\frac{1}{c}X_{\zeta,\omega},\frac{1}{c}X'_{\zeta,\omega'}\rangle = 0. \quad (44)$$

So if $\omega^2 \neq (\omega')^2$, the inner product must be zero.

This is almost enough to define the transform and its inverse. The idea is that when $\omega = \omega'$, the two functions $X_{\zeta,\omega}$, $X'_{\zeta,\omega}$ are nearly coherent, and their product, over $\mathbb{R}$, will integrate to infinity. (Certainly this is the case if we choose $X = X'$ as in the symmetric transform instance.) By introducing a proper scaling, we expect the functions to integrate to Dirac delta functions. There is still work to be done here, which we will explore further with examples.

CONCLUSIONS

We have given a framework for solving a 1D and 2D wave equation exactly, in the case of a transversely inhomogeneous medium. The key step is to define a multidimensional transform, analogous to the Fourier transform, that expands any function as a linear combination of fundamental solutions to the Sturm-Liouville differential equation that appears in separating variables in the wave equation. An example of the transform is given in the 1D case, for the example of a two-block velocity profile. Steps towards a 2D transform are indicated, but not completely solved.

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